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Property (ω_1) and Single Valued Extension Property

Chen Hui SUN, Xiao Hong CAO*, Lei DAI
College of Mathematics and Information Science, Shaanxi Normal University,
Shaanxi 710062*, P. R. China*

Abstract In this note we study the property (ω_1), a variant of Weyl's theorem by means of the single valued extension property, and establish for a bounded linear operator defined on a Banach space the necessary and sufficient condition for which property (ω_1) holds. As a consequence of the main result, the stability of property (ω_1) is discussed.

Keywords property (ω_1); single valued extension property.

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1. Introduction

Weyl [1] examined the spectra of all compact perturbations of a hermitian operator on Hilbert space and found in 1909 that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. This " Weyl's theorem " has been considered by many authors. Variants have been discussed by Harte and Lee [2] and Rakočevič [3, 4]. In this note, we study a new variant of Weyl's theorem which is called property (ω_1) by means of the single valued extension property and establish for a bounded linear operator defined on a Banach space the necessary and sufficient conditions for which property (ω_1) holds. Also, the stability of property (ω_1) is discussed.

Throughout this paper, X denotes an infinite dimensional complex Banach space, and $B(X)$ ($K(X)$) denotes the algebra of all bounded linear operators (compact operators) on X. For an operator $T \in B(X)$ we shall denote by $n(T)$ the dimension of the kernel $N(T)$, and by $d(T)$ the codimension of the range $R(T)$. We call $T \in B(X)$ an upper semi-Fredholm operator if $n(T) < \infty$ and $R(T)$ is closed; But if $d(T) < \infty$ and $R(T)$ is closed, T is a lower semi-Fredholm operator. An operator $T \in B(X)$ is said to be Fredholm if $R(T)$ is closed and both the deficiency induces $n(T)$ and $d(T)$ are finite. If $T \in B(X)$ is an upper (or a lower) semi-Fredholm operator, the index of T, $\mathrm{ind}(T)$, is defined to be $\mathrm{ind}(T) = n(T) - d(T)$. The ascent of T, $\mathrm{asc}(T)$, is the least non-negative integer n such that $N(T^n) = N(T^{n+1})$ and the descent, $\mathrm{des}(T)$, is the least

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* Corresponding author
E-mail address: xiaohongcao@snnu.edu.cn (X. H. CAO)

non-negative integer n such that $R(T^n) = R(T^{n+1})$. The operator T is Weyl if it is Fredholm of index zero, and T is said to be Browder if it is Fredholm "of finite ascent and descent". The upper semi-Fredholm spectrum $\sigma_{SF_+}(T)$ is defined by: $\sigma_{SF_+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not upper semi-Fredholm$\}$. Let $\rho(T)$ denote the resolvent set of the operator T and $\sigma(T) = \mathbb{C}\backslash\rho(T)$ denote the usual spectrum of T. And let $\sigma_a(T)$ denote the approximate point spectrum of the operator $T \in B(X)$, $\rho_a(T) = \mathbb{C}\backslash\sigma_a(T)$. The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined as $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Weyl $\}$; $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not Browder $\}$. Let $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$ and $SF_+^-(X) = \{T \in B(X)$, T is upper semi-Fredholm operators and $\text{ind}(T) \leq 0\}$. Let $\sigma_{ab}(T) = \cap\{\sigma_a(T+K) : K \in K(X) \cap \text{comm}(T)\}$, it is well known that $\lambda \notin \sigma_{ab}(T)$ if and only if $T - \lambda I$ is upper semi-Fredholm and $T - \lambda I$ has finite ascent. If

$$\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T),$$

then Weyl's theorem holds for T, where $\pi_{00}(T)$ denotes the set of isolated points such that $\dim N(T - \lambda I) < \infty$; the Browder's theorem holds for T if

$$\sigma_w(T) = \sigma_b(T).$$

Let $\pi_{00}^a(T)$ be the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim N(T - \lambda I) < \infty$, T satisfies a-Weyl's theorem if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}^a(T).$$

We can prove that a-Weyl's theorem $\Longrightarrow$ Weyl's theorem $\Longrightarrow$ Browder's theorem, but the converse is generally false.

$T \in B(X)$ is said to satisfy property (ω) (see [5]) if

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}(T).$$

If T has property (ω), then Weyl's and Browder's theorem hold for T. Many fundamental results and theory of property (ω) and its stability were established by Ainena in [5] and [6].

An operator $T \in B(X)$ has single valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at $\lambda_0 \in \mathbb{C}$ for short, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$, which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$. Trivially, every operator T has SVEP at every point of the resolvent $\rho(T)$ and $\mathbb{C}\backslash\sigma_a(T)$; also T has the SVEP at $\lambda \in \partial\sigma(T)$. We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$ (see [7]).

Weyl type theorems for operators satisfying SVEP have been studied by numerous authors, see for example [8, 9]. The rest of this paper is organized as follows. In Section 2, we give the definition of property (ω_1) and a necessary and sufficient condition for T such that property (ω_1) holds. Then we study the property (ω_1) for an operator T on a Banach space such that T^* has SVEP, where T^* denotes the adjoint of T. At last, the stability of property (ω_1) is discussed.

2. Property (ω_1) and SVEP

The property (ω_1) is defined as follows:

Definition 2.1 *Property (ω_1) holds for T if*

$$\sigma_a(T)\backslash\sigma_{ea}(T) \subseteq \pi_{00}(T).$$

If T satisfies property (ω_1), then Browder's theorem holds for T. If T or T^* has SVEP, then T satisfies Browder's theorem [8]. For property (ω_1), we have:

Theorem 2.1 *$T \in B(X)$ satisfies property (ω_1) $\Longleftrightarrow$ T^* has SVEP at all $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$.*

Proof Suppose T has property (ω_1). Let $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T-\lambda I$ is Browder. Thus $T^*-\lambda I$ is Browder, which means that $\lambda \in \operatorname{iso}\sigma(T^*) \cup \rho(T^*)$. Hence T^* has SVEP at λ.

Conversely, suppose that T^* has SVEP at all $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Let $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T-\lambda I \in SF_+^-(X)$. Therefore, $T^*-\lambda I$ is lower semi-Fredholm and $\operatorname{ind}(T^*-\lambda I) \geq 0$. Since T^* has SVEP at λ, it follows that $\operatorname{asc}(T^*-\lambda I) < \infty$ (see [7, Theorem 15]). Then $\operatorname{ind}(T^*-\lambda I) \leq 0$. Thus $T^*-\lambda I$ is Weyl. The fact that T^* has SVEP at λ tells us that $T^*-\lambda I$ is Browder. Hence $\lambda \in \pi_{00}(T)$ and T satisfies property (ω_1). $\Box$

Remark 2.1 T has SVEP cannot imply property (ω_1) holds for T. For example, if $T \in B(\ell^2)$ is defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, x_3, x_4, \ldots),$$

then $\sigma(T) = \{\lambda \in \mathbb{C} : 0 \leq |\lambda| \leq 1\}$, $\pi_{00}(T) = \emptyset$. Since $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$ and $asc(T) = 1$, we know that T has SVEP. But $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_a(T)\backslash\sigma_{ea}(T) = \{0\}$. Hence T does not have property (ω_1). However, by using Theorem 2.1, T^* satisfies property (ω_1).

Theorem 2.2 *The following statements are equivalent:*

(1) $T \in B(X)$ has property (ω_1) and $\sigma_a(T) = \sigma(T)$;

(2) T^* has SVEP at all $\lambda \in \sigma(T)\backslash\sigma_{ea}(T)$;

(3) $\sigma_w(T) = \sigma_b(T)$ and for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\operatorname{ind}(T-\lambda I) \geq 0$;

(4) T has SVEP at all $\lambda \notin \sigma_{ea}(T)$ and $\sigma_{ab}(T) = \sigma_b(T)$.

Proof (1) $\Leftrightarrow$ (2). Using Theorem 2.1, we know that (1) implies (2).

For the converse, we need to prove that $\sigma_a(T) = \sigma(T)$. Let $\lambda \notin \sigma_a(T)$. Then $T-\lambda I$ is bounded from below and $T^*-\lambda I$ is surjective. If $\lambda \in \sigma(T)$, then $\lambda \in \sigma(T)\backslash\sigma_{ea}(T)$, which means that T^* has SVEP λ. Thus $\operatorname{asc}(T^*-\lambda I) < \infty$ and $T^*-\lambda I$ is invertible. Therefore $T-\lambda I$ is invertible, which is in contradiction to the fact that $\lambda \in \sigma(T)$. Hence $\sigma_a(T) = \sigma(T)$.

(1)$\Leftrightarrow$(3). Suppose T satisfies property (ω_1) and $\sigma_a(T) = \sigma(T)$. If there exists a $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, such that $\operatorname{ind}(T-\lambda I) < 0$. If $\lambda \notin \sigma_a(T)$, since $\sigma_a(T) = \sigma(T)$ we know that $T-\lambda I$ is invertible. It is in contradiction to the fact that $\operatorname{ind}(T-\lambda I) < 0$. If $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$, the condition T satisfies property (ω_1) tells us that $T-\lambda I$ is Browder. Then $\operatorname{ind}(T-\lambda I) = 0$, it is a contradiction again. Hence for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\operatorname{ind}(T-\lambda I) \geq 0$. Since property (ω_1) implies Browder's theorem, we know that $\sigma_w(T) = \sigma_b(T)$.

Conversely, let $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T-\lambda I \in SF_+^-(X)$. Since for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\operatorname{ind}(T-\lambda I) \geq 0$, it follows that $T-\lambda I$ is Weyl. The fact $\sigma_w(T) = \sigma_b(T)$ tells us that $T-\lambda I$ is

Browder. Then $\lambda \in \pi_{00}(T)$, which means that T has property (ω_1). If $\lambda \notin \sigma_a(T)$, then $T-\lambda I$ is bounded from below, $\text{ind}(T-\lambda I) \leq 0$. The condition implies that $\text{ind}(T-\lambda I) \geq 0$. Thus $T-\lambda I$ is invertible and $\lambda \notin \sigma(T)$. Hence $\sigma_a(T) = \sigma(T)$.

(1)$\Leftrightarrow$(4). Suppose T has property (ω_1) and $\sigma_a(T) = \sigma(T)$. Let $\lambda \notin \sigma_{ea}(T)$. If $T-\lambda I$ is bounded from below, then T has SVEP at λ. If $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$, since T has property (ω_1), we know that $\lambda \in \text{iso}\,\sigma(T)$. Thus T also has SVEP at λ. In what follows we will prove that $\sigma_{ab}(T) = \sigma_b(T)$. Let $\lambda \notin \sigma_{ab}(T)$. Then $T-\lambda I$ is upper semi-Fredholm, $\text{ind}(T-\lambda I) \leq 0$ and $asc(T-\lambda I) < \infty$. From (3), we know that $T-\lambda I$ is Browder.

For the converse, suppose T has SVEP at all $\lambda \notin \sigma_{ea}(T)$ and $\sigma_{ab}(T) = \sigma_b(T)$. Let $\lambda \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T-\lambda I \in SF_+^-(X)$. Since T has SVEP at λ, we know that $asc(T-\lambda I) < \infty$. Thus $\lambda \notin \sigma_{ab}(T)$, which implies that $T-\lambda I$ is Browder. Hence T satisfies property (ω_1). If $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{ab}(T)$. Thus $\lambda \notin \sigma_b(T)$ and $T-\lambda I$ is invertible. Hence $\sigma_a(T) = \sigma(T)$. □

It is well known that Browder's theorem holds for T if and only if T^* satisfies Browder's theorem. But for property (ω_1), similar consequence is generally false. For example, if $T \in B(\ell^2)$ is defined by

$$T(x_1, x_2, x_3, \ldots) = (x_1, 0, x_4, x_5, \ldots),$$

then

$$T^*(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, x_3, x_4, \ldots).$$

From Remark 2.1, we know that T has property (ω_1), but T^* doesn't satisfy property (ω_1).

Corollary 2.1 *Suppose T^* has SVEP and $\sigma_w(T^*) = \sigma_a(T^*)$, then both T and T^* have property (ω_1).*

Proof Using Theorem 2.1, we only need to prove that T^* satisfies property (ω_1). Let $\lambda_0 \in \sigma_a(T^*)\backslash\sigma_{ea}(T^*)$. Then $T^* - \lambda_0 I \in SF_+^-(X)$ and $0 < n(T^* - \lambda_0 I) < \infty$. Since T^* has SVEP, it follows that $asc(T^* - \lambda_0 I) < \infty$ ([7]). Thus there exists $\epsilon > 0$ such that $T^* - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0| < \epsilon$. For this λ, the fact $\sigma_w(T^*) = \sigma_a(T^*)$ tells us that $T^* - \lambda I$ is Weyl. Since T^* has SVEP, we know that $T^* - \lambda I$ is Browder. Then $T^* - \lambda I$ is invertible. Hence $\lambda_0 \in \text{iso}\,\sigma(T^*)$ and $\lambda_0 \in \pi_{00}(T^*)$. Therefore T^* has property (ω_1). □

Example 2.1 Let $T^* \in B(\ell^2)$ be defined by

$$T^*(x_1, x_2, x_3, \ldots) = (0, 0, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, \ldots).$$

Then T^* is quasinilpotent, hence T^* has SVEP and $\sigma_w(T^*) = \sigma_a(T^*) = \{0\}$. Using Corollary 2.1, we know that both T and T^* satisfy property (ω_1).

Remark 2.2 (1) If T^* has SVEP, the consequences (1)–(4) in Theorem 2.2 are valid.

(2) Using Theorem 2.2, we know that T^* has SVEP implies that for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$. Hence the spectral mapping theorem holds for $\sigma_{ea}(T)$. In addition, $\sigma_{ea}(T) = \sigma_w(T)$. In fact, let $\lambda \notin \sigma_{ea}(T)$. Then $T-\lambda I$ is upper semi-Fredholm and $\text{ind}(T-\lambda I) \leq 0$. Since for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\text{ind}(T-\lambda I) \geq 0$, it follows that $T-\lambda I$ is Weyl. Hence $\sigma_{ea}(T) = \sigma_w(T)$.

In the following, let $H(T)$ be the class of all complex-valued functions which are analytic on a neighborhood of $\sigma(T)$ and are not constant on any component of $\sigma(T)$. If T^* has SVEP, then for any $f \in H(T)$, $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ and $\sigma_w(f(T)) = f(\sigma_w(T))$.

Corollary 2.2 *If $T \in B(X)$, then*

(1) T^* has SVEP at all $\lambda \in \sigma(T)\backslash\sigma_{ea}(T) \Longleftrightarrow$ for any $f \in H(T)$, property (ω_1) holds for $f(T)$ and $\sigma_a(T) = \sigma(T)$.

(2) If T^* has SVEP and $\sigma_w(T^*) = \sigma_a(T^*)$, then for any $f \in H(T)$, both $f(T)$ and $f(T^*)$ satisfy property (ω_1).

Proof (1) Using Theorem 2.2, we only need to prove that if T^* has SVEP at all $\lambda \in \sigma(T)\backslash\sigma_{ea}(T)$, $f(T)$ has property (ω_1). Let $\mu_0 \in \sigma_a(f(T))\backslash\sigma_{ea}(f(T))$. Then $f(T) - \mu_0 I \in SF_+^-(X)$. Let

$$f(T) - \mu_0 I = (T - \lambda_1 I)^{n_1}(T - \lambda_2 I)^{n_2} \cdots (T - \lambda_k I)^{n_k} g(T),$$

where $\lambda_i \neq \lambda_j$ and $g(T)$ is invertible. Then $T - \lambda_i I$ is upper semi-Fredholm and $\sum_{i=1}^k \text{ind}[(T - \lambda_i I)^{n_i}] = \text{ind}(f(T) - \mu_0 I) \leq 0$. If $T - \lambda_i I$ is invertible, then $T - \lambda_i I$ is Browder. If $\lambda_i \in \sigma(T)$, since T^* has SVEP at all $\lambda \in \sigma(T)\backslash\sigma_{ea}(T)$ implies that for any $\lambda \in \mathbb{C}\backslash\sigma_{SF_+}(T)$, $\text{ind}(T - \lambda I) \geq 0$, it follows that $T - \lambda_i I$ is Weyl. Thus $T^* - \lambda_i I$ is Weyl. The fact that T^* has SVEP at λ_i tells us that $T^* - \lambda_i I$ is Browder. Then $T - \lambda_i I$ is Browder. Therefore $f(T) - \mu_0 I$ is Browder, which means that $f(T)$ has property (ω_1).

(2) If T^* has SVEP, then let $f \in H(T)$, $f(T^*) = f(T)^*$ has SVEP [10, Theorem 3.3.9]. Using Remark 2.2, we know that $\sigma_w(f(T^*)) = \sigma_w(f(T)) = f(\sigma_w(T)) = f(\sigma_w(T^*))$. The fact that $\sigma_a(f(T^*)) = f(\sigma_a(T^*))$ and Corollary 2.1 imply that both $f(T)$ and $f(T^*)$ have property (ω_1). $\square$

The Weyl's theorem for T is not sufficient for the Weyl's theorem for $T + F$ with finite rank [11]. So does a-Weyl's theorem [12]. But if $\sigma_a(T) = \sigma(T)$, we have:

Theorem 2.3 *$T \in B(X)$, if F is a compact operator commuting with T and $\sigma_a(T) = \sigma(T)$, then $T + F$ satisfies property (ω_1) if and only if property (ω_1) holds for T.*

Proof Suppose $T + F$ has property (ω_1). Let $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$. Then $T - \lambda_0 I \in SF_+^-(X)$. Thus $T + F - \lambda_0 I \in SF_+^-(X)$. If $T + F - \lambda_0 I$ is bounded from below, then $asc(T + F - \lambda_0 I) < \infty$. Thus $\text{asc}(T - \lambda_0 I) < \infty$ (see [13]). There exists $\epsilon > 0$ such that $T - \lambda I$ is bounded from below if $0 < |\lambda - \lambda_0| < \epsilon$. Since $\sigma_a(T) = \sigma(T)$, we know that $T - \lambda I$ is invertible. Then $\lambda_0 \in \text{iso}\,\sigma(T)$, which means that $T - \lambda_0 I$ is Browder. If $\lambda_0 \in \sigma_a(T + F)\backslash\sigma_{ea}(T + F)$, the fact that $T + F$ has property (ω_1) tells us that $T + F - \lambda_0 I$ is Browder. Thus $T - \lambda_0 I$ is Browder and $\lambda_0 \in \pi_{00}(T)$. Hence T satisfies property (ω_1).

Conversely, suppose property (ω_1) holds for T. Let $\lambda_0 \in \sigma_a(T + F)\backslash\sigma_{ea}(T + F)$. Then $T + F - \lambda_0 I \in SF_+^-(X)$. Thus $T - \lambda_0 I \in SF_+^-(X)$. If $T - \lambda_0 I$ is bounded from below, since $\sigma_a(T) = \sigma(T)$, we know that $T - \lambda_0 I$ is invertible. Therefore $T + F - \lambda_0 I$ is Browder. If $\lambda_0 \in \sigma_a(T)\backslash\sigma_{ea}(T)$, the condition that T has property (ω_1) implies that $T - \lambda_0 I$ is Browder. Thus property (ω_1) holds for $T + F$. $\square$

Corollary 2.3 *Suppose $T \in B(X)$, then for every compact operator F commuting with T, $T+F$ satisfies property (ω_1) and $\sigma_a(T)=\sigma(T) \Longleftrightarrow T^*$ has SVEP at all $\lambda \in \sigma(T)\backslash\sigma_{ea}(T)$.*

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