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A New Iterative Method for Finding Common Solutions of Generalized Equilibrium Problem and Fixed Point Problem in Hilbert Spaces

Min LIU*, **Shi Sheng ZHANG**
Department of Mathematics, Yibin University, Sichuan 644007, P. R. China

Abstract In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions for a generalized equilibrium problems and the set of fixed points for nonexpansive mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend and improve some recent results.

Keywords nonexpansive mapping; generalized equilibrium problem; variational inequality; continuous monotone mapping.

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1. Introduction

Throughout this paper we assume that H is a real Hilbert space and C is a nonempty closed convex subset of H. Let F be an equilibrium bifunction from $C \times C$ into R and let $A : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem: find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP, i.e.,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}.$$

In the case of $A \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is denoted by $VI(C, A)$.

A mapping $S : C \to H$ is said to be nonexpansive, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In the sequel, we denote the set of fixed points of S by $F(S)$. Recently Tada and Takahashi [2], and Takahashi and Takahashi [3] considered iterative methods for finding an element of $EP(F) \cap F(S)$. On the other hand, Takahashi and Toyoda [4] introduced an iterative method

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* Corresponding author
E-mail address: liuminybsc@yahoo.com.cn (M. LIU); changss@yahoo.cn (S. S. ZHANG)

for finding an element of $VI(C,A) \cap F(S)$, where $A : C \to H$ is an inverse-strongly monotone mapping. Very recently, Takahashi and Takahashi [1] introduced an iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is an inverse-strongly monotone mapping and then proved a strong convergence theorem.

In this paper, motivated by Takahashi and Takahashi [1], we introduce a new iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is a continuous monotone mapping and then prove a strong convergence theorem. Moreover, the method of proof adopted in the paper is different from that of [1].

2. Preliminaries

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Let H be a Hilbert space, C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C.$$

Such a mapping P_C from H onto C is called the metric projection.

Remark 1 It is well-known that the metric projection P_C has the following properties:

i) $P_C : H \to C$ is nonexpansive;

ii) P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle, \quad x, y \in H;$$

iii) For each $x \in H$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \ge 0, \quad \forall y \in C.$$

It is also known that H satisfies Opial's condition [5], that is, for any sequence$\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \ne x$.

We know that if $S : C \to C$ is a nonexpansive mapping, then the set $F(S)$ of fixed points of S is closed and convex. Further, if C is bounded, closed and convex, then $F(S)$ is nonempty.

The following is Suzuki's lemma [6] which was proved in a Banach space.

Lemma 2.1 ([6]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

For solving the equilibrium problem for bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

(A_1) $F(x,x) = 0$ for all $x \in C$;

(A_2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;

(A_3) For each $x,y,z\in C$,

$$\lim_{t\downarrow 0} F(tz+(1-t)x,y)\leq F(x,y);$$

(A_4) For each $x\in C$, $y\mapsto F(x,y)$ is a convex and lower semicontinuous.

If an equilibrium bifunction $F: C\times C\to R$ satisfies conditions (A_1)–(A_4), then we have the following two important results.

Lemma 2.2 ([7, 8]) *Let C be a nonempty closed convex subset of H and let F be an equilibrium bifunction $F: C\times C\to R$ satisfying conditions (A_1)–(A_4). Let $r>0$ and $x\in C$. Then there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in C.$$

Lemma 2.3 ([8]) *Let F be the same as given in Lemma 2.2. For given $r>0$ and $x\in C$ define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}.$$

Then the following conclusions hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\| T_rx-T_ry\|^2\leq\langle T_rx-T_ry,x-y\rangle;$$

(3) $F(T_r)=EP(F)$;

(4) $EP(F)$ is a closed and convex set.

We can also obtain the following lemma.

Lemma 2.4 ([1]) *Let C, H, F and $T_r(x)$ be as in Lemma 2.3. Then the following holds:*

$$\| T_sx-T_tx\|^2\leq\frac{s-t}{s}\langle T_sx-T_tx,T_sx-x\rangle$$

for all s, $t>0$ and $x\in H$.

Lemma 2.5 ([9]) *Assume $\{a_n\}$, is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq(1-\gamma_n)a_n+\delta_n,\quad \forall n\geq n_0,$$

where γ_n is a sequence in $(0,1)$ and δ_n is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n=\infty$;

(2) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\leq 0$ or $\sum_{n=1}^{\infty}|\delta_n|=\infty$,

then, $\lim_{n\to\infty}a_n=0$.

3. The main results

In this section, we prove a strong convergence theorem which is the main result in the paper.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H and A*

be a continuous monotone mapping from C into H. Let F be a bifunction from $C\times C$ to R which satisfies (A_1)–(A_4) and let S be a nonexpansive mappings of C into itself such that $F(S)\bigcap EP\neq\emptyset$, f be a contraction on C with a coefficient h $(0<h<1)$. Let $x_1\in C$, $\{z_n\}\subset C$ and $\{x_n\}\subset C$ be the sequence generated by

$$\begin{cases} F(z_n,y)+\langle Az_n,y-z_n\rangle+\dfrac{1}{r_n}\langle y-z_n,z_n-x_n\rangle\geq 0,\ \forall\ y\in C,\\ x_{n+1}=\beta_nx_n+(1-\beta_n)Sy_n,\\ y_n=\alpha_nf(x_n)+(1-\alpha_n)z_n,\end{cases}\qquad \forall\ n\in N, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$. If the following conditions are satisfied

(C_1) $\lim_{n\to\infty}\alpha_n=0$;

(C_2) $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(C_3) $\lim_{n\to\infty}(r_{n+1}-r_n)=0$; $\liminf_{n\to\infty}r_n>0$,

(C_4) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,

then $\{x_n\}$ converges strongly to p, where $p=P_{F(S)\cap EP}f(p)$.

Proof We define a bifunction $G:C\times C\to R$ by

$$G(z,y)=F(z,y)+\langle Az,y-z\rangle,\quad \forall z,y\in C.$$

Next, we prove that the bifunction G satisfies conditions (A_1)–(A_4):

(A_1) $G(x,x)=0$ for all $x\in C$.

Since $G(x,x)=F(x,x)+\langle Ax,0\rangle=F(x,x)=0$, for all $x\in C$.

(A_2) G is monotone, i.e., $G(z,y)+G(y,z)\leq 0$ for all $y,z\in C$.

Since A is monotone, from the definition of G we have

$$\begin{aligned} G(z,y)+G(y,z)&=F(z,y)+\langle Az,y-z\rangle+F(y,z)+\langle Ay,z-y\rangle\\ &=F(z,y)+F(y,z)+\langle Az,y-z\rangle-\langle Ay,y-z\rangle\\ &\leq 0+\langle Az-Ay,y-z\rangle=-\langle Ay-Az,y-z\rangle\leq 0.\end{aligned}$$

(A_3) For each $x,y,z\in C$,

$$\lim_{t\downarrow 0}G(tz+(1-t)x,y)\leq G(x,y).$$

Since A is continuous, we have

$$\begin{aligned} &\lim_{t\downarrow 0}G(tz+(1-t)x,y)\\ &\quad=\lim_{t\downarrow 0}F(tz+(1-t)x,y)+\lim_{t\downarrow 0}\langle A(tz+(1-t)x),y-(tz+(1-t)x)\rangle\\ &\quad\leq F(x,y)+\langle Ax,y-x\rangle=G(x,y).\end{aligned}$$

(A_4) For each $x\in C$, $y\mapsto G(x,y)$ is a convex and lower semicontinuous.

For each $x\in C$, $\forall t\in(0,1)$ and $\forall y,z\in C$, since F satisfies (A_4), we have

$$\begin{aligned} G(x,ty+(1-t)z)&=F(x,ty+(1-t)z)+\langle Ax,ty+(1-t)z-x\rangle\\ &\leq t[F(x,y)+\langle Ax,y-x\rangle]+(1-t)[F(x,z)+\langle Ax,z-x\rangle]\end{aligned}$$

$$= tG(x,y) + (1-t)G(x,z).$$

So, $y \mapsto G(x,y)$ is convex.

Similarly, we can prove that $y \mapsto G(x,y)$ is lower semi-continuous.

Therefore, the generalized equilibrium problem (1.1) is equivalent to the following equilibrium problem: find $z \in C$ such that

$$G(z,y) \geq 0, \quad \forall y \in C,$$

and (3.1) can be written as:

$$\begin{cases} G(z_n,y) + \dfrac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, & \forall\, n \in N \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)z_n. \end{cases} \tag{3.2}$$

Let $Q = P_{F(S)\cap EP(G)}$. Note that f is a contraction with coefficient $h \in (0,1)$. Then, we have

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq h\|x - y\|$$

for all $x,\ y \in C$. Therefore, $Q(f)$ is a contraction of C into itself, which implies that there exists a unique element $p \in C$ such that $p = Qf(p) = P_{F(S)\cap EP(G)}f(p)$.

Since the bifunction G satisfies conditions (A_1)–(A_4), from Lemma 2.3, for given $r > 0$ and $x \in C$, we can define a mapping $W_r : H \to C$ as follows:

$$W_r(x) = \{z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}.$$

Moreover, W_r satisfies the conclusions in Lemma 2.3.

We divide the proof of Theorem 3.1 into six steps:

Step 1. First prove the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$ and $\{W_{r_n}x_n\}$ are bounded.

(a) Pick $p \in F(S) \cap EP(G)$. Since $z_n = W_{r_n}x_n$ and $p = W_{r_n}p$, we have

$$\|z_n - p\| = \|W_{r_n}x_n - W_{r_n}p\| \leq \|x_n - p\|. \tag{3.3}$$

(b) From (3.2) and (3.3), we have

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)z_n - p\| \\ &= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\| \\ &\leq \alpha_n\|f(x_n) - f(p) + f(p) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\leq \alpha_n h\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= (1 - \alpha_n(1-h))\|x_n - p\| + \alpha_n\|f(p) - p\|. \end{aligned} \tag{3.4}$$

Form (3.2), (3.3) and (3.4) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(x_n - p) + (1-\beta_n)(Sy_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\{(1-\alpha_n(1-h))\|x_n - p\| + \alpha_n\|f(p) - p\|\} \\ &= (1-\alpha_n(1-h))\|x_n - p\| + \alpha_n(1-\beta_n)\|f(p) - p\| \end{aligned}$$

$$\begin{aligned}&\le (1-\alpha_n(1-h))\|x_n-p\|+\alpha_n\|f(p)-p\|\\&\le \max\{\|x_n-p\|,\frac{1}{1-h}\|f(p)-p\|\}\\&\quad\vdots\\&\le \max\{\|x_1-p\|,\frac{1}{1-h}\|f(p)-p\|\}.\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H. Therefore $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$ and $\{W_{r_n}x_n\}$ are all bounded.

Step 2. Next we prove that

$$\|x_{n+1}-x_n\|\to 0,\quad n\to\infty. \tag{3.5}$$

Since $z_n=W_{r_n}x_n$, we have

$$\begin{aligned}y_{n+1}-y_n&=\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})z_{n+1}-[\alpha_nf(x_n)+(1-\alpha_n)z_n]\\&=\alpha_{n+1}f(x_{n+1})-\alpha_nf(x_n)+(1-\alpha_{n+1})z_{n+1}-(1-\alpha_n)z_n\\&=\alpha_{n+1}(f(x_{n+1})-f(x_n))+f(x_n)(\alpha_{n+1}-\alpha_n)+\\&\quad(1-\alpha_{n+1})[W_{r_{n+1}}x_{n+1}-W_{r_{n+1}}x_n+W_{r_{n+1}}x_n-W_{r_n}x_n+W_{r_n}x_n]-(1-\alpha_n)W_{r_n}x_n.\end{aligned}$$

So, we have

$$\begin{aligned}\|y_{n+1}-y_n\|\le&\alpha_{n+1}h\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\cdot\|f(x_n)\|+(1-\alpha_{n+1})\|x_{n+1}-x_n\|+\\&(1-\alpha_{n+1})\|W_{r_{n+1}}x_n-W_{r_n}x_n\|+|\alpha_{n+1}-\alpha_n|\cdot\|W_{r_n}x_n\|.\end{aligned}$$

Then

$$\begin{aligned}\|Sy_{n+1}-Sy_n\|&\le\|y_{n+1}-y_n\|\\&\le(1-\alpha_{n+1}(1-h))\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\cdot\|f(x_n)\|+\\&\quad|\alpha_{n+1}-\alpha_n|\cdot\|W_{r_n}x_n\|+(1-\alpha_{n+1})\|W_{r_{n+1}}x_n-W_{r_n}x_n\|.\end{aligned}$$

Let $s=r_{n+1}$, $t=r_n$ and $x=x_n$. From Lemma 2.4, we have

$$\begin{aligned}&\|Sy_{n+1}-Sy_n\|\\&\le\|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\cdot\|f(x_n)\|+|\alpha_{n+1}-\alpha_n|\cdot\|W_{r_n}x_n\|+\\&\quad(1-\alpha_{n+1})\{\frac{|r_{n+1}-r_n|}{r_{n+1}}|\langle W_{r_{n+1}}x_n-W_{r_n}x_n,\ W_{r_{n+1}}x_n-x_n\rangle|\}^{\frac{1}{2}}.\end{aligned}$$

From $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and conditions (C_2), (C_3) it follows

$$\limsup_{n\to\infty}(\|Sy_{n+1}-Sy_n\|-\|x_{n+1}-x_n\|)\le 0.$$

From Lemma 2.1, we get

$$Sy_n-x_n\to 0. \tag{3.6}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|Sy_n-x_n\|=0.$$

Step 3. Next we prove that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.7}$$

For each $p \in F(S) \cap EP(G)$, since $z_n = W_{r_n} x_n$, we have

$$\begin{aligned}\|z_n - p\|^2 &= \|W_{r_n} x_n - W_{r_n} p\|^2 \leq \langle x_n - p, z_n - p\rangle \\ &= \frac{1}{2}\{\|x_n - p\|^2 + \|z_n - p\|^2\} - \|x_n - z_n\|^2.\end{aligned}$$

So, we get

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - 2\|x_n - z_n\|^2. \tag{3.8}$$

Then, from (3.2) and (3.8), it follows

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(Sy_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|\alpha_n(f(x_n) - p) + (1-\alpha_n)(z_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\{\alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|z_n - p\|^2\} \\ &\leq \beta_n\|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 + (1-\beta_n)\{\|x_n - p\|^2 - 2\|x_n - z_n\|^2\} \\ &= \|x_n - p\|^2 + \alpha_n\|f(x_n) - p\|^2 - 2(1-\beta_n)\|x_n - z_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}2(1-\beta_n)\|x_n - z_n\|^2 &\leq (\|x_n - p\| - \|x_{n+1} - p\|)\cdot(\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\|f(x_n) - p\|^2 \\ &\leq \|x_{n+1} - x_n\|\cdot(\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\|f(x_n) - p\|^2.\end{aligned}$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and condition (C_4) we have

$$\|x_n - z_n\| \to 0.$$

Step 4. Next we prove that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0, \quad \lim_{n\to\infty}\|y_n - x_n\| = 0 \text{ and } \lim_{n\to\infty}\|Sy_n - y_n\| = 0.$$

Since $y_n = \alpha_n f(x_n) + (1-\alpha_n)z_n$, we have $y_n - z_n = \alpha_n(f(x_n) - z_n)$. Hence

$$\|y_n - z_n\| = \alpha_n\|f(x_n) - z_n\| \to 0, \text{ and so } \|y_n - x_n\| \to 0. \tag{3.9}$$

Since

$$\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - z_n\| + \|z_n - y_n\|,$$

from (3.6), (3.7) and (3.9) we have

$$\|Sy_n - y_n\| \to 0. \tag{3.10}$$

Step 5. Next we prove that

$$\limsup_{n\to\infty}\langle f(p) - p, y_n - p\rangle \leq 0, \text{ where } p = P_{F(S)\cap EP(G)}f(p). \tag{3.11}$$

For the purpose, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(p) - p, y_n - p\rangle = \lim_{i\to\infty}\langle f(p) - p, y_{n_i} - p\rangle. \tag{3.12}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence of $\{y_{n_i}\}$, without loss of generality, we still denote it by $\{y_{n_i}\}$ such that $y_{n_i} \rightharpoonup w$.

Now we show that $w \in F(S) \cap EP(G)$.

(a) First we prove that $w \in EP(G)$.

From (3.9), we have $z_{n_i} \rightharpoonup w$. Since $z_n = W_{r_n} x_n$, we have

$$G(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

By condition (A_2)

$$\langle y - z_n, \frac{z_n - x_n}{r_n}\rangle \geq -G(z_n, y) \geq G(y, z_n).$$

Hence we have

$$\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq G(y, z_{n_i}).$$

Since $\frac{z_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, $z_{n_i} \rightharpoonup w$, by condition ($A_4$), we have $G(y, w) \leq 0$ for all $y \in C$. For any t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$, we see $G(y_t, w) \leq 0$. From conditions (A_1) and (A_4), we have

$$0 = G(y_t, y_t) \leq tG(y_t, y) + (1-t)G(y_t, w) \leq tG(y_t, y).$$

This implies that $G(y_t, y) \geq 0$. Hence from condition (A_3), we have $G(w, y) \geq 0$ for all $y \in C$, and hence $w \in EP(G)$.

(b) Now we prove that $w \in F(S)$.

If not, we have $w \neq Sw$. From Opial's Lemma [6] and (3.10), we have

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Sw\| = \liminf_{i\to\infty} \|y_{n_i} - Sy_{n_i} + Sy_{n_i} - Sw\| \\ &\leq \liminf_{i\to\infty} \|y_{n_i} - w\|.\end{aligned}$$

This is a contradiction. So, we have $w \in F(S)$. Since $w \in F(S) \cap EP(G)$, from (3.12) and the property of metric projection, we have

$$\limsup_{n\to\infty}\langle f(p) - p, y_n - p\rangle = \lim_{i\to\infty}\langle f(p) - p, y_{n_i} - p\rangle = \langle f(p) - p, w - p\rangle \leq 0.$$

It follows from (3.9) and (3.11) that

$$\limsup_{n\to\infty}\langle f(p) - p, x_n - p\rangle \leq \limsup_{n\to\infty}\langle f(p) - p, x_n - y_n\rangle + \limsup_{n\to\infty}\langle f(p) - p, y_n - p\rangle \leq 0. \quad (3.13)$$

Step 6. Finally, we prove that

$$x_n \to p = P_{F(S)\cap EP(G)} f(p). \quad (3.14)$$

Indeed, from (3.2), (3.3) and (3.4), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Sy_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\{(1-\alpha_n)^2\|z_n - p\|^2 + 2\alpha_n\langle f(x_n) - p, y_n - p\rangle\} \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)(1 - 2\alpha_n + \alpha_n^2)\|x_n - p\|^2 + 2\alpha_n(1-\beta_n)\langle f(x_n) - p, y_n - x_n\rangle +\end{aligned}$$

$$
\begin{aligned}
& \quad 2\alpha_n(1-\beta_n)\langle f(x_n)-p, x_n-p\rangle \\
&= (1-2\alpha_n(1-\beta_n))\|x_n-p\|^2 + (1-\beta_n)\alpha_n\{\alpha_n\|x_n-p\|^2 + 2\langle f(x_n)-p, y_n-x_n\rangle\}+ \\
& \quad 2\alpha_n(1-\beta_n)\langle f(x_n)-p, x_n-p\rangle. \qquad (3.15)
\end{aligned}
$$

Since

$$\langle f(x_n)-p, x_n-p\rangle = \langle f(x_n)-f(p)+f(p)-p, x_n-p\rangle \le h\|x_n-p\|^2 + \langle f(p)-p, x_n-p\rangle, \quad (3.16)$$

substituting (3.16) into (3.15), after simplifying, we have

$$\|x_{n+1}-p\|^2 \le (1-2\alpha_n(1-\beta_n)(1-h))\|x_n-p\|^2 + \delta_n \quad \forall n \ge 0,$$

where $\delta_n = \alpha_n(1-\beta_n)\{\alpha_n M + 2\langle f(x_n)-p, y_n-x_n\rangle + 2\langle f(p)-p, x_n-p\rangle\}$ with $M = \sup_{n\ge 0}\|x_n - p\|^2$. Let $\gamma_n = 2\alpha_n(1-\beta_n)(1-h) \in (0,1)$. By the assumptions, we have $\sum_{n=0}^{\infty}\gamma_n = \infty$. Since $\alpha_n \to 0$ and $\|y_n - x_n\| \to 0$, from (3.13), it follows

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} &= \limsup_{n\to\infty}\frac{1}{2(1-h)}\{\alpha_n M + 2\langle f(x_n)-p, y_n-x_n\rangle + 2\langle f(p)-p, x_n-p\rangle\} \\
&\le 0.
\end{aligned}
$$

By Lemma 2.5, it yields $\|x_n - p\| \to 0$. i.e., $x_n \to p$.

This completes the proof of Theorem 3.1. □

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R which satisfies (A_1)–(A_4) and let S be a nonexpansive mapping of C into itself such that $F(S)\bigcap EP(F) \ne \emptyset$, f be a contraction on C with a coefficient h $(0 < h < 1)$. Let $x_1 \in C$, $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be the sequence generated by*

$$
\begin{cases}
F(z_n, y) + \dfrac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \ \forall\, y \in C, \\
x_{n+1} = \beta_n x_n + (1-\beta_n)Sy_n, & \forall\, n \in N, \\
y_n = \alpha_n f(x_n) + (1-\alpha_n)z_n,
\end{cases} \qquad (3.17)
$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. If the following conditions are satisfied

(C_1) $\lim_{n\to\infty}\alpha_n = 0$;

(C_2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C_3) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$; $\liminf_{n\to\infty} r_n > 0$;

(C_4) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

then $\{x_n\}$ converges strongly to p, where $p = P_{F(S)\cap EP(F)}f(p)$.

Proof Taking $A \equiv 0$ in Theorem 3.1 gives $F = G$. Hence the conclusion of Theorem 3.2 can be obtained form Theorem 3.1 immediately.

Theorem 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a continuous monotone mapping from C into H and let S be a nonexpansive mapping of C into itself such that $F(S)\bigcap VI(C,A) \ne \emptyset$, f be a contraction on C with a coefficient h $(0 < h < 1)$.*

Let $x_1 \in C$, $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be the sequence generated by

$$\begin{cases} \langle Az_n, y - z_n\rangle + \dfrac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall\, y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S y_n, \\ y_n = \alpha_n f(x_n) + (1-\alpha_n) z_n, \end{cases} \quad \forall\, n \in N, \tag{3.18}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$. If the following conditions are satisfied

(C_1) $\lim_{n\to\infty} \alpha_n = 0$;

(C_2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C_3) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$; $\liminf_{n\to\infty} r_n > 0$;

(C_4) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to p, where $p = P_{F(S)\cap VI(C,A)} f(p)$.

Proof Taking $F \equiv 0$ in Theorem 3.1 gives

$$\langle Az_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \ \forall n \in N.$$

Hence the conclusion of Theorem 3.3 can be obtained form Theorem 3.1 immediately. □

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