Journal of Mathematical Research & Exposition
Nov., 2010, Vol. 30, No. 6, pp. 1078–1082
DOI:10.3770/j.issn:1000-341X.2010.06.017
Http://jmre.dlut.edu.cn

Incompleteness of Complex Exponential System in L^p_α Space

Xie Quan FAN, Guan Tie DENG*
Key Laboratory of Mathematics and Complex Systems, Ministry of Education, China,
School of Mathematical Sciences, Beijing Normal University, Beijing 100875*, P. R. China*

Abstract A necessary and sufficient condition is obtained for the incompleteness of complex exponential system in the weighted Banach space $L^p_\alpha=\{f:\int_{-\infty}^{+\infty}|f(t)e^{-\alpha(t)}|^p\mathrm{d}t<+\infty\}$, where $1\le p<+\infty$ and $\alpha(t)$ is a weight on $\mathbf{R}$.

Keywords incompleteness; exponential system; approximation.

Document code A
MR(2000) Subject Classification 31A10; 31A15
Chinese Library Classification O174.5; O174.52

1. Introduction

A system $E=\{e_k:k=1,2,\ldots\}$ of elements of a Banach space B is called *incomplete* if $\overline{\mathrm{span}}E$ does not coincide with the whole B, where $\mathrm{span}E$ is the linear span of the system E and the $\overline{\mathrm{span}}E$ is the closure of $\mathrm{span}E$ in B.

Suppose $\alpha(t)$ is a nonnegative continuous function, called a weight, on $\mathbf{R}$, and satisfies

$$\lim_{t\to+\infty}t^{-1}\alpha(t)=+\infty,\quad a_0=\limsup_{t\to-\infty}|t|^{-1}\alpha(t)<+\infty. \tag{1}$$

Given a weight $\alpha(t)$, we take the weighted Banach space C_α consisting of complex continuous functions $f(t)$ defined on $\mathbf{R}$ with $f(t)\exp(-\alpha(t))$ vanishing at infinity and the norm $\|f\|_\alpha=\sup\{|f(t)e^{-\alpha(t)}|:t\in\mathbf{R}\}$. Suppose $L^p_\alpha=\{f:||f||=(\int_{-\infty}^{+\infty}|f(t)e^{-\alpha(t)}|^p\mathrm{d}t)^{\frac{1}{p}}<+\infty\}$, $1\le p<+\infty$. Then L^p_α is also a Banach space. Let $\Lambda=\{\lambda_n:n=1,2,\ldots\}$ be a sequence of distinct complex numbers in the half plane $\mathbf{C}_{a_0}=\{z=x+iy:x>a_0\}$ satisfying

$$a_1=\sup_n|\theta_n|<\frac{\pi}{2}. \tag{2}$$

Let $M=\{m_n:n=1,2,\ldots\}$ be a sequence of positive integers and suppose that there exists an increasing positive function $q(r)$ on $[0,\infty)$ satisfying

$$a_2=\limsup_{r\to+\infty}q(r)r^{-1}\log r<+\infty; \tag{3}$$

Received October 22, 2008; Accepted September 15, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10671022) and the Research Fund for the Doctoral of Higher Education (Grant No. 20060027023).
* Corresponding author
E-mail address: fanxiequan@hotmail.com (X. Q. FAN); denggt@bnu.edu.cn (G. T. DENG)

$$D(q)=\limsup_{r\to+\infty}\frac{n(r+q(r))-n(r)}{q(r)}<+\infty, \tag{4}$$

where $n(t)=\sum_{|\lambda_n|\le t}m_n$. We denote the system of complex exponentials by

$$E(\Lambda,M)=\{t^{k-1}e^{\lambda_n t}:k=1,2,\ldots,m_n;n=1,2,\ldots\}.$$

The condition (1) guarantees that $E(\Lambda,M)$ is a subset of C_α and L^p_α. In the article [1], the author has obtained some results on incompleteness of $E(\Lambda,M)$ in C_α. Now we ask whether $E(\Lambda,E)$ is incomplete in L^p_α in the norm $\|\ \|$. The similar results were obtained in the articles [2], [3] and [4].

Theorem A ([1]) *Let $\alpha(t)$ be continuous on $\boldsymbol{R}$ and convex on $[t_0,\infty)$ for some constant t_0, and satisfy* (1). *Suppose that $\Lambda=\{\lambda_n=|\lambda_n|e^{i\theta_n}:n=1,2,\ldots\}$ is a sequence of distinct complex numbers in $\boldsymbol{C}_{a_0}$ satisfying (2) and $M=\{m_n:n=1,2,\ldots\}$ is a sequence of positive integers. If there exists a positive and increasing function $q(r)$ on $[0,\infty)$ such that (3) and (4) hold, then $E(\Lambda,M)$ is incomplete in C_α if and only if there exists $a\in\boldsymbol{R}$ such that*

$$J(a)=\int_0^{+\infty}\frac{\alpha(\lambda(t)+a)}{1+t^2}\mathrm{d}t<+\infty, \tag{5}$$

where

$$\lambda(r)=\begin{cases}2\displaystyle\sum_{|\lambda_n|\le r}\frac{m_n\cos\theta_n}{|\lambda_n|}\mathrm{d}t, & \text{if } r\ge|\lambda_1|,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 1 *Assume $\alpha(t)$, Λ and M satisfy the same conditions as Theorem A. If there exists a positive and increasing function $q(r)$ on $[0,\infty)$ such that (3) and (4) hold, then $E(\Lambda,M)$ is incomplete in L^p_α if and only if there exists $a\in\boldsymbol{R}$ such that (5) holds.*

2. Lemmas and Proof of Theorem 1

In order to prove Theorem 1, we need the following technical lemmas.

Lemma 1 ([5]) *Let $\beta(x)$ be a convex function on $[0,\infty)$ and assume that*

$$\beta^*(t)=\sup\{xt-\beta(x):x\ge0\},\quad t\in\boldsymbol{R} \tag{7}$$

is the Legendre transform (or the Young dual function) ([6]) of $\beta(x)$. Suppose that $\lambda(r)$ is an increasing function on $[0,\infty)$ satisfying

$$\lambda(R)-\lambda(r)\le A(\log R-\log r+1),\quad R>r>1. \tag{8}$$

Then there exists an analytic function $f(z)\not\equiv0$ in $\boldsymbol{C}_0=\{z=x+iy:x>0\}$ satisfying

$$|f(z)|\le A\exp\{Ax+\beta(x)-x\lambda(|z|)\},\quad z=x+iy\in\boldsymbol{C}_0), \tag{9}$$

if and only if there exists $a\in\boldsymbol{R}$ such that

$$\int_1^{+\infty}\frac{\beta^*(\lambda(t)+a)}{1+t^2}\mathrm{d}t<+\infty. \tag{10}$$

Remark We denote a positive constant by A, not necessarily the same at each occurrence.

Lemma 2 ([1]) *Suppose that $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers satisfying (2) and $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of positive integers. If there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (3) and (4) hold, then for each $b > 0$, the function*

$$G_b(z) = Q_b(z) \prod_{\mathrm{Re}\lambda_n > b} \Big(\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\bar{\lambda}_n}} \Big)^{m_n} \exp\Big(\frac{2zm_n \cos\theta_n}{|\lambda_n|} \Big) \tag{11}$$

is meromorphic and analytic in the half-plane $\boldsymbol{C}_{-b} = \{z = x + iy : x > -b\}$ with zeros of orders m_n at each points λ_n $(n = 1, 2, \ldots)$ and satisfies the following inequality

$$|G_b(z)| \le \exp\{|x|\lambda(2r) + A|x| + A\}, \quad z \in \boldsymbol{C}_{-b}, \tag{12}$$

where

$$Q_b(z) = \prod_{|\mathrm{Re}\lambda_n| \le b} (\frac{z - \lambda_n}{z + b + 1})^{m_n}.$$

Moreover, for each positive constant A_0 and $\varepsilon_0 > 0$,

$$|G_b(z)| \ge \exp\{x\lambda(r) - A|x| - A\}, \quad z \in C(A_0, \varepsilon_0)), \tag{13}$$

where $C(A_0, \varepsilon_0) = \{z \in \boldsymbol{C}_{-b} : |z - \lambda_n| \ge \delta_n, n = 1, 2, \ldots\}$, $\delta_n = \epsilon_0|\lambda_n|^{-A_0}, n = 1, 2, \ldots$.

Proof If the system $E(\Lambda, M)$ is incomplete in L^p_α, then by Hahn-Banach Theorem, there exists a bounded linear function T on L^p_α such that

$$\| T \| = 1 \;\text{ and }\; T(t^{k-1}e^{\lambda_n t}) = 0, \quad k = 1, 2, \ldots, m_n;\; \lambda_n \in \Lambda.$$

So by Riesz representation theorem, there exists a $g \in L^q_{-\alpha}$, such that $\|T\| = \|g\|_{q,-\alpha}$ and $T(f) = \int_{-\infty}^{+\infty} f(t)g(t)\mathrm{d}t$ $(f \in L^p_\alpha)$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$L^q_{-\alpha} = \{g : \|g\|_{q,-\alpha} = (\int_{-\infty}^{+\infty} |g(t)e^{\alpha(t)}|^q \mathrm{d}t)^{\frac{1}{q}} < +\infty\};$$

$$L^\infty_{-\alpha} = \{g : \|g\|_{\infty,-\alpha} = \text{ess sup}\{|g(t)|e^{\alpha(t)} : t \in \mathbf{R}\} < +\infty\}.$$

For each $b > a_0 + 1$, the function

$$f(z) = \frac{1}{G_b(z)} \int_{-\infty}^{+\infty} e^{t(z+b)} g(t)\mathrm{d}t, \quad x > a_0$$

is analytic in $\mathbf{C}_{-1} = \{z = x + iy : x > -1\}$, where $G_b(z)$ is defined by (11) with zeros $\lambda_n - b : n = 1, 2, \ldots$. By the Lemma 2, we have

$$|f(z)| \le A\exp\{\tilde{\beta}(x) - x\lambda(|z|) + Ax\}, \quad x > 0,$$

where $\tilde{\beta}(x) = \sup\{xt - \frac{\alpha(t)}{2} : t \in \mathbf{R}\}$. Then (3) and (4) imply that for any $D_1 > D(q)$ and $A_1 > a_2$, there exists $r_0 > b + 1$ such that

$$n(r + q(r)) - n(r) \le D_1 q(r), \quad r \ge r_0;$$

$$q(r) \le A_1 r(\log r)^{-1} \le 2^{-1}r, \quad r \ge r_0.$$

These imply that

$$n(t) - n(r) \le D_1(t + q(t) - r), \quad t > r \ge r_0;$$
$$\lambda(R) - \lambda(r) \le 2D_1(1 + A_1)(\log R - \log r + 1), \quad R > r \ge r_0.$$

In fact, if $r \ge r_0$, let $p_0(r) = r$, $p_{k+1}(r) = p_k(r) + q(p_k(r))$ $(k = 0, 1, 2, \ldots)$. Then $p_{k+1}(r) \ge p_k(r) + q(r)$ and $p_k(r) \ge r + kq(r)$ $(k = 0, 1, 2, \ldots)$. So if $l \ge 0$ is an integer such that $p_l(r) \le t < p_{l+1}(r)$, then

$$\begin{aligned} n(t) - n(r) &\le \sum_{k=0}^{l}(n(p_{k+1}(r)) - n(p_k(r))) \le D_1 \sum_{k=0}^{l}(p_{k+1}(r) - p_k(r)) \\ &= D_1(p_l(r) + q(p_l(r)) - r) \le D_1(t + q(t) - r). \end{aligned}$$

Since

$$\lambda(R) - \lambda(r) \le \int_r^R \frac{\mathrm{d}n(t)}{t}, \quad R > r \ge r_0,$$

integrating by parts gives

$$\lambda(R) - \lambda(r) \le 2D_1(1 + A_1)(\log R - \log r + 1), \quad R > r \ge r_0.$$

By Lemma 1, there exists $a \in \mathbf{R}$ such that (5) holds.

Suppose that there exists a real number a such that (5) holds. If $\lambda(r)$ is bounded, then (5) holds for any real number a. So we may think that $\lambda(r)$ is unbounded on $r \ge 0$. Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\lambda(t) + a)$ for $t \ge 0$ and let $u(z)$ be the Poisson integral of $2\varphi(t)$, i.e.,

$$u(z) = \frac{x}{\pi}\int_{-\infty}^{+\infty} \frac{2\varphi(t)}{x^2 + (y - t)^2}\mathrm{d}t. \tag{14}$$

Then $u(x+iy)$ is harmonic in the half-plane $\mathbf{C}_0 = \{z = x+iy : x > 0\}$ and there exists a positive constant $A > 0$ such that

$$u(z) \ge \frac{x}{\pi}\int_{-\infty}^{+\infty} \frac{2(\varphi(|z|) - A)}{x^2 + (y - t)^2}\mathrm{d}t = \varphi(|z|) - A, \quad x > 0.$$

Therefore, there exists an analytic function $g(z)$ on $\mathbf{C}_0$ such that $\mathrm{Re}g(z) = u(z) \ge \varphi(|z|) - A$ $(x > 0)$. For $b > a_0 + 2$, let

$$\varphi_b(z) = \frac{G_b(z)}{(1 + z + b)^4}\exp\{-g(z + b)\}, \tag{15}$$

where $G_b(z)$ is defined by (11). Then $\varphi_b(z)$ is analytic in $\mathbf{C}_{-b} = \{z = x + iy : x > -b\}$. By (12) there exists a positive constant A_2 such that

$$|\varphi_b(z)| \le \frac{A}{1 + y^2}\exp\{\alpha^*(x - 1) + A_2 x\}, \quad x > -b. \tag{16}$$

Let

$$h_b(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi_b(iy)e^{-iy(t+A_2)}\mathrm{d}y \tag{17}$$

be the Fourier transform of $\varphi_b(iy)e^{-iyA_2}$. Then $h_b(t)$ is bounded and continuous on $\mathbf{R}$. By Cauchy's formula,

$$h_b(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \varphi_b(x + iy)e^{-(x+iy)(t+A_2)}\mathrm{d}y, \quad x > -b. \tag{18}$$

By (16), (18) and the formula of the Legendre transform for $\alpha(t)$, we see that $|h_b(t)| \le A\exp(-\alpha(t) - t)$ $(t \ge t_0)$, and that (by taking $x = -b+1$ in (18)) $|h_b(t)| \le A\exp((b-1)t)$ $(t \le t_0)$. Therefore, by (1), if $b > a_0 + 3$, $|h_b(t)| \le A\exp\{-\alpha(t) - |t|\}$ $(t \in \mathbf{R})$. By (18) and the inverse Fourier transform formula,

$$\varphi_0(z)e^{-A_2 z} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h_0(t)e^{tz}\mathrm{d}t, \quad x > a_0.$$

Therefore the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} h_b(t)h(t)\mathrm{d}t, \quad h \in L^p_\alpha$$

satisfies $T(t^{k-1}e^{\lambda_n t}) = 0$ $(k = 1, 2, \ldots, m_n; \lambda_n \in \Lambda)$, and

$$\|T\| = \frac{1}{\sqrt{2\pi}}\Big(\int_{-\infty}^{+\infty} |h_b(t)e^{\alpha(t)}|^q\mathrm{d}t\Big)^{\frac{1}{q}} > 0.$$

By the Riesz representation theorem, the space $E(\Lambda, M)$ is incomplete in L^p_α. This completes the proof of Theorem 1. □

References

[1] DENG Guantie. *Incompleteness and minimality of complex exponential system* [J]. Sci. China Ser. A, 2007, **50**(10): 1467–1476.

[2] YAN Feng, DENG Guantie. *Completeness of complex exponential system in L^p_α space* [J]. J. Math. Res. Exposition, 2008, **28**(3): 691–694.

[3] DENG Guantie. *Completeness of complex exponential systems* [J]. Chinese Ann. Math. Ser. A, 2005, **26**(4): 537–542. (in Chinese)

[4] DENG Guantie. *Weighted exponential polynomial approximation* [J]. Sci. China Ser. A, 2003, **46**(2): 280–287.

[5] MALLIAVIN P. *Sur quelques procédés d'extrapolation* [J]. Acta Math., 1955, **93**: 179–255. (in French)

[6] ROCKAFELLAR R T. *Convex Analysis* [M]. Princeton University Press, Princeton, N.J. 1970.