Journal of Mathematical Research & Exposition
Nov., 2010, Vol. 30, No. 6, pp. 1091–1098
DOI:10.3770/j.issn:1000-341X.2010.06.019
Http://jmre.dlut.edu.cn

Weakly KKM Map, Intersection Theorems and Minimax Inequalities on Abstract Convex Spaces

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Abstract In this paper, we introduce the concept of weakly KKM map on an abstract convex space without any topology and linear structure, and obtain Fan's matching theorem and intersection theorem under very weak assumptions on abstract convex spaces. Finally, we give several minimax inequality theorems as applications. These results generalize and improve many known results in recent literature.

Keywords abstract convex space; map class $\mathfrak{K}$; $\mathfrak{KC}$; $\mathfrak{KO}$; weakly KKM map; matching theorem.

Document code A
MR(2000) Subject Classification 47H10; 54H25
Chinese Library Classification O189.1; O177.91

1. Introduction

For a nonempty set X, $\langle X\rangle$ denotes the class of all nonempty finite subsets of X.

Let X and Y be two nonempty sets, $T : X \multimap Y$ denote the function from X into the power set 2^Y of Y, and $T(A) := \bigcup_{x\in A} T(x)$ for each $A \subset X$. And in view of T, define a map $T^- : Y \multimap X$ by $T^-(y) = \{x \in X : y \in T(x)\}$ for each $y \in Y$.

The famous KKM theorem [1] and its generalizations are of fundamental importance in modern nonlinear analysis. Many authors have studied lots of KKM theorems and their equivalent forms and discussed the properties of corresponding KKM mapping.

In 1996, Chang and Yen [2] made a systematic study of the class $\mathrm{KKM}(X, Y)$ which is defined as follows:

Let X be a convex subset of a vector space and Y a topological space. If $S, T : X \multimap Y$ are two multimaps such that $T(\mathrm{co}\, A) \subset S(A)$ for any $A \in \langle X\rangle$, then S is called a generalized KKM mapping with respect to T. A mapping $T : X \multimap Y$ is said to have the KKM property if for any generalized KKM mapping $S : X \multimap Y$ with respect to T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property.

In 1998, Lin, Ko and Park [3] extended the results of Chang and Yen to generalized convex space as follows:

Let $(X, D; \Gamma)$ be a G-convex space [4], Y a nonempty set, $T : X \multimap Y$ and $S : D \multimap Y$ two maps. We say that S is a generalized G-KKM mapping with respect to T if for each $A \in \langle D\rangle$,

Received November 11, 2008; Accepted January 5, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10361005).
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$T(\Gamma(A)) \subset S(A)$. If Y is a topological space, $T : X \multimap Y$ is said to have the G-KKM property if for any generalized G-KKM mapping $S : D \multimap Y$ with respect to T, the family $\{\overline{S(z)} : z \in D\}$ has the finite intersection property.

Later, there have appeared some generalizations of generalized KKM map with respect to a map, for examples, s-KKM map or S-KKM map with respect to a map for convex subset of a vector space, H-space and G-convex space, respectively [5–7].

Recently, Balaj [8] introduced the concept of weakly G-KKM mapping with respect to a map which includes the generalized G-KKM mapping with respect to a map as special case. The concept is following:

Let $(X, D; \Gamma)$ be a G-convex space, Y a nonempty set, $T : X \multimap Y$ and $S : D \multimap Y$ two maps. We say that S is a weakly G-KKM mapping with respect to T if for each $N \in \langle D \rangle$ and any $x \in \Gamma(N)$, $T(x) \cap S(N) \neq \emptyset$.

In 2007, Tang, Zhang and Cheng [9], Deng and Xia [10] extended Balaj's definition from G-convex space to FC-space [11] and general topological space respectively.

In this paper, we will introduce the concept of weakly KKM map with respect to a map on an abstract convex space without any topology which generalizes and improves all the definitions mentioned above, and obtain Fan type matching theorem and intersection theorem. At the same time, we give some minimax inequality theorems as applications of intersection theorem. These results generalize and improve many known results in recent literature.

2. Preliminaries

Recently, Park [12] introduced a new concept of abstract convex spaces without any topology and a map class $\mathfrak{K}$ having certain KKM property which are adequate to establish the KKM theory. With this new concept, he obtained some basic results on abstract convex spaces [12, 13]. These results generalize and simplify the corresponding results in the theory on convex spaces [14], H-spaces [15], G-convex spaces [4, 16], and others.

Definition 1 ([12]) *An abstract convex space $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$.*

The examples of an abstract convex space are convex space in the sense of Lassonde [14], C-space (or H-space) due to Hovarth [15], G-convex space of Park [4, 16], and many other known spaces. For details, see [12].

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-convex if for each $A \in \langle X \cap D \rangle$, we have $\Gamma_A \subset X$. In case $E = D$, let $(E; \Gamma) := (E \supset D; \Gamma)$.

An abstract convex space with any topology is called an abstract convex topological space (Simply, A-convex spaces).

Definition 2 ([12]) *Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a multimap*

$F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) = \bigcup_{y \in A} G(y) \quad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a KKM map with respect to F. A KKM map $G : D \multimap E$ is a map with respect to 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{K}$-map if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is } \ \text{a } \mathfrak{K} \text{ map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Now, we introduce the concept of weakly KKM map with respect to a map on an abstract convex space without any topology and linear structure as follows.

Definition 3 *Let $(E, D; \Gamma)$ be an abstract convex space, Y a nonempty set, $S : D \multimap Y$ and $T : E \multimap Y$ two maps. S is said to be a weakly KKM map with respect to T, if there exists a KKM map $H : D \multimap E$ with respect to i_E such that for each $N \in \langle D \rangle$ and any $x \in H(N)$, $T(x) \cap S(N) \neq \emptyset$. And H is said to be the companion of (S, T).*

Remark Definition 3 generalizes and improves the corresponding definitions in [8–10] for G-convex space and FC-space and general topological space, respectively.

Let X and Y be two topological spaces and $\mathbb{R}$ a real space.

A map $T : X \multimap Y$ is said to be upper [resp. lower] semicontinuous, if $\{x \in X : T(x) \cap C \neq \emptyset\}$ is closed [resp. open] for each closed [resp. open] subset of Y.

A function $f : X \to \mathbb{R}$ is said to be upper [resp. lower] semicontinuous if for each $\lambda \in \mathbb{R}$, the set $\{x \in X : f(x) < \lambda\}$ [resp. $\{x \in X : f(x) > \lambda\}$] is open.

3. Matching and intersection results

We begin with the following known result.

Theorem 1 ([12]) *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a multimap. Then $F \in \mathfrak{K}(E, Z)$ if and only if for any multimap $G : D \multimap Z$ satisfying $F(\Gamma_N) \subset G(N)$ for each $N \in \langle D \rangle$, we have $F(E) \cap \bigcap\{G(y) | y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.*

Remark ([12]) If Z has any topology and $F \in \mathfrak{KO}(E, Z)$ [resp. $F \in \mathfrak{KC}(E, Z)$], then we have to assume that G is open-valued [resp. closed-valued].

From Theorem 1, we obtain the next result.

Theorem 2 *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ and $G, T : D \multimap Z$ three multimaps and $F \in \mathfrak{K}(E, Z)$. If* *(i) G is a KKM map with respect to F; and*

(ii) for each $N \in \langle D \rangle$, $G(N) \subset T(N)$.

Then $F(E) \cap \bigcap\{T(y)|y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Remark If Z has any topology and $F \in \mathfrak{KO}(E, Z)$ [resp. $F \in \mathfrak{KC}(E, Z)$], then we have to assume that T is open-valued [resp. closed-valued].

From Theorem 2, we obtain the following particular form.

Theorem 3 *Let $(E, D; \Gamma)$ be an abstract convex space, $i_E \in \mathfrak{K}(E, E)$. If two maps $G, T : D \multimap E$ satisfy*

(i) G is a KKM map with respect to i_E; and

(ii) for each $N \in \langle D \rangle$, $G(N) \subset T(N)$.

Then $\{T(y)|y \in D\}$ has the finite intersection property.

Furthermore, if E has any topology and $\bigcap_{z \in M} \overline{T(z)}$ is compact for some $M \in \langle D \rangle$, then $\bigcap_{y \in D} \overline{T(y)} \neq \emptyset$.

Remark If E has any topology and $i_E \in \mathfrak{KO}(E, E)$ [resp. $i_E \in \mathfrak{KC}(E, E)$], then we have to assume that T is open-valued [resp. closed-valued].

Theorem 4 *Let $(E, D; \Gamma)$ be an abstract convex space and $i_E \in \mathfrak{K}(E, E)$ and $G : D \multimap E$ a multimap. If there exist $A \in \langle D \rangle$ and a family of subsets $\{M_z : z \in A\}$ of E such that $E = \bigcup\{M_z|z \in A\}$ and G is a KKM map with respect to i_E. Then there exists $B \in \langle A \rangle$ such that $G(B) \cap \bigcap\{M_z|z \in B\} \neq \emptyset$.*

Proof Suppose that the result is not true. Then $G(B) \cap \bigcap\{M_z|z \in B\} = \emptyset$ for each $B \in \langle A \rangle$, hence $G(B) \subset \bigcup\{{M_z}^C|z \in B\}$ for each $B \in \langle A \rangle$. Define $F : A \multimap E$ by $F(z) = {M_z}^C$ for each $z \in A$ and let $G' = G|_A$. Then it is easy to check that F and G' satisfy all requirements of Theorem 3 on new abstract convex space $(E, A; \Gamma)$. Hence $\{F(y)|y \in A\}$ has the finite intersection property, particularly $\bigcap\{{M_z}^C|z \in A\} = \bigcap\{F(y)|y \in A\} \neq \emptyset$, which implies that $\bigcup\{M_z|z \in A\} \neq E$, leading to a contradiction. □

Remark If E has any topology and $i_E \in \mathfrak{KO}(E, E)$ [resp. $i_E \in \mathfrak{KC}(E, E)$], then we have to assume that $\{M_z : z \in A\}$ is a family of closed subsets [resp. open subsets] of E.

Next, we will obtain intersection theorems.

Theorem 5 *Let $(E, D; \Gamma)$ be an abstract convex space, Y a nonempty set, $i_E \in \mathfrak{K}(E, E)$, $S : D \multimap Y$ and $T : E \multimap Y$ two maps. If S is a weakly KKM map with respect to T, then for any $A \in \langle D \rangle$, there exists an $x_0 \in E$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in A$. Furthermore, if T is a single valued map, then $\bigcap_{z \in A} S(z) \neq \emptyset$.*

Proof Suppose that the conclusion is not true. Then there exists an $A \in \langle D \rangle$ such that for any $x \in E$ there exists a $z_0 \in A$ such that $T(x) \cap S(z_0) = \emptyset$.

Let $M_z = \{x \in E : T(x) \cap S(z) = \emptyset\}$ for each $z \in A$. Then $E = \bigcup\{M_z : z \in A\}$ by assumption. Since S is a weakly KKM map with respect to T, there exists the companion H of (S,T). Since H is a KKM map with respect to i_E and $i_E \in \mathfrak{K}(E,E)$, there exists a $B \in \langle A \rangle$ such that $H(B) \cap \bigcap\{M_z : z \in B\} \neq \emptyset$ by Theorem 4. Take $y \in H(B) \cap \bigcap\{M_z : z \in B\}$, then $y \in H(B)$ implies $T(y) \cap S(B) \neq \emptyset$ by Definition 3. On the other hand, $y \in \bigcap\{M_z : z \in B\}$ implies $T(y) \cap S(z) = \emptyset$ for all $z \in B$, hence $T(y) \cap S(B) = \emptyset$. This is a contradiction. □

Remark If E has any topology and $i_E \in \mathfrak{KO}(E,E)$ [resp. $i_E \in \mathfrak{KC}(E,E)$], then we have to assume that for $z \in D$, the set $\{x \in E : T(x) \cap S(z) \neq \emptyset\}$ is open [resp. closed].

Theorem 6 *Let $(E,D;\Gamma)$ be a compact A-convex space, $i_E \in \mathfrak{KC}(E,E)$, Y a nonempty set, $S : D \multimap Y$ and $T : E \multimap Y$ two maps. If*

(i) S is a weakly KKM map with respect to T;

(ii) for each $z \in D$, $\{x \in E : T(x) \cap S(z) \neq \emptyset\}$ is closed.

Then there exists an $x_0 \in E$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in D$.

Proof Suppose that the conclusion is not true. Then for any $x \in E$ there exists a $z_0 \in D$ such that $T(x) \cap S(z_0) = \emptyset$.

Let $M_z = \{x \in E : T(x) \cap S(z) = \emptyset\}$ for each $z \in D$. Then M_z is open by (ii) for each $z \in D$ and $E = \bigcup\{M_z : z \in D\}$ by assumption. Hence there exists an $A \in \langle D \rangle$ such that $E = \bigcup\{M_z : z \in A\}$ by the compactness of E. The remainder proof follows from the proof of Theorem 5 and its Remark. □

Remark If Y is a topological space, then the condition (ii) of Theorem 6 can be replaced by S having closed values and T being an upper semicontinuous mapping.

3. Ky Fan type minimax inequalities

As applications of intersection theorems, we give Ky Fan type minimax inequalities on abstract convex spaces.

In the next theorems, as in the other minimax theorems established, we shall suppose $\inf_x \sup_y f(x,y) > -\infty$ in our paper. As to the case $\inf_x \sup_y f(x,y) = -\infty$, all these results remain true evidently.

Theorem 7 *Let $(X,D;\Gamma)$ be an abstract convex space, Y a nonempty set, $i_X \in \mathfrak{K}(X,X)$, $H : D \multimap X$ a KKM map with respect to i_X and $T : X \multimap Y$ a map. Let $\psi : D \times Y \to \mathbb{R}$ and $\varphi : X \times Y \to \mathbb{R}$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} \varphi(x,y)$. Suppose that for any $\lambda < \beta$ and $y \in T(X)$, each $A \in \langle\{z \in D : \psi(z,y) < \lambda\}\rangle$ implies $H(A) \subset \{x \in X : \varphi(x,y) < \lambda\}$, then*

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x,y) \leqq \inf_{A\in\langle D\rangle} \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z,y).$$

Proof Let $\lambda < \beta$ be fixed and define a map $S : D \multimap Y$ as follows

$$S(z) = \{y \in Y : \psi(z,y) \geqq \lambda\}, \quad \forall z \in D.$$

Now we prove that S is a weakly KKM map with respect to T and H is the companion of (S,T).

In fact, otherwise, there exits $N \in \langle D \rangle$ and $\overline{x} \in H(N)$ such that $T(\overline{x}) \cap S(N) = \emptyset$. Hence for each $y \in T(\overline{x})$, $N \subset \{z \in D : \psi(z, y) < \lambda\}$. Therefore $\overline{x} \in H(N) \subset \{x \in X : \varphi(x.y) < \lambda\}$ for all $y \in T(\overline{x})$, this is, $\varphi(\overline{x}, y) < \lambda$ for all $y \in T(\overline{x})$, which implies that $\beta = \inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \lambda$. This is a contradiction with $\lambda < \beta$.

S and T satisfy all requirements of Theorem 5. Hence for each $A \in \langle D \rangle$, there exists an $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in A$. For any $z \in A$, there exists $y_z \in T(x_0)$ such that $y_z \in S(z)$. Hence $\psi(z, y_z) \geqq \lambda$, which implies that $\lambda \leqq \inf_{z\in A} \sup_{y\in T(x_0)} \psi(z, y)$, and therefore $\lambda \leqq \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y)$. Let $\lambda \to \beta$. Then $\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y)$ for each $A \in \langle D \rangle$. Hence

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \inf_{A\in\langle D\rangle} \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y). \quad \square$$

Theorem 8 *Let $(X, D; \Gamma)$ be an A-convex space, Y a topological space, $i_X \in \mathfrak{KC}(X, X)$ [resp. $i_X \in \mathfrak{KO}(X, X)$], $H : D \multimap X$ a KKM map with respect to i_X, and $T : X \multimap Y$ an upper semicontinuous map [resp. lower semicontinuous] map. Let $\psi : D \times Y \to \mathbb{R}$ and $\varphi : X \times Y \to \mathbb{R}$ be two functions and $\beta = \inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y)$. Suppose that*

(i) for each $z \in D$, $\psi(z, \cdot)$ is upper semicontinuous [resp. lower semicontinuous] on Y;

(ii) for any $\lambda < \beta$ and $y \in T(X)$, each $A \in \langle\{z \in D : \psi(z, y) < \lambda\}\rangle$ [resp. $A \in \langle\{z \in D : \psi(z, y) \leqslant \lambda\}\rangle$] implies $H(A) \subset \{x \in X : \varphi(x, y) < \lambda\}$ [resp. $H(A) \subset \{x \in X : \varphi(x, y) \leqslant \lambda\}$].

Then

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \inf_{A\in\langle D\rangle} \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y).$$

Proof Let $\lambda < \beta$ be fixed and define a map $S : D \multimap Y$ as follows

$$S(z) = \{y \in Y : \psi(z, y) \geqq \lambda\}[\text{resp. } S(z) = \{y \in Y : \psi(z, y) > \lambda\}], \quad \forall z \in D.$$

Then $S(z)$ is closed [resp. open] by (i), and therefore, $\{x \in X : T(x) \cap S(z) \neq \emptyset\}$ is closed [resp. open] by upper [resp. lower] semicontinuity of T.

Now we prove that S is a weakly KKM map with respect to T and H is the companion of (S, T). In fact, otherwise, there exists $N \in \langle D \rangle$ and $\overline{x} \in H(N)$ such that $T(\overline{x}) \cap S(N) = \emptyset$. Hence for each $y \in T(\overline{x})$, $N \subset \{z \in D : \psi(z, y) < \lambda\}$[resp. $N \in \langle\{z \in D : \psi(z, y) \leqslant \lambda\}\rangle$]. Therefore $\overline{x} \in H(N) \subset \{x \in X : \varphi(x.y) < \lambda\}$ [resp. $\overline{x} \in H(N) \subset \{x \in X : \varphi(x, y) \leqslant \lambda\}$] for all $y \in T(\overline{x})$ by (ii), that is, $\varphi(\overline{x}, y) < \lambda$ [resp. $\varphi(\overline{x}, y) \leqslant \lambda$] for all $y \in T(\overline{x})$, which implies that $\beta = \inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \lambda$, a contradiction with $\lambda < \beta$.

S and T satisfy all requirements of Theorem 5 and its Remark. Hence for each $A \in \langle D \rangle$, there exists an $x_0 \in E$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in A$. For any $z \in A$, there exists $y_z \in T(x_0)$ such that $y_z \in S(z)$. Hence $\psi(z, y_z) \geqq \lambda$, which implies that $\lambda \leqq \inf_{z\in A} \sup_{y\in T(x_0)} \psi(z, y)$, and therefore $\lambda \leqq \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y)$. Let $\lambda \to \beta$. Then $\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)}\psi(z, y)$ for each $A \in \langle D \rangle$. Hence

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \inf_{A\in\langle D\rangle} \sup_{x\in X} \inf_{z\in A} \sup_{y\in T(x)} \psi(z, y). \quad \square$$

Theorem 9 *Let $(X, D; \Gamma)$ be a compact A-convex space, Y a topological space, $i_X \in \mathfrak{KC}(X, X)$,*

$H : D \multimap X$ a KKM map with respect to i_X and $T : X \multimap Y$ an upper semicontinuous map. Let $\psi : D \times Y \to \mathbb{R}$ and $\varphi : X \times Y \to \mathbb{R}$ be two functions and $\beta = \inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y)$. Suppose that

(i) for each $z \in D$, $\psi(z, \cdot)$ is upper semicontinuous on Y;

(ii) for any $\lambda < \beta$ and $y \in T(X)$, each $A \in \langle\{z \in D : \psi(z, y) < \lambda\}\rangle$ implies $H(A) \subset \{x \in X : \varphi(x, y) < \lambda\}$.

Then

(a) $\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$.

(b) If T has compact values, then there exists an $x_0 \in X$ such that

$$\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y).$$

Proof Let $\lambda < \beta$ be fixed and define a map $S : D \multimap Y$ as follows

$$S(z) = \{y \in Y : \psi(z, y) \geqq \lambda\}, \quad \forall z \in D.$$

Then by (i), $S(z)$ is closed for each $z \in D$, and since T is upper semicontinuous, $\{x \in X : T(x) \cap S(z) \neq \emptyset\}$ is closed for each $z \in D$. On the other hand, from (ii) and the proof of Theorem 7, we know that S is a weakly KKM map with respect to T and H is the companion of (S, T). Hence there exists an $x_0 \in E$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in D$ by Theorem 6. For each $z \in D$, there exists $y_z \in T(x_0)$ such that $y_z \in S(z)$. Hence $\psi(z, y_z) \geqq \lambda$, which implies that $\lambda \leqq \inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y)$, and therefore $\lambda \leqq \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$. Let $\lambda \to \beta$. Then we have $\inf_{x\in X} \sup_{y\in T(x)} \varphi(x, y) \leqq \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$. This completes (a).

Further, if T has compact values, then $x \to \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$ is upper semicontinuous on X because T is upper semicontinuous on X and $\psi(z, \cdot)$ is upper continuous on Y (see [17, Proposition 3.1.21]). Since X is compact, there exists an $x_0 \in X$ such that $\inf_{z\in D} \sup_{y\in T(x_0)} \psi(z, y) = \sup_{x\in X} \inf_{z\in D} \sup_{y\in T(x)} \psi(z, y)$. And therefore conclusion (b) follows from (a). This completes our proof. □

Remark Theorems 7–9 generalize and improve the corresponding results in [8–10] for G-convex space, FC-space and general topological space respectively. Hence many known Fan type minimax inequality theorems become special cases of ours.

References

[1] KNASTER B, KURATOWSKI K, MAZURKIEWICZ B. *Ein Beweis des Fixpunksatzes Für n-dimensionale Simplexe* [J]. Fund. Math., 1929, **14**: 132–137.

[2] CHANG T H, YEN C L. *KKM property and fixed point theorems* [J]. J. Math. Anal. Appl., 1996, **203**(1): 224–235.

[3] LIN L J, KO C J, PARK S. *Coincidence theorems for set-valued maps with G-KKM property on generalized convex space* [J]. Discuss. Math. Differential Incl., 1998, **18**(1-2): 69–85.

[4] PARK S. *The KKM, matching, and fixed point theorems in generalized convex spaces* [J]. Nonlinear Funct. Anal. Appl., 2006, **11**(1): 139–154.

[5] CHANG T H, HUANG Y Y, JENG J C, et al. *On S-KKM property and related topics* [J]. J. Math. Anal. Appl., 1999, **229**(1): 212–227.

[6] LIU Xinge, CAI Haitao. *H-space and vector quasi-variational inequalities* [J]. Math. Theory Appl. (Changsha), 2004, **24**(3): 1–5.

[7] PIAO Yongjie. *Almost fixed point theorems and fixed point theorems on locally G-convex uniform spaces* [J]. J. Math. Res. Exposition, 2007, **27**(4): 859–862. (in Chinese)

[8] BALAJ M. *Weakly G-KKM mappings, G-KKM property, and minimax inequalities* [J]. J. Math. Anal. Appl., 2004, **294**(1): 237–245.

[9] TANG Gusheng, ZHANG Qingbang, CHENG Caozong. *W-G-F-KKM mapping, intersection theorems and minimax inequalities in FC-space* [J]. J. Math. Anal. Appl., 2007, **334**(2): 1481–1491.

[10] DENG Lei, YANG Mingge. *Weakly R-KKM mappings—intersection theorems and minimax inequalities in topological spaces* [J]. Appl. Math. Mech. (English Ed.), 2007, **28**(1): 103–109.

[11] DING Xieping. *Himmelberg type fixed point theorem in locally FC-spaces* [J]. Sichuan Shifan Daxue Xuebao Ziran Kexue Ban, 2005, **28**(2): 127–130.

[12] PARK S. *On generalizations of the KKM principle on abstract convex spaces* [J]. Nonlinear Anal. Forum, 2006, **11**(1): 67–77.

[13] PARK S. *Fixed point theorems on $\mathfrak{KC}$-maps in abstract convex spaces* [J]. Nonlinear Anal. Forum, 2006, **11**(2): 117–127.

[14] LASSONDE M. *On the use of KKM multifunctions in fixed point theory and related topics* [J]. J. Math. Anal. Appl., 1983, **97**(1): 151–201.

[15] HORVATH C D. *Contractibility and generalized convexity* [J]. J. Math. Anal. Appl., 1991, **156**(2): 341–357.

[16] PARK S. *Elements of the KKM theory for generalized convex spaces* [J]. Korean J. Comput. Appl. Math., 2000, **7**(1): 1–28.

[17] Aubin J P, Ekeland I. *Applied Nonlinear Analysis* [M]. Wiley, New York, 1984.