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Exponential Attractor of Strong Dissipative KDV Type Equation

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Abstract In this paper, we consider the strong dissipative KDV type equation on an unbounded domain R^1. By applying the theory of decomposing operator and the method of constructing some compact operator in weighted space, the existence of exponential attractor in phase space $H^2(\mathrm{R}^1)$ is obtained.

Keywords unbounded domain; KDV type equation; exponential attractor.

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1. Introduction

The KDV type of equations have been an important class of nonlinear evolution equations with numerous applications in physical sciences and engineering fields [7, 8]. In recent years, there has been a considerable interest in the attractor of a class of KDV equations [9–11]. Our aim in this work is to study the existence of the exponential attractor of the following type of strong dissipative KDV equation [1] in the phase space $H^2(\mathrm{R}^1)$.

$$u_t + \alpha u u_x + u_{xxx} + \gamma u_{xxxx} - u_{xx} + \beta u = f, \tag{1}$$

$$u(x, 0) = u_0(x), \tag{2}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ are constants. The existence of the compact global attractor of (1)–(2) in the phase space $H^2(\mathrm{R}^1)$ has been proved in [1].

For the existence of exponential attractors, there are many classical results in a bounded domain such as [6, 9, 12]. But as far as the case of unbounded domains is concerned [2, 3, 13], it is difficult to do research on this respect. First, the Laplace operator in the corresponding equation is neither continuous nor compact, and its spectrum is not discrete. Secondly, since $H^s(\mathrm{R}^1) \hookrightarrow H^{s_1}(\mathrm{R}^1)$ $(s > s_1)$ is not compact, it is difficult to prove the compactness of absorbing sets and the squeezing property. In this paper, we borrow the ideas from Babin in [2,3] and use

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the theory of decomposing operator and the method of constructing some compact operator in weighted space to prove the existence of exponential attractor of (1)–(2) in $H^2(\mathrm{R}^1)$.

2. Preliminaries

For convenience, we introduce some notations. We rewrite (1)–(2) as

$$u_t + \alpha u\nabla u + \nabla\Delta u + \gamma\Delta^2 u - \Delta u + \beta u = f, \tag{3}$$

$$u(x,0) = u_0(x), \tag{4}$$

where $-\Delta = -\frac{\partial^2}{\partial x^2}$, $\nabla = \frac{\partial}{\partial x}$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm of the space $L^2(\mathrm{R}^1)$, respectively, and $\|\cdot\|_s$ as the norm of the space $H^s(\mathrm{R}^1)$. In particular, we denote K as all the real constants, C as all the positive constants, R as all the real constants depending on time t. From [1], we get the bounded absorbing set B of (1)–(2). Namely, B=$\{u \in H^2(\mathrm{R}^1), \|u\|_2 \le \rho_1\}$. Let $M = \{u \in H^3(\mathrm{R}^1), \|u\|_3 \le \rho_2\}$. Then $M \subset B$, and $S(t)M \subset M$.

Lemma 2.1 ([2]) *Let X be a closed invariant set in a Hilbert space. If*

1) There exists a covering of X by a finite number of balls of radius 1;

2) The operator of semigroup $S(t)$ has a global attractor on X;

3) For a fixed $t > 0$, $S(t)$ has a strong squeezing property and is uniformly Lipschitzian on X,

then $S(t)$ has an exponential attractor on X.

Lemma2.2 ([3, 4]) *Let s, s_1 be integers and $s > s_1$. Then $H^s(\mathrm{R}^n)\bigcap H^{s_1}(\mathrm{R}^n;(1+x^2)\mathrm{d}x) \hookrightarrow H^{s_1}(\mathrm{R}^n)$ is compact.*

From [1], we find

Lemma 2.3 ([1]) *Let $u(t)$ be a solution of (3)–(4). Then*

$$\|u\|_\infty, \|\nabla u\|_\infty, \|\Delta u\|_\infty, \|\nabla\triangle u\|_\infty \le C,$$

where $C > 0$ is a general constant.

Lemma 2.4 ([5]) (Gagliardo-Nirenberg inequality)

$$\|D^j u\|_p \le \ C\|u\|_q^{1-\lambda}\|D^m u\|_r^\lambda, \quad u \in L^q \cap H^{m,r}(\mathrm{R}^n),$$

where $\frac{1}{p} = \frac{j}{n} + \lambda(\frac{1}{r} - \frac{m}{n}) + (1-\lambda)\frac{1}{q}$, $1 \le q$, $r \le \infty$, j is integer, $0 \le j \le m$, $\frac{j}{m} \le \lambda \le 1$. If $m - j - \frac{n}{r}$ is non-negative, when $\frac{j}{m} \le \lambda < 1$, the conclusion is satisfied.

3. Main results

Theorem 3.1 *Assume $f \in H^2(\mathrm{R}^1)$, $S(t)$ is the operator of semigroup derived by (3), then $S(t)$ has an exponential attractor in $M \subset H^2(\mathrm{R}^1)$.*

The proof is based on the Lemma 2.1 and we shall prove the following Propositions.

Proposition 3.1 *The operators $S(t)$ are Lipschitzian on $M \subset B$.*

Proof Let $u(t)$, $v(t)$ be solutions of (3)–(4) with initial value u_0, v_0, respectively, and let $u(t)-v(t)=w(t)$, $u_0-v_0=w_0$. Then we obtain the equation

$$w_t+\alpha w\nabla u+\alpha v\nabla w+\nabla\Delta w+\gamma\Delta^2 w-\Delta w+\beta w=0, \tag{5}$$

$$w(x,0)=w_0(x). \tag{6}$$

Multipling (5) by $(-\triangle^3 w+w)$, and integrating on R^1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\triangle w\|^2+\|w\|^2)+\gamma(\|\nabla\triangle^2 w\|^2+\|\triangle w\|^2)+\|\triangle^2 w\|^2+\|\nabla w\|^2+\beta(\|\nabla\triangle w\|^2+\\ &\quad\|w\|^2)=-\alpha(w\nabla u,-\triangle^3 w)-\alpha(w\nabla u,w)-\alpha(v\nabla w,-\triangle^3 w)-\alpha(v\nabla w,w).\end{aligned}$$

Controlling the right-hand side as

$$\begin{aligned}|\alpha(w\nabla u,-\triangle^3 w)|&\le|\alpha(\nabla w\nabla u,\nabla\triangle^2 w)|+|\alpha(w\triangle u,\nabla\triangle^2 w)|\\ &\le\alpha\|\nabla u\|_\infty\|\nabla w\|\|\nabla\triangle^2 w\|+\alpha\|\triangle u\|_\infty\|w\|\|\nabla\triangle^2 w\|\\ &\le\alpha\|\nabla u\|_\infty\|\ w\|^{\frac{4}{5}}\|\nabla\triangle^2 w\|^{\frac{6}{5}}+\alpha\|\triangle u\|_\infty\|w\|\|\nabla\triangle^2 w\|\\ &\le C\|w\|^2+\frac{\gamma}{4}\|\nabla\triangle^2 w\|^2+C\|w\|^2+\frac{\gamma}{4}\|\nabla\triangle^2 w\|^2\\ &\le C\|w\|^2+\frac{\gamma}{2}\|\nabla\triangle^2 w\|^2,\\ |\alpha(w\nabla u,w)|&\le\alpha\|\nabla u\|_\infty\|w\|\|w\|\le C\|w\|^2,\\ |\alpha(v\nabla w,-\triangle^3 w)|&\le|\alpha(\nabla v\nabla w,\nabla\triangle^2 w)|+|\alpha(v\triangle\ w,\nabla\triangle^2 w)|\\ &\le C\|w\|^2+\frac{\gamma}{2}\|\nabla\triangle^2 w\|^2,\\ |\alpha(v\nabla w,w)|&\le\alpha\|v\|_\infty\|\nabla w\|\|\ w\|\le C\|w\|^2+\gamma\|\triangle w\|^2.\end{aligned}$$

Thus, it leads to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\triangle w\|^2+\|w\|^2)\le K(\|w\|^2+\|\nabla\triangle w\|^2).$$

By the Gronwall Lemma, we find

$$\|\nabla\triangle w\|^2+\|w\|^2\le(\|\nabla\triangle w(0)\|^2+\|w(0)\|^2)e^{Kt},\quad t\in[0,T].$$

The proposition is proved. □

Proposition 3.2 *Let $M\subset B$, and $S(t)$ be the operators of semigroup acting on this set. Then for any $\delta\in(0,\frac{1}{4})$, there exists $t>0$ such that $S(t)$ has squeezing property on M.*

The proof of this proposition is based on the next three technical Lemmas. Before proving these Lemmas, we need some assumptions as follows.

Decomposing the operators of solution $S(t)$ corresponding to (5)–(6) as $S(t)=S_1(t)+S_2(t)$. Let

$$\lambda_L(x)=\begin{cases}1, & |x|\le L,\\ 0, & |x|>1+L.\end{cases}$$

Then $\forall\eta\in(0,1)$, $\exists L(\eta)>0$, such that

$$\|u-u_\eta\|\le\eta,\ u_\eta=u\cdot\lambda_L(x);\ \|v-v_\eta\|\le\eta,\ v_\eta=v\cdot\lambda_L(x).$$

Let $w_1(t)$ be a solution of the following equation

$$w_{1t} + \nabla\Delta w_1 + \gamma\Delta^2 w_1 - \Delta w_1 + \beta w_1 = -\alpha w_1\nabla(u-u_\eta) - \alpha(v-v_\eta)\nabla w_1, \tag{7}$$

$$w_1(x,0) = w(0). \tag{8}$$

Let $w_2(t)$ be a solution of the following equation

$$w_{2t} + \nabla\Delta w_2 + \gamma\Delta^2 w_2 - \Delta w_2 + \beta w_2 = -\alpha w_2\nabla u - \alpha w_1\nabla u_\eta - \alpha v\nabla w_2 - \alpha v_\eta\nabla w_1, \tag{9}$$

$$w_2(x,0) = 0. \tag{10}$$

Lemma 3.2 *Let $w_1(t)$ be a solution of (7)–(8). Then*

$$\|w_1\|, \|\nabla w_1\|, \|\triangle w_1\|, \|\nabla\triangle w_1\| \le Ce^{-ct}, t>0.$$

Proof Multiplying (7) by $(-\triangle w_1 + w_1)$, and integrating on R^1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w_1\|^2 + \|w_1\|^2) + \gamma(\|\nabla\triangle w_1\|^2 + \|\triangle w_1\|^2) + \|\triangle w_1\|^2 + \|\nabla w_1\|^2 + \beta(\|\nabla w_1\|^2 + \|w_1\|^2)\\ &\quad = -\alpha(w_1\nabla(u-u_\eta), -\triangle w_1) - \alpha(w_1\nabla(u-u_\eta), w_1) - \alpha((v-v_\eta)\nabla w_1, -\triangle w_1) -\\ &\qquad \alpha((v-v_\eta)\nabla w_1, w_1).\end{aligned}$$

Control the right-hand side as

$$\begin{aligned}|\alpha(w_1\nabla(u-u_\eta), -\triangle w_1)| &\le \alpha\|\nabla(u-u_\eta)\|_\infty\|w_1\|\|\triangle w_1\| \le \frac{\alpha^2\eta^2}{2\gamma}\|w_1\|^2 + \frac{\gamma}{2}\|\triangle w_1\|^2,\\ |\alpha(w_1\nabla(u-u_\eta), w_1)| &\le \alpha\|\nabla(u-u_\eta)\|_\infty\|w_1\|^2 \le \alpha\eta\|w_1\|^2,\\ |\alpha((v-v_\eta)\nabla w_1, -\triangle w_1)| &\le \alpha\|v-v_\eta\|_\infty\|\nabla w_1\|\|\triangle w_1\| \le \frac{\alpha^2\eta^2}{2\gamma}\|\nabla w_1\|^2 + \frac{\gamma}{2}\|\triangle w_1\|^2,\\ |\alpha((v-v_\eta)\nabla w_1, w_1)| &\le \alpha\|v-v_\eta\|_\infty\|\nabla w_1\|\|w_1\| \le \frac{\alpha^2\eta^2}{2\beta}\|\nabla w_1\|^2 + \frac{\beta}{2}\|w_1\|^2.\end{aligned}$$

Note that $\forall\eta\in(0,1)$, choose η such that

$$\max\{\frac{\alpha^2\eta^2}{2\gamma}, \alpha\eta, \frac{\alpha^2\eta^2}{2\beta}\} \le \frac{\beta}{8}.$$

We come to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w_1\|^2 + \|w_1\|^2) \le -c(\|\nabla w_1\|^2 + \|w_1\|^2),$$

where $c>0$ depends on β. By the Gronwall Lemma, we find

$$\|\nabla w_1\|^2 + \|w_1\|^2 \le Ce^{-ct}, \quad t>0.$$

In addition, multiplying $(\triangle^2 w_1 + w_1)$ and $(-\triangle^3 w_1 + w_1)$ with (7), respectively, and integrating on R^1, like the above proof, we come to the similar conclusion . That is $\|\Delta w_1\|^2 + \|w_1\|^2 \le Ce^{-ct}$, $t>0$; $\|\nabla\Delta w_1\|^2 + \|w_1\|^2 \le Ce^{-ct}$, $t>0$.

Lemma 3.3 *Let $w_2(t)$ be a solution of (9)–(10). Then*

$$\|w_2\|, \|\nabla w_2\|, \|\triangle w_2\|, \|\nabla\triangle w_2\| \le R, \quad t>0.$$

Proof Multiplying (9) by $(-\triangle w_2 + w_2)$, and integrating on R^1, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w_2\|^2+\|w_2\|^2)+\gamma(\|\nabla\triangle w_2\|^2+\|\triangle w_2\|^2)+\|\triangle w_2\|^2+\|\nabla w_2\|^2+\beta(\|\nabla w_2\|^2+\|w_2\|^2)\\&=-\alpha(w_2\nabla u,-\triangle w_2)-\alpha(w_2\nabla u,w_2)-\alpha(w_1\nabla u_\eta,-\triangle w_2)-\alpha(w_1\nabla u_\eta,w_2)-\\&\quad\alpha(v\nabla w_2,-\triangle w_2)-\alpha(v\nabla w_2,w_2)-\alpha(v_\eta\nabla w_1,-\triangle w_2)-\alpha(v_\eta\nabla w_1,w_2).\end{aligned}$$

Control the right-hand side as

$$\begin{aligned}|\alpha(w_2\nabla u,-\triangle w_2)|&\le\alpha\|\nabla u\|_\infty\|w_2\|\|\triangle w_2\|\le C\|w_2\|^2+\frac{\gamma}{4}\|\triangle w_2\|^2,\\|\alpha(w_2\nabla u,w_2)|&\le\alpha\|\nabla u\|_\infty\|w_2\|^2\le C\|w_2\|^2,\\|\alpha(w_1\nabla u_\eta,-\triangle w_2)|&\le\alpha\|\nabla u_\eta\|\|w_1\|_\infty\|\triangle w_2\|\le C+\frac{\gamma}{4}\|\triangle w_2\|^2,\\|\alpha(w_1\nabla u_\eta w_2)|&\le\alpha\|w_1\|_\infty\|\nabla u_\eta\|\|w_2\|\le C+C\|w_2\|^2,\\|\alpha(v\nabla w_2,-\triangle w_2)|&\le\alpha\|v\|_\infty\|\nabla w_2\|\|\triangle w_2\|\le C\|\nabla w_2\|^2+\frac{\gamma}{4}\|\triangle w_2\|^2,\\|\alpha(v\nabla w_2,w_2)|&\le\alpha\|v\|_\infty\|\nabla w_2\|\|w_2\|\le C\|\nabla w_2\|^2+C\|w_2\|^2,\\|\alpha(v_\eta\nabla w_1,-\triangle w_2)|&\le\alpha\|\nabla w_1\|_\infty\|v_\eta\|\|\triangle w_2\|\le C+\frac{\gamma}{4}\|\triangle w_2\|^2,\\|\alpha(v_\eta\nabla w_1,w_2)|&\le\alpha\|\nabla w_1\|_\infty\|v_\eta\|\|w_2\|\le C+C\|w_2\|^2.\end{aligned}$$

Thus, from the above estimates we come to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w_2\|^2+\|w_2\|^2)\le K(\|\nabla w_2\|^2+\|w_2\|^2).$$

By the Gronwall lemma, we conclude

$$\|\nabla w_2\|^2+\|w_2\|^2\le Ce^{-Kt}=:R,\quad t\in[0,T].$$

In addition, multiplying $(\triangle^2 w_2+w_2)$ and $(-\triangle^3 w_2+w_2)$ with (9), respectively, and integrating on R^1, as the above proof, we achieve the similar conclusion. That is $\|\Delta w_2\|^2+\|w_2\|^2\le R$, $t\in[0,T]$; $\|\nabla\Delta w_2\|^2+\|w_2\|^2\le R$, $t\in[0,T]$.

Lemma 3.4 *Let $w_2(t)$ be a solution of (9)–(10). Then*

$$\|xw_2\|,\ \|x\nabla w_2\|,\ \|x\triangle w_2\|\le R,\quad t\in[0,T].$$

Proof Multiplying (9) by x^2w_2, and integrating on R^1, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|xw_2\|^2+\gamma\|x\triangle w_2\|^2+\beta\|xw_2\|^2\\&=3(w_2,x\triangle w_2)-4\gamma(\triangle w_2,x\nabla w_2)+2\gamma\|\nabla w_2\|^2+(\triangle w_2,x^2w_2)-\alpha(w_2\nabla u,x^2w_2)-\\&\quad\alpha(w_1\nabla u_\eta,x^2w_2)-\alpha(v\nabla w_2,x^2w_2)-\alpha(v_\eta\nabla w_1,x^2w_2).\end{aligned}$$

Control the right-hand side as

$$\begin{aligned}|3(w_2,x\triangle w_2)|&\le3\|w_2\|\|x\triangle w_2\|\le R+\frac{\gamma}{3}\|x\triangle w_2\|^2,\\|4\gamma(\triangle w_2,x\nabla w_2)|&\le4\gamma\|x\triangle w_2\|\|\nabla w_2\|\le\frac{\gamma}{3}\|x\triangle w_2\|^2+R,\\2\gamma\|w_2\|^2&\le R,\end{aligned}$$

$$\begin{aligned}
|(\triangle w_2, x^2 w_2)| &\le \|xw_2\|\|x\triangle w_2\| \le C\|xw_2\|^2 + \frac{\gamma}{3}\|x\triangle w_2\|^2,\\
|\alpha(w_2\nabla u, x^2 w_2)| &\le \alpha\|\nabla u\|_\infty \|xw_2\|^2 \le C\|xw_2\|^2,\\
|\alpha(w_1\nabla u_\eta, x^2 w_2)| &\le \alpha\|w_1\|_\infty \|x\nabla u_\eta\|\|xw_2\| \le C + \|xw_2\|^2,\\
|\alpha(v\nabla w_2, x^2 w_2)| &\le \frac{1}{2}\|\nabla v\|_\infty \|xw_2\|^2 + \|w_2\|\|v\|_\infty\|xw_2\| \le C\|xw_2\|^2 + R,\\
|\alpha(v_\eta\nabla w_1, x^2 w_2)| &\le \alpha\|\nabla w_1\|_\infty \|xv_\eta\|\|xw_2\| \le C + \|xw_2\|^2.
\end{aligned}$$

Thus, from the above estimates it leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|xw_2\|^2 \le R\|xw_2\|^2 + R.$$

By the Gronwall lemma, we get

$$\|xw_2\|^2 \le R.$$

Furthermore, using the operator ∇ acting on (9) and multiplying (9) by $x^2\nabla w_2$, and integrating on R^1, we have $\|x\nabla w_2\|^2 \le R$; using the operator $\triangle$ acting on (9) and multiplying (9) by $x^2\triangle w_2$, and integrating on R^1, it follows that $\|x\triangle w_2\|^2 \le R$.

Proof of Proposition 3.2 Let t be large enough. $\forall\delta \in (0, \frac{1}{4})$, due to Lemma 3.2, we obtain

$$\|\triangle w_1(t)\| \le \frac{\delta}{8}\|w(0)\|.$$

According to Lemmas 3.3 and 3.4, we conclude

$$\|\nabla\triangle w_2\|^2 + \|x\triangle w_2\|^2 \le R. \tag{11}$$

The left-hand side of (11) can be written in the form (Lw_2, w_2), where

$$Lw_2 = -\triangle^3 w_2 + x^2\triangle^2 w_2 - 2\triangle w_2.$$

According to Lemma 2.2, the set B_0 defined by (11) is compactly embeded into $H^2(\mathrm{R}^1)$. Hence L^{-1} is compact. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in $H^2(\mathrm{R}^1)$, where the corresponding eigenvalue is λ_j, and

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \lambda_j \to +\infty, j \to +\infty.$$

Let

$$B_0 = \Big\{w_2(t) | \sum_{j=1}^{\infty}\lambda_j (w_2(t), e_j)^2 \le R^2\Big\}.$$

Take N large enough such that

$$\lambda_N \ge \frac{16R^2}{\delta^2\|w(0)\|^2}. \tag{12}$$

Let $E_N = \mathrm{span}\{e_1, e_2, \dots, e_N\}$, P_N be the orthoprojector onto E_N. Obviously, if $w_2(t) \in B_0$, then

$$\|(I - P_N)w_2(t)\|_2^2 = \sum_{j=N+1}^{\infty}(w_2, e_j)^2 \le \frac{R^2}{\lambda_N} \le \frac{\delta^2}{16}\|w(0)\|^2,$$

that is,

$$\|(I - P_N)w_2(t)\|_2 \le \frac{\delta}{4}\|w(0)\|. \tag{13}$$

Let

$$B_1 = \{u \in B_0 | \|P_N u\|_2 \geq \frac{3\delta}{4}\|u(0)\|\}, \quad B_2 = B_0 \setminus B_1.$$

Let $w_2(t) \in B_1$. Then

$$\begin{aligned}\|(I-P_N)w(t)\|_2 &\leq \|(I-P_N)w_1\|_2 + \|(I-P_N)w_2\|_2 \leq \|w_1\|_2 + \|(I-P_N)w_2\|_2 \\ &\leq \frac{\delta}{8}\|w(0)\| + \frac{\delta}{4}\|w(0)\| \leq \frac{\delta}{2}\|w(0)\|, \end{aligned} \tag{14}$$

$$\begin{aligned}\|P_N w\|_2 &= \|P_N w_1 + P_N w_2\|_2 \geq \|P_N w_2\|_2 - \|P_N w_1\|_2 \\ &\geq \frac{3\delta}{4}\|w(0)\| - \frac{\delta}{8}\|w(0)\| \geq \frac{\delta}{2}\|w(0)\|. \end{aligned} \tag{15}$$

Hence

$$\|P_N w\|_2 > \|(I-P_N)w\|_2. \tag{16}$$

Now let $w_2(t) \in B_2$. We have

$$\|w_2\|_2^2 \leq \|P_N w_2\|_2^2 + \|(I-P_N)w_2\|_2^2 \leq \frac{10\delta^2}{16}\|w(0)\|^2, \tag{17}$$

and

$$\|w\|_2 \leq \|w_1\|_2 + \|w_2\|_2 \leq \frac{\delta}{8}\|w(0)\| + \frac{\sqrt{10}\delta}{4}\|w(0)\| \leq \delta\|w(0)\|. \tag{18}$$

Take into account that $u_0 - v_0 = w_0, u(t) - v(t) = w(t) = S(t)u_0 - S(t)v_0$, we deduce from (16) and (18) that either

$$\|(I-P_N)(S(t)u_0 - S(t)v_0)\| \leq \|P_N(S(t)u_0 - S(t)v_0)\|$$

or

$$\|S(t)u_0 - S(t)v_0\| \leq \delta\|u_0 - v_0\|$$

and the Proposition 3.2 is proved.

Proposition 3.3 *There exists an invariant set $M \subset B$, which can be covered by a finite number of balls of radius ε.*

Proof It is proved in Proposition 3.2 that, $\forall u \in M$, $u \in H^3(\mathrm{R}^n) \bigcap H^2(\mathrm{R}^1; (1+x^2)\mathrm{d}x)$, we have $M \hookrightarrow H^2(\mathrm{R}^1)$ is compact. Since M is bounded, M is compact set in $H^2(\mathrm{R}^1)$. Hence, M can be covered by an infinite number of unit balls, and there exists a finite number of balls of radius ε covering M.

Proof of Theorem 3.1 According to Propositions 3.1, 3.2 and 3.3, all assumptions of Lemma 2.1 are fulfilled. Therefore, $S(t)$ possesses an exponential attractor on $M \subset H^2(\mathrm{R}^1)$.

References

[1] ZHANG Wenbing. *Attractor of cauchy problem in dissipative kdv type equation* [J]. Int. J. Nonlinear Sci., 2006, **1**(3): 155–163.

[2] BABIN A V, NICOLAEMKO B. *Exponential attractor of reaction-diffusion systems in an unbounded domain* [J]. J. Dynam. Differential Equations, 1995, **7**(4): 567–590.

[3] BABIN A V, VISHIK M I. *Attractors of partial differential evolution equation in an unbounded domain* [J]. Proc. Roy. Soc. Edinburgh Sect. A, 1990, **116**(3-4): 221–243.

[4] GUO Boling, LI Yongsheng. *Attractor for dissipative Klein-Gordon-Schrödinger equations in R^3* [J]. J. Differential Equations, 1997, **136**(2): 356–377.

[5] FRIEDMAN A. *Partial Differential Equations* [M]. Pure Appl. Math., New York, 1969.

[6] EDEN A, MILANI A J. *Exponential attractors for extensible beam equations* [J]. Nonlinearity, 1993, **6**(3): 457–479.

[7] OSBORNE A R. *The inverse scattering transform: tools for the nonlinear Fourier analysis and filtering of ocean surface waves* [J]. Chaos Solitons Fractals, 1995, **5**(12): 2623–2637.

[8] OSTROVSKY L, STEPANYANTS YU A. *Do interal solutions exist in the ocean* [J]. Rev Geophys., 1989, **27**: 23–37.

[9] DAI Zhengde, ZHU Zhiwei. *The inertial fractal set of weakly damped forced Korteweg-de Vries equation* [J]. Appl. Math. Mech. (English Ed.), 1995, **16**(1): 37–45.

[10] LAURENGOT PH. *Compact attractor for weakly damped driven Korteweg-de Vries equations on the real line* [J]. Czechoslovak Math. J., 1998, **48**(1): 85–94.

[11] GOUBET O R, RICARDO M S. *Asymptotic smoothing and the global attractor of a weakly damped KdV equation on the real line* [J]. J. Differential Equations, 2002, **185**(1): 25–53.

[12] GUO Boling, WANG Bixiang. *Exponential attractors for the generalized Ginzburg-Landau equation* [J]. Acta Math. Sin. (Engl. Ser.), 2000, **16**(3): 515–526.

[13] DAI Zhengde, JIANG Murong. *Exponential attractors of the Ginzburg-Landau-BBM equations in an unbounded domain* [J]. Acta Math. Appl. Sinica (English Ser.), 2001, **17**(4): 484–493.