Journal of Mathematical Research & Exposition
Jan., 2011, Vol. 31, No. 1, pp. 165–172
DOI:10.3770/j.issn:1000-341X.2011.01.019
Http://jmre.dlut.edu.cn

On the Reduced Minimum Modulus of Projections and the Angle between Two Subspaces

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Abstract Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of a Hilbert space $\mathcal{H}$, and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the orthogonal projections on $\mathcal{M}$ and $\mathcal{N}$, respectively. In this note, an exact representation of the angle and the minimum gap of $\mathcal{M}$ and $\mathcal{N}$ is obtained. In addition, we study relations between the angle, the minimum gap of two subspaces $\mathcal{M}$ and $\mathcal{N}$, and the reduced minimum modulus of $(I-P_{\mathcal{N}})P_{\mathcal{M}}$.

Keywords subspace; angle between two subspaces; reduced minimum modulus.

Document code A
MR(2010) Subject Classification 46C05; 46C07
Chinese Library Classification O177.2

1. Introduction

Throughout this note, a subspace is a closed linear manifold of a separable Hilbert space $\mathcal{H}$ with inner product and norm denoted by $\langle x,y\rangle$ and $\|x\|=\sqrt{\langle x,x\rangle}$, respectively. If $\mathcal{M}$ is a subspace of $\mathcal{H}$, the orthogonal complement of $\mathcal{M}$ is denoted by $\mathcal{M}^{\perp}$ and the orthogonal projection on $\mathcal{M}$ is denoted by $P_{\mathcal{M}}$. In recent years, the variety of quantities involving two subspaces have been studied by a number of researchers in the wide literatures [2–11]. In this note, using the technique of block-operators, some results about minimum gap and the angle between two closed subspaces of a Hilbert space are improved which are obtained by Deng in [3] and other results concerning two subspaces of a Hilbert space are obtained. The angle [6, 7, 9] between $\mathcal{M}$ and $\mathcal{N}$ is an angle in $[0,\frac{\pi}{2}]$ whose cosine is defined by

$$c(\mathcal{M},\mathcal{N})=\sup\{|\langle x\rangle y|: x\in\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}, y\in\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^{\perp} \text{ and } \|x\|=\|y\|=1\}. \quad (1)$$

By this formula $c(\mathcal{M},\mathcal{N})$ is defined only when $\mathcal{M}$ is not a subspace of $\mathcal{N}$ and $\mathcal{N}$ is not a subspace of $\mathcal{M}$. If $\mathcal{M}\subseteq\mathcal{N}$ or $\mathcal{N}\subseteq\mathcal{M}$, we let $c(\mathcal{M},\mathcal{N})=0$. The minimal angle [7] between $\mathcal{M}$ and $\mathcal{N}$ is an angle in $[0,\frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{M},\mathcal{N})=\sup\{|\langle x\rangle y|: x\in\mathcal{M}, y\in\mathcal{N} \text{ and } \|x\|=\|y\|=1\}. \quad (2)$$

Received December 20, 2008; Accepted June 30, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10871224) and the Fundamental Research Funds for the Central Universities (Grant No. GK 200902049).
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Recall that the minimum gap $\gamma(\mathcal{M},\mathcal{N})$ between two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space has been defined [3, 9] by

$$\gamma(\mathcal{M},\mathcal{N}) = \inf_{x\in\mathcal{M},x\notin\mathcal{N}} \frac{\operatorname{dist}(x,\mathcal{N})}{\operatorname{dist}(x,\mathcal{M}\cap\mathcal{N})}. \tag{3}$$

By this formula $\gamma(\mathcal{M},\mathcal{N})$ is defined only when $\mathcal{M}$ is not a subspace of $\mathcal{N}$. If $\mathcal{M}\subseteq\mathcal{N}$, we set $\gamma(\mathcal{M},\mathcal{N})=1$. Obviously, $\gamma(\mathcal{M},\mathcal{N})=1$, if $\mathcal{N}\subseteq\mathcal{M}$.

Before proving the main results in this paper, let us introduce some notations and terminology which are used in the later. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For an operator $A\in\mathcal{B}(\mathcal{H})$, the adjoint, the range, the null-space and the spectrum of A are denoted by A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\sigma(A)$, respectively. An operator $A\in\mathcal{B}(\mathcal{H})$ is said to be self-adjoint if $A=A^*$. An operator $A\in\mathcal{B}(\mathcal{H})$ is said to be positive if $(Ax,x)\geq 0$ for $x\in\mathcal{H}$. If A is a positive operator, the unique square root of A is denoted by $A^{\frac{1}{2}}$. An operator A is said to be a contraction (strict contraction) if $\| A \|\leq 1$ ($\| A \|< 1$). The reduced minimum modulus $\gamma(A)$ of $A\in B(\mathcal{H})$ (see [1, 9]) is defined by

$$\gamma(A)=\begin{cases}\inf\{\|Ax\| : \operatorname{dist}(x,\mathcal{N}(A))=1\}, & A\neq 0;\\ 0, & A=0.\end{cases}$$

It is well known that for $A\neq 0$, $\mathcal{R}(A)$ is closed if and only if $\gamma(A)>0$.

2. Main results

Lemma 1 ([4, 8]) *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. If $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the orthogonal projections on $\mathcal{M}$ and $\mathcal{N}$, respectively, then $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices*

$$P_{\mathcal{M}} = I_1\oplus I_2\oplus 0I_3\oplus 0I_4\oplus I_5\oplus 0I_6 \tag{4}$$

and

$$P_{\mathcal{N}} = I_1\oplus 0I_2\oplus I_3\oplus 0I_4\oplus \begin{pmatrix} Q & Q^{\frac{1}{2}}(I_5-Q)^{\frac{1}{2}}D \\ D^*Q^{\frac{1}{2}}(I_5-Q)^{\frac{1}{2}} & D^*(I_5-Q)D \end{pmatrix} \tag{5}$$

with respect to the space decomposition $\mathcal{H}=\bigoplus_{i=1}^{6}\mathcal{H}_i$, respectively, where $\mathcal{H}_1=\mathcal{M}\cap\mathcal{N}$, $\mathcal{H}_2=\mathcal{M}\cap\mathcal{N}^{\perp}$, $\mathcal{H}_3=\mathcal{M}^{\perp}\cap\mathcal{N}$, $\mathcal{H}_4=\mathcal{M}^{\perp}\cap\mathcal{N}^{\perp}$, $\mathcal{H}_5=\mathcal{M}\ominus(\mathcal{H}_1\oplus\mathcal{H}_2)$ and $\mathcal{H}_6=\mathcal{H}\ominus(\bigoplus_{j=1}^{5}\mathcal{H}_j)$, Q is a positive contraction on $\mathcal{H}_5$, 0 and 1 are not eigenvalues of Q, and D is a unitary from $\mathcal{H}_6$ onto $\mathcal{H}_5$. I_i is the identity on $\mathcal{H}_i$, $i=1,\ldots,5$.

For convenience, in the sequel, we always assume that $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (4) and (5), also the zero operator on $\mathcal{H}_i$ is denoted by $0I_i$, $i=1,\ldots,6$.

First, we give some necessary and sufficient conditions for $\mathcal{H}_5=\mathcal{H}_6=\{0\}$.

Lemma 2 *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. The following statements are equivalent:*

(a) $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ commute: $P_{\mathcal{M}}P_{\mathcal{N}}=P_{\mathcal{N}}P_{\mathcal{M}}$;

(b) $P_{\mathcal{M}}P_{\mathcal{N}}=P_{\mathcal{N}\cap\mathcal{M}}$;

(c) $P_{\mathcal{M}}P_{\mathcal{N}}$ is an orthogonal projections;

(d) $P_{\mathcal{M}}P_{\mathcal{N}}$ is an idempotent;

(e) $P_{\mathcal{M}}P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{N}}P_{\mathcal{M}}P_{\mathcal{N}}$;

(f) $\mathcal{H} = \mathcal{M}\cap\mathcal{N} \oplus \mathcal{M}\cap\mathcal{N}^{\perp} \oplus \mathcal{M}^{\perp}\cap\mathcal{N} \oplus \mathcal{N}^{\perp}\cap\mathcal{M}^{\perp}$;

(g) $\mathcal{M} = \mathcal{M}\cap\mathcal{N} \oplus \mathcal{M}\cap\mathcal{N}^{\perp}$.

Proof Since $\|P_{\mathcal{M}}P_{\mathcal{N}}\| \le 1$, it is clear that (c)$\Longleftrightarrow$(d).

(c)$\Longrightarrow$(f). By Lemma 1, $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (4) and (5). It is easy to calculate that

$$P_{\mathcal{M}}P_{\mathcal{N}} = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q & Q^{\frac{1}{2}}(I_5-Q)^{\frac{1}{2}}D \\ 0 & 0 \end{pmatrix}$$

and

$$(P_{\mathcal{M}}P_{\mathcal{N}})^2 = I_1 \oplus 0I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q^2 & Q^{\frac{3}{2}}(I_5-Q)^{\frac{1}{2}}D \\ 0 & 0 \end{pmatrix}.$$

Therefore, if $\mathcal{H}_5 \neq \{0\}$, then $Q^2 = Q$, so $\sigma(Q) = \{0,1\}$, hence 0 and 1 are eigenvalues of Q. It is a contradiction to Lemma 1, so $\mathcal{H}_5 = \{0\}$, then $\mathcal{H}_6 = \{0\}$. Thus $\mathcal{H} = \mathcal{M}\cap\mathcal{N} \oplus \mathcal{M}\cap\mathcal{N}^{\perp} \oplus \mathcal{M}^{\perp}\cap\mathcal{N} \oplus \mathcal{N}^{\perp}\cap\mathcal{M}^{\perp}$.

(e)$\Longrightarrow$(f). By a similar calculation as above, we have $P_{\mathcal{M}}P_{\mathcal{N}}P_{\mathcal{M}} = P_{\mathcal{N}}P_{\mathcal{M}}P_{\mathcal{N}}$ which implies $\mathcal{H}_5 = \mathcal{H}_6 = \{0\}$.

It is obvious that (f)$\Longrightarrow$ (g) $\Longrightarrow$(a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c). $\square$

The following lemma was obtained in [1].

Lemma 3 ([1]) *Let $T \in B(\mathcal{H})$. Then*

$$\gamma(T) = \gamma(T^*) = (\inf\{\sigma(TT^*) \setminus \{0\}\})^{\frac{1}{2}} = (\inf\{\sigma(T^*T) \setminus \{0\}\})^{\frac{1}{2}}.$$

From above lemmas, we give the specific representation of $\gamma(P_{\mathcal{M}}P_{\mathcal{N}})$.

Theorem 4 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

$$\gamma(P_{\mathcal{M}}P_{\mathcal{N}}) = \begin{cases} 1, & \textit{if } P_{\mathcal{M}}P_{\mathcal{N}} \textit{ is a nonzero orthogonal projection;} \\ 0, & \textit{if } P_{\mathcal{M}}P_{\mathcal{N}} = 0; \\ (1-\|I_5-Q\|)^{\frac{1}{2}}, & \textit{if } P_{\mathcal{M}}P_{\mathcal{N}} \textit{ is not an orthogonal projection}. \end{cases}$$

Proof If $P_{\mathcal{M}}P_{\mathcal{N}}$ is not an orthogonal projection, then $\mathcal{H}_5 \neq \{0\}$ and $\mathcal{H}_6 \neq \{0\}$. It follows from Lemma 1 that

$$P_{\mathcal{M}}P_{\mathcal{N}}(P_{\mathcal{M}}P_{\mathcal{N}})^* = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4 \oplus \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}.$$

Since 0 is not an eigenvalue of Q, 0 is not an isolated point of $\sigma(Q)$. Hence by Lemma 2, $\gamma(P_{\mathcal{M}}P_{\mathcal{N}}) = (\inf\{\sigma(Q) \setminus \{0\}\})^{\frac{1}{2}} = (\inf\{\sigma(Q)\})^{\frac{1}{2}}$. Thus

$$\begin{aligned}\gamma(P_{\mathcal{M}}P_{\mathcal{N}}) &= \inf\{\lambda \in \mathbb{C} : \lambda \in \sigma(Q)\}^{\frac{1}{2}} \\ &= (1 - \sup\{\lambda \in \mathbb{C} : \lambda \in \sigma(I_5 - Q)\})^{\frac{1}{2}} = (1-\|I_5-Q\|)^{\frac{1}{2}}. \ \square\end{aligned}$$

It is well-known that

$$\sup\{\sigma(T)\} = \lim_{n\to\infty} \|(T^n)\|^{\frac{1}{n}},$$

and if T is invertible, then

$$\lim_{k\to\infty}\gamma(T^k)^{\frac{1}{k}}=\inf\{\sigma(T)\}.$$

Similarly, we have following conclusion for $T=P_{\mathcal{M}}P_{\mathcal{N}}$.

Theorem 5 *Let $T\neq 0$ be product of two orthogonal projections. Then*

$$\lim_{k\to\infty}\gamma(T^k)^{\frac{1}{k}}=\inf\{\sigma(T)\setminus\{0\}\}.$$

Proof If T is an orthogonal projection, then $\lim_{k\to\infty}\gamma(T^k)^{\frac{1}{k}}=1$, the conclusion is clear.

If T is not an orthogonal projection, then let $T=P_{\mathcal{M}}P_{\mathcal{N}}$, where $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ have the operator matrices (4) and (5). By Lemma 1, $\mathcal{H}_5\neq\{0\}$ and $\mathcal{H}_6\neq\{0\}$. It is easy to calculate that

$$(P_{\mathcal{M}}P_{\mathcal{N}})^k=I_1\oplus 0I_2\oplus I_3\oplus 0I_4\oplus\begin{pmatrix} Q^k & Q^{\frac{2k-1}{2}}(I_5-Q)^{\frac{1}{2}}D\\ 0 & 0\end{pmatrix}$$

and

$$(P_{\mathcal{M}}P_{\mathcal{N}})^k((P_{\mathcal{M}}P_{\mathcal{N}})^*)^k=I_1\oplus 0I_2\oplus I_3\oplus 0I_4\oplus\begin{pmatrix} Q^{2k-1} & 0\\ 0 & 0\end{pmatrix}.$$

Thus

$$\begin{aligned}\gamma((P_{\mathcal{M}}P_{\mathcal{N}})^k)&=\inf\{\sigma(Q^{2k-1})\setminus\{0\}\}^{\frac{1}{2}}\\&=(\inf\{\sigma(Q)^{2k-1}\setminus\{0\}\})^{\frac{1}{2}}(\text{ by Spectra Mapping Theorem})\\&=(\inf\{\sigma(Q)^{2k-1}\})^{\frac{1}{2}}(\text{ since 0 is not an eigenvalues of } Q^{2k-1})\\&=(\inf\{\sigma(Q)\})^{\frac{2k-1}{2}}.\end{aligned}$$

Hence $\lim_{k\to\infty}\gamma(T^k)^{\frac{1}{k}}=\inf\{\sigma(Q)\}$. It is easy to see that

$$\sigma(Q)\subseteq\sigma(T)=\sigma(P_{\mathcal{M}}P_{\mathcal{N}})\subseteq\{1,0\}\cup\sigma(Q),$$

so $\inf\{\sigma(T)\setminus\{0\}\}=\inf\{\sigma(Q)\setminus\{0\}\}=\inf\{\sigma(Q)\}$. Therefore,

$$\lim_{k\to\infty}\gamma(T^k)^{\frac{1}{k}}=\inf\{\sigma(T)\setminus\{0\}\}.\quad\square$$

In Lemma 2.10 of [7], the following results were obtained. To make this work complete, we include a proof.

Lemma 6 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

$$c_0(\mathcal{M},\mathcal{N})=\|P_{\mathcal{M}}P_{\mathcal{N}}\|=\|P_{\mathcal{M}}P_{\mathcal{N}}P_{\mathcal{M}}\|^{\frac{1}{2}},$$

and

$$c(\mathcal{M},\mathcal{N})=\|P_{\mathcal{M}}P_{\mathcal{N}}P_{(\mathcal{M}\cap\mathcal{N})^\perp}\|.$$

Proof

$$\begin{aligned}c_0(\mathcal{M},\mathcal{N})&=\sup\{|\langle x\rangle y|:x\in\mathcal{M},y\in\mathcal{N}\text{ and }\|x\|=\|y\|=1\}\\&=\sup\{|\langle P_{\mathcal{M}}x,P_{\mathcal{N}}y\rangle|:x,y\in\mathcal{H}\text{ and }\|x\|=\|y\|=1\}\\&=\sup\{|\langle x,P_{\mathcal{M}}P_{\mathcal{N}}y\rangle|:x,y\in\mathcal{H}\text{ and }\|x\|=\|y\|=1\}\end{aligned}$$

$$
\begin{aligned}
&= \|P_{\mathcal{M}}P_{\mathcal{N}}\|. \\
c(\mathcal{M},\mathcal{N}) &= c_0(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}, \mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}) \\
&= \|P_{\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}}P_{\mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^{\perp}}\| \\
&= \|P_{\mathcal{M}}P_{(\mathcal{M}\cap\mathcal{N})^{\perp}}P_{\mathcal{N}}P_{(\mathcal{M}\cap\mathcal{N})^{\perp}}\| \\
&= \|P_{\mathcal{M}}P_{\mathcal{N}}P_{(\mathcal{M}\cap\mathcal{N})^{\perp}}\|. \quad \square
\end{aligned}
$$

The following is an extension of Theorem 4 in [3].

Corollary 7 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

$$
c(\mathcal{M},\mathcal{N}) = \begin{cases} 0, & \text{if } P_{\mathcal{M}}P_{\mathcal{N}} \text{ is an orthogonal projection;} \\ \|Q\|^{\frac{1}{2}}, & \text{if } P_{\mathcal{M}}P_{\mathcal{N}} \text{ is not an orthogonal projection,} \end{cases}
$$

and

$$
c_0(\mathcal{M},\mathcal{N}) = \begin{cases} 1, & \text{if } \mathcal{M}\cap\mathcal{N}\neq\{0\} \text{ or } P_{\mathcal{M}}P_{\mathcal{N}} \text{ is a nonzero orthogonal projection;} \\ 0, & \text{if } P_{\mathcal{M}}P_{\mathcal{N}} = 0; \\ \|Q\|^{\frac{1}{2}}, & \text{otherwise .} \end{cases}
$$

Proof From Lemmas 6 and 2, it is easy to see that if $P_{\mathcal{M}}P_{\mathcal{N}}$ is an orthogonal projection, then $c(\mathcal{M},\mathcal{N}) = 0$. If $P_{\mathcal{M}}P_{\mathcal{N}}$ is not an orthogonal projection, then $\mathcal{H}_5 \neq 0$ and $\mathcal{H}_6 \neq 0$. Note that $P_{(\mathcal{M}\cap\mathcal{N})^{\perp}} = 0I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_6$, so

$$
c(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}}P_{\mathcal{N}}P_{(\mathcal{M}\cap\mathcal{N})^{\perp}}\| = \|P_{\mathcal{M}}P_{\mathcal{N}}P_{(\mathcal{M}\cap\mathcal{N})^{\perp}}P_{\mathcal{N}}P_{\mathcal{M}}\|^{\frac{1}{2}} = \|Q\|^{\frac{1}{2}}.
$$

From the relation of $c(\mathcal{M},\mathcal{N})$ and $c_0(\mathcal{M},\mathcal{N})$, the expression of $c_0(\mathcal{M},\mathcal{N})$ is clear. $\square$

The following theorem is one of our main results.

Theorem 8 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

(a) If $\mathcal{M}\subseteq\mathcal{N}$, then $\gamma(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}) = c(\mathcal{M},\mathcal{N}) = 0$.

(b) If $\mathcal{M}\nsubseteq\mathcal{N}$, then $\gamma^2(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}) + c^2(\mathcal{M},\mathcal{N}) = 1$.

Proof (a) is clear.

(b) Case 1. If $P_{\mathcal{M}}P_{\mathcal{N}}$ is an orthogonal projection, then $P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}$ is a nonzero orthogonal projection, so $\gamma(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}) = 1$ and $c(\mathcal{M},\mathcal{N}) = 0$, by Lemma 7.

Case 2. If $P_{\mathcal{M}}P_{\mathcal{N}}$ is not an orthogonal projection, it follows from Lemma 2 that $\mathcal{H}_5 \neq \{0\}$, then $\mathcal{H}_6 \neq \{0\}$. By Lemma 1, it is easy to calculate that

$$
P_{\mathcal{N}^{\perp}}P_{\mathcal{M}} = 0I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} 1-Q & 0 \\ D^*(I_5-Q)^{\frac{1}{2}}Q^{\frac{1}{2}} & 0 \end{pmatrix}
$$

and

$$
P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}})^* = 0I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4 \oplus \begin{pmatrix} 1-Q & 0 \\ 0 & 0 \end{pmatrix}.
$$

Thus by Lemma 3,

$$
\gamma(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}) = \inf\{\sigma(1-Q)\setminus\{0\}\}^{\frac{1}{2}} = (1-\sup\{\lambda\in\mathbb{C} : \lambda\in\sigma(Q)\})^{\frac{1}{2}}
$$

$$= (1 - \|Q\|)^{\frac{1}{2}}.$$

It follows from Corollary 7 that $\gamma^2(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) + c^2(\mathcal{M},\mathcal{N}) = 1$. □

Consequently, we obtain some results of [2] and [7].

Corollary 9 ([7, Theorems 2.15, 2.16]) *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

(a) $c(\mathcal{M},\mathcal{N}) = c(\mathcal{M}^\perp,\mathcal{N}^\perp)$;

(b) If $\mathcal{M}\cap\mathcal{N} = \{0\}$ and $\mathcal{M}+\mathcal{N} = \mathcal{H}$, then $c_0(\mathcal{M},\mathcal{N}) = c_0(\mathcal{M}^\perp,\mathcal{N}^\perp)$.

Proof (a) Case 1. If $\mathcal{M}\subseteq\mathcal{N}$, then $\mathcal{N}^\perp\subseteq\mathcal{M}^\perp$, so by Lemma 7, $c(\mathcal{M},\mathcal{N}) = c(\mathcal{M}^\perp,\mathcal{N}^\perp) = 0$.

Case 2. If $\mathcal{M}\nsubseteq\mathcal{N}$, then $\mathcal{N}^\perp\nsubseteq\mathcal{M}^\perp$, by Theorem 8,

$$\gamma^2(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) + c^2(\mathcal{M},\mathcal{N}) = 1 \text{ and } \gamma^2(P_{\mathcal{M}}P_{\mathcal{N}^\perp}) + c^2(\mathcal{N}^\perp,\mathcal{M}^\perp) = 1.$$

By Lemma 3, $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) = \gamma(P_{\mathcal{M}}P_{\mathcal{N}^\perp})$, so $c(\mathcal{M},\mathcal{N}) = c(\mathcal{N}^\perp,\mathcal{M}^\perp) = c(\mathcal{M}^\perp,\mathcal{N}^\perp)$.

(b) Since $\mathcal{M}\cap\mathcal{N} = \{0\}$, it is obvious that $c_0(\mathcal{M},\mathcal{N}) = c(\mathcal{M},\mathcal{N})$. It follows from $\mathcal{M}+\mathcal{N} = \mathcal{H}$ that $\mathcal{M}^\perp\cap\mathcal{N}^\perp = \{0\}$, so $c_0(\mathcal{M}^\perp,\mathcal{N}^\perp) = c(\mathcal{M}^\perp,\mathcal{N}^\perp)$. According to (a), $c_0(\mathcal{M},\mathcal{N}) = c_0(\mathcal{M}^\perp,\mathcal{N}^\perp)$. □

Corollary 10 ([7, Theorem 2.13]) *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then the following statements are equivalent:*

(a) $c(\mathcal{M},\mathcal{N}) < 1$;

(b) $\mathcal{M}+\mathcal{N}$ *is closed;*

(c) $\mathcal{M}^\perp+\mathcal{N}^\perp$ *is closed.*

Proof If $\mathcal{M}\subseteq\mathcal{N}$, then the conclusion is clear. In the following proof, we assume $\mathcal{M}\nsubseteq\mathcal{N}$. It is easy to see that $\mathcal{M}+\mathcal{N} = \mathcal{N}+P_{\mathcal{N}^\perp}(\mathcal{M})$, where $P_{\mathcal{N}^\perp}(\mathcal{M}) := \{P_{\mathcal{N}^\perp}y : y\in\mathcal{M}\}$. Hence $\mathcal{M}+\mathcal{N}$ is closed if and only if $\mathcal{R}(P_{\mathcal{N}^\perp}(\mathcal{M}))$ is closed. It is well-known that $\mathcal{R}(P_{\mathcal{N}^\perp}(\mathcal{M}))$ is closed if and only if $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}}) > 0$. Therefore, it follows from Theorem 8 that (a)$\Longleftrightarrow$ (b). From Corollary 9, $c(\mathcal{M},\mathcal{N}) < 1 \Longleftrightarrow c(\mathcal{M}^\perp,\mathcal{N}^\perp) < 1 \Longleftrightarrow \mathcal{M}^\perp+\mathcal{N}^\perp$ is closed. □

The following result is fundamental in [7]. A technical proof has been given in [7]. Here, we give a simple proof.

Corollary 11 ([7, Lemma 2.14]) *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. If $c_0(\mathcal{M},\mathcal{N}) < 1$, then for any closed subspace X of $\mathcal{H}$ which contains $\mathcal{M}+\mathcal{N}$, we have*

$$c_0(\mathcal{M},\mathcal{N}) \le c_0(\mathcal{M}^\perp\cap X,\mathcal{N}^\perp\cap X).$$

Proof For convenience, we divide proof into three steps.

Step 1. If $\mathcal{M}^\perp\cap\mathcal{N}^\perp\cap X \ne \{0\}$, then $c_0(\mathcal{M}^\perp\cap X,\mathcal{N}^\perp\cap X) = 1$, so $c_0(\mathcal{M},\mathcal{N}) \le c_0(\mathcal{M}^\perp\cap X,\mathcal{N}^\perp\cap X)$.

Step 2. Let $X = \mathcal{H}$. Since $c_0(\mathcal{M},\mathcal{N}) < 1$, we have $\mathcal{M}\cap\mathcal{N} = \{0\}$ and $\mathcal{M}+\mathcal{N}$ is closed. If $\mathcal{M}^\perp\cap\mathcal{N}^\perp \ne \{0\}$, then $c_0(\mathcal{M}^\perp\cap X,\mathcal{N}^\perp\cap X) = c_0(\mathcal{M}^\perp,\mathcal{N}^\perp) = 1$, so $c_0(\mathcal{M},\mathcal{N}) \le c_0(\mathcal{M}^\perp\cap X,\mathcal{N}^\perp\cap X)$.

If $\mathcal{M}^\perp \cap \mathcal{N}^\perp = \{0\}$, then $\mathcal{M}+\mathcal{N} = \mathcal{H}$, since $\mathcal{M}+\mathcal{N}$ is closed. It follows from Corollary 9 that $c_0(\mathcal{M},\mathcal{N}) = c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X)$.

Step 3. If $\mathcal{M}^\perp \cap \mathcal{N}^\perp \cap X = \{0\}$, then $(\mathcal{M}+\mathcal{N})^\perp \cap X = \{0\}$. Therefore, $X = \mathcal{M}+\mathcal{N}$, since $X \supseteq \mathcal{M}+\mathcal{N}$ and $\mathcal{M}+\mathcal{N}$ is closed. It is easy to see that $c_0(\mathcal{M},\mathcal{N}) = c_0(\mathcal{M}\cap X, \mathcal{N}\cap X)$, then we may replace $\mathcal{H}$ by X. It follows from Step 2 that $c_0(\mathcal{M},\mathcal{N}) \le c_0(\mathcal{M}^\perp \cap X, \mathcal{N}^\perp \cap X)$. $\Box$

The following result has been proved in [2, 7]. As an application of Theorem 8, we give an alternative proof.

Corollary 12 ([2, 7]) *If A and B are bounded operators on $\mathcal{H}$ with closed ranges, then the following statements are equivalent:*

(a) AB has closed range;

(b) $c(\mathcal{R}(B), \mathcal{N}(A)) < 1$;

(c) $\mathcal{R}(B)+\mathcal{N}(A)$ is closed.

Proof If $AB = 0$, then the conclusion is clear. In the following proof, assume that $AB \neq 0$. Since $\mathcal{R}(A)$ is closed, we have

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{N}(A)^\perp \oplus \mathcal{N}(A) \to \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp,$$

where A_1 is invertible from $\mathcal{N}(A)^\perp$ onto $\mathcal{R}(A)$. It is easy to see that

$$\mathcal{R}(AB) = \mathcal{R}(A_1 P_{\mathcal{N}(A)^\perp} P_{\mathcal{R}(B)}).$$

Since A_1 is invertible, $\mathcal{R}(AB)$ is closed $\Longleftrightarrow \mathcal{R}(P_{\mathcal{N}(A)^\perp} P_{\mathcal{R}(B)})$ is closed $\Longleftrightarrow \gamma(P_{\mathcal{N}(A)^\perp} P_{\mathcal{R}(B)}) > 0$ $\Longleftrightarrow c(\mathcal{R}(B), \mathcal{N}(A)) < 1$, by Theorem 8. $\Box$

The following theorem is our another main result which is an extension of Theorem 8 of [3].

Theorem 13 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

$$\gamma(\mathcal{M},\mathcal{N}) = \begin{cases} 1, & \text{if } P_\mathcal{M} P_\mathcal{N} \text{ is an orthogonal projection;} \\ (1-\|Q\|)^{\frac{1}{2}}, & \text{if } P_\mathcal{M} P_\mathcal{N} \text{ is not an orthogonal projection.} \end{cases}$$

Especially, $\mathcal{M}+\mathcal{N}$ is closed if and only if $\gamma(\mathcal{M},\mathcal{N}) > 0$.

Proof If $P_\mathcal{M} P_\mathcal{N}$ is an orthogonal projection, then $\mathcal{H}_5 = \mathcal{H}_6 = 0$, so

$$P_\mathcal{M} = I_1 \oplus I_2 \oplus 0I_3 \oplus 0I_4, \text{ and } P_\mathcal{N} = I_1 \oplus 0I_2 \oplus I_3 \oplus 0I_4.$$

Case 1 If $\mathcal{M}\cap\mathcal{N}^\perp = \{0\}$, then $\mathcal{M} \subseteq \mathcal{N}$, by the definition of $\gamma(\mathcal{M},\mathcal{N})$, we have $\gamma(\mathcal{M},\mathcal{N}) = 1$.

Case 2 If $\mathcal{M}^\perp\cap\mathcal{N} = \{0\}$, then $\mathcal{M} \supseteq \mathcal{N}$, so $\gamma(\mathcal{M},\mathcal{N}) = 1$.

Case 3 If $\mathcal{M}^\perp\cap\mathcal{N} \neq \{0\}$ and $\mathcal{M}\cap\mathcal{N}^\perp \neq \{0\}$, let $x \in \mathcal{M}\setminus\mathcal{N}$. Then $x = x_1 + x_2$, where $x_1 \in \mathcal{M}\cap\mathcal{N}$ and $0 \neq x_2 \in \mathcal{M}\cap\mathcal{N}^\perp$, since $P_\mathcal{M} P_\mathcal{N}$ is an orthogonal projection. It is easy to see that

$$\operatorname{dist}(x,\mathcal{N}) = \inf\{\|x-y\| : y \in \mathcal{N}\} = \inf\{\|x_2-y\| : y \in \mathcal{N}\} = \|x_2\|,$$

$$\operatorname{dist}(x,\mathcal{M}\cap\mathcal{N}) = \inf\{\|x-y\| : y \in \mathcal{M}\cap\mathcal{N}\} = \inf\{\|x_2-y\| : y \in \mathcal{M}\cap\mathcal{N}\} = \|x_2\|.$$

Hence $\gamma(\mathcal{M},\mathcal{N})=1$.

If $P_{\mathcal{M}}P_{\mathcal{N}}$ is not an orthogonal projection, then $\mathcal{H}_5\neq 0$ and $\mathcal{H}_6\neq 0$. For a vector $x\in\mathcal{M}\setminus\mathcal{N}$, x has the decomposition $x=x_1+x_2+x_5$ with $x_i\in\mathcal{H}_i, i=1,2,5$, then $\|x_2\|^2+\|x_5\|^2\neq 0$, so

$$\begin{aligned}
\gamma(\mathcal{M},\mathcal{N}) &= \inf_{x\in\mathcal{M},x\notin\mathcal{N}}\frac{\operatorname{dist}(x,\mathcal{N})}{\operatorname{dist}(x,\mathcal{M}\cap\mathcal{N})}=\inf_{x\in\mathcal{M},x\notin\mathcal{N}}\sqrt{\frac{\|x_2\|^2+\|(I_5-Q)^{\frac{1}{2}}x_5\|^2}{\|x_2\|^2+\|x_5\|^2}}\\
&=\inf_{x\in\mathcal{M},x\notin\mathcal{N}}\frac{\|(I_5-Q)^{\frac{1}{2}}x_5\|}{\|x_5\|}=\inf_{x_5\in\mathcal{H}_5\setminus\{0\}}\frac{\|(I_5-Q)^{\frac{1}{2}}x_5\|}{\|x_5\|}\\
&\quad\text{(note that } x_5\in\mathcal{H}_5 \text{ implies } x_5\in\mathcal{M}\setminus\mathcal{N})\\
&=\gamma((I_5-Q)^{\frac{1}{2}}),
\end{aligned}$$

since $\mathcal{N}(I_5-Q)=\{0\}$. It follows from Lemma 3 that

$$\begin{aligned}
\gamma((I_5-Q)^{\frac{1}{2}}) &= (\inf\{\sigma(I_5-Q)\setminus\{0\}\})^{\frac{1}{2}}=(\inf\{\sigma(I_5-Q)\})^{\frac{1}{2}}\\
&=(1-\sup\{\lambda\in\mathbb{C}:\lambda\in\sigma(Q)\})^{\frac{1}{2}}=(1-\|Q\|)^{\frac{1}{2}}.
\end{aligned}$$

By Corollary 10 and Corollary 7, $\mathcal{M}+\mathcal{N}$ is closed $\Longleftrightarrow c(\mathcal{M},\mathcal{N})<1\Longleftrightarrow\|Q\|<1\Longleftrightarrow\gamma(\mathcal{M},\mathcal{N})>0$. □

Combining Theorem 8 and Theorem 13, we obtain the following result.

Corollary 14 *Let $\mathcal{M}$ and $\mathcal{N}$ be nonzero subspaces of $\mathcal{H}$. Then*

(a) If $\mathcal{M}\subseteq\mathcal{N}$, then $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}})=0$ and $\gamma(\mathcal{M},\mathcal{N})=1$;

(b) If $\mathcal{M}\nsubseteq\mathcal{N}$, then $\gamma(P_{\mathcal{N}^\perp}P_{\mathcal{M}})=\gamma(\mathcal{M},\mathcal{N})$;

(c) $\gamma(\mathcal{M},\mathcal{N})=\gamma(\mathcal{N}^\perp,\mathcal{M}^\perp)$.

References

[1] APOSTOL C. *The reduced minimum modulus* [J]. Michigan Math. J., 1985, **32**(3): 279–294.
[2] BOULDIN R. *The product of operators with closed range* [J]. Tôhoku Math. J., 1973, **25**(2): 359–363.
[3] DENG Chunyuan. *On the minimum gap and the angle between two subspaces* [J]. J. Comput. Appl. Math., 2007, **206**(2): 625–630.
[4] DENG Chunyuan, DU Hongke. *Common complements of two subspaces and an answer to GroB's question* [J]. Acta Math. Sinica (Chin. Ser.), 2006, **49**(5): 1099–1112.
[5] DU Hongke, DENG Chunyuan. *A new characterization of gaps between two subspaces* [J]. Proc. Amer. Math. Soc., 2005, **133**(10): 3065–3070.
[6] DUGGAL B P. *Subspace gaps and range-kernel orthogonality of an elementary operator* [J]. Linear Algebra Appl., 2004, **383**: 93–106.
[7] DEUTSCH F. *The Angle between Subspaces of a Hilbert Space* [M]. Kluwer Acad. Publ., Dordrecht, 1995.
[8] HALMOS P R. *Two subspaces* [J]. Trans. Amer. Math. Soc., 1969, **144**: 381–389.
[9] KATO T. *Perturbation Theorty for Linear Operators* [M]. Springer, New York, 1966.
[10] TANG Waishing. *Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces* [J]. Proc. Amer. Math. Soc., 2000, **128**(2): 463–473.
[11] YAO Xiyan. *A note on regular pairs of orthogonal projections* [J]. Adv. Math. (China), 2005, **34**(5): 631–636.