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Applying Multiquadric Quasi-Interpolation to Solve KdV Equation

Min Lu XIAO, Ren Hong WANG, Chun Gang ZHU*
School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024, *P. R. China*

Abstract Quasi-interpolation is very useful in the study of approximation theory and its applications, since it can yield solutions directly without the need to solve any linear system of equations. Based on the good performance, Chen and Wu presented a kind of multiquadric (MQ) quasi-interpolation, which is generalized from the $\mathscr{L}_{\mathscr{D}}$ operator, and used it to solve hyperbolic conservation laws and Burgers' equation. In this paper, a numerical scheme is presented based on Chen and Wu's method for solving the Korteweg-de Vries (KdV) equation. The presented scheme is obtained by using the second-order central divided difference of the spatial derivative to approximate the third-order spatial derivative, and the forward divided difference to approximate the temporal derivative, where the spatial derivative is approximated by the derivative of the generalized $\mathscr{L}_{\mathscr{D}}$ quasi-interpolation operator. The algorithm is very simple and easy to implement and the numerical experiments show that it is feasible and valid.

Keywords KdV equation; multiquadric(MQ) quasi-interpolation; numerical solution.

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1. Introduction

Since Hardy proposed in 1968 the multiquadric(MQ) which is a kind of radial basis function (RBF), it has been investigated thoroughly. Hardy [1] summarized the achievements of study of multiquadric (MQ) from 1968 to 1988 and showed that MQ can be applied in hydrology, geodesy, photogrammetry, surveying and mapping, geophysics and crustal movement, geology and mining and so on. Now, the RBFs have found wider and wider applications. Since Kansa [2, 3] successfully modified MQ for solving partial differential equation (PDE), more and more researchers have been attracted by this meshless, scattered data approximation scheme. In most of the known methods of solving differential equations using multiquadric, one must resolve a system of linear equations at each time step. Hon and Wu [4], Wu [5, 6] and others have provided some successful examples using MQ quasi-interpolation to solve differential equations.

Beaston and Powell [7] proposed three univariate multiquadric quasi-interpolations, namely, $\mathscr{L}_{\mathscr{A}}$, $\mathscr{L}_{\mathscr{B}}$, $\mathscr{L}_{\mathscr{C}}$. Wu and Schaback [8] presented the univariate multiquadric quasi-interpolation

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* Corresponding author
E-mail address: cgzhu@dlut.edu.cn (C. G. ZHU)

$\mathscr{L}_{\mathscr{D}}$ and proved that the scheme is shape preserving and convergent. In [9–11], Chen and Wu generalized the MQ quasi-interpolant $\mathscr{L}_{\mathscr{D}}$ and used it to solve second-order differential equations. The numerical schemes using generalized $\mathscr{L}_{\mathscr{D}}$ to solve Burgers' equation and hyperbolic conservation laws were presented in [9, 11] and [10], respectively. Moreover, Chen, Han and Wu [12] have done more work and showed the relation between the generalized $\mathscr{L}_{\mathscr{D}}$ in [9–11] and $\mathscr{L}_{\mathscr{D}}$. Besides, Ma and Wu [13] studied the approximation properties to the k-th derivatives by multiquadric quasi-interpolation and showed successful examples in [14].

Thanks to so many people who are introduced before, the MQ quasi-interpolation are wider and wider used to solve PDEs. Based on Chen and Wu's method [9–11], the numerical scheme with the generalized $\mathscr{L}_{\mathscr{D}}$ for solving Kortewege-de Vries (KdV) equations with the third-order spatial derivative variable is presented in this paper.

Kortewege-de Vries (KdV) equation is a nonlinear partial differential equation, which is given by:

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0,$$

where ε and μ are positive real constants. This equation shows both dispersion and nonlinearity. Gardner et al. [15] have shown the existence and uniqueness of solutions of the KdV equation are necessary for various boundary and initial conditions to model many physical events. So there has been a considerable interest in the numerical solution of a class of KdV equation. Many well-known numerical techniques such as finite-difference scheme, finite-element schemes, Fourier spectral methods and meshfree radial basis functions collation method [16–20] have been used to solve the KdV equation. A small time solution for the KdV equation has been found by heat balance integral (HBIM) method [21], which has been suggested to be used to initialize some other numerical methods at some small time when an exact solution of the KdV equation does not exist [22]. A higher accuracy method than the classical explicit finite difference (CFDM) and HBIM in getting the numerical solution of the KdV equation at small-time was given by Bahadir [23], which is exponential finite-difference method (EFDM).

In this paper, we use the generalized $\mathscr{L}_{\mathscr{D}}$ quasi-interpolation to solve the KdV equation numerically. So we do not have to solve any system of linear equations and thus we do not meet the question of the ill-condition of the matrix in [20]. So the computational time is saved. In our methods, we use the derivative of the MQ quasi-interpolation to approximate the first-order spatial derivative and employ a frist order forward divided difference to approximate the temporal derivative as Chen and Wu did in [9–11]. Besides, we use the second-order central divided difference of the first-order spatial derivative to approximate the third-order spatial derivative since the performance of approxiamtion of high order derivatives by MQ quasi-interpolation is not good.

The rest of this paper is organized as follows. In Section 2, we introduce the theory of the univariate MQ quasi-interpolation briefly. The numerical scheme using MQ quasi-interpolation to solve the KdV equation is presented in Section 3. In Section 4, we apply the method on the third-order nonlinear equation, namely, KdV equation, with three different initial values. The results are compared with the analytical solutions and the results in [20, 21]. We can find that the results are also acceptable, so the scheme is valid. Finaly, we make a brief conclusion and give remaks for the resulting scheme and further works in Section 5.

2. Univariate multiquadric quasi-interpolation

According to the papers [9–11], we introduce the univariate multiquadric quasi-interpolation. Beaston and Powell [7] proposed three univariate multiquadric quasi-interpolations, namely, $\mathscr{L}_{\mathscr{A}}$, $\mathscr{L}_{\mathscr{B}}$, $\mathscr{L}_{\mathscr{C}}$, to approximate a function $\{f(x), x_0 \leqslant x \leqslant x_m\}$ from the space that is spanned by the multiquadrics $\{\phi_j(x) = \sqrt{(x-x_j)^2+c^2}, x \in \mathbb{R}, j = 0, \dots, m\}$ and linear function, where c is positive constant and the centers $\{x_j : j = 0, \dots, m\}$ are given distinct points in interval $[x_0, x_m]$. In 1994, Wu and Schaback [8] proposed the univariate multiquadric quasi-interpolation $\mathscr{L}_{\mathscr{D}}$ on $[x_0, x_m]$ and proved that the scheme is shape preserving and convergent.

Given $\{(x_j, y_j)\}_{j=0}^m$, where $x_0 < x_1 < \cdots < x_m$, the univariate quasi-interpolation is of the form

$$f^*(x) = \sum_{j=0}^{m} f_j \Psi_j(x), \tag{1}$$

where

$$\Psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1}-x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad 0 \le j \le m. \tag{2}$$

The definition of $\phi_j(x)$ will be given whereafter.

Now we introduce some definitions and theorems related to quasi-interpolation .

Definition 2.1 *If the quasi-interpolation $f^*(x)$ possesses the property*

$$f^*(x) \equiv C \quad \text{if} \quad f_0 = f_1 = \cdots = f_m = C, \tag{3}$$

where C is any real constant, we say that the quasi-interpolation is constant reproducing on $[x_0, x_m]$.

Definition 2.2 *We say that the quasi-interpolation $f^*(x)$ possesses linear reproducing property on $[x_0, x_m]$, if $f^*(x) = px + q$ as $f_j = px_j + q$, $j = 0, \dots, m$, for all $p, q \in \mathbb{R}$.*

Remark 2.1 It is obvious that if a quasi-interpolation $f^*(x)$ possesses linear reproducing property on $[x_0, x_m]$, then it must be constant reproducing.

Definition 2.3 *If the quasi-interpolation $f^*(x)$ is monotone increasing (decreasing) for monotone increasing (decreasing) data f_j, $j = 0, \dots, m$, then we say that it possesses preserving monotonicity on $[x_0, x_m]$.*

Given $\{(x_j, y_j)\}_{j=0}^m$, where $x_0 < x_1 < \cdots x_m$, Wu and Shacback defined in [8] the univariate multiquadric quasi-interpolation $\mathscr{L}_{\mathscr{D}}$ as follows:

$$(\mathscr{L}_{\mathscr{D}} f)(x) = f_0\alpha_0(x) + f_1\alpha_1(x) + \sum_{j=2}^{n-2} f_j \Psi_j(x) + f_{n-1}\alpha_{n-1}(x) + f_n\alpha_n(x), \tag{4}$$

where

$$\alpha_0(x) = \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\alpha_1(x) = \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},$$

$$\alpha_{n-1}(x) = \frac{x_n - x - \phi_{n-1}(x)}{2(x_n - x_{n-1})} - \frac{\phi_{n-1}(x) - \phi_{n-2}(x)}{2(x_{n-1} - x_{n-2})},$$

$$\alpha_n(x) = \frac{1}{2} + \frac{\phi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})}.$$

And $\phi_j(x)$, $j = 1, \ldots, n-1$ and $\Psi_j(x)$, $j = 2, \ldots, n-2$ are defined as follows.

Definition 2.4 ([8–11]) *For the initial data $\{(x_j, f_j)\}_{j=0}^m$, $f_j = f(x_j)$, we define $f^*(x)$ on $[x_0, x_m]$ with (1), (2),*

$$\phi_m(x) = \phi_0(x) - 2x + x_m + x_0, \tag{5}$$

$$\begin{cases} \phi_{-1}(x) = \phi_0(x) + x_0 - x_{-1}, \\ \phi_{m+1}(x) = \phi_m(x) + x_{m+1} - x_m \end{cases} \tag{6}$$

and

$$\phi_j(x) = \sqrt{(x - x_j)^2 + c^2}, \quad 0 \le j \le m-1, \ c \in \mathbb{R}. \tag{7}$$

Then $f^*(x)$ is univariate multiquadric quasi-interpolation $(\mathscr{L}_{\mathscr{D}} f)(x)$.

Theorem 2.1 ([8–11]) *The quasi-interpolation $f^*(x)$, defined by Definition 2.4, possesses linear reproducing property and preserving monotonicity on $[x_0, x_m]$. Meantime, on $[x_0, x_m]$, $f^*(x)$ can be rewritten as follows:*

$$\begin{aligned} f^*(x) =& \frac{1}{2} \sum_{j=1}^{m-1} \left(\frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}\right) f_j + \\ & \frac{1}{2}\left(1 + \frac{\phi_1(x) - \phi_0(x)}{(x_1 - x_0)}\right) f_0 + \frac{1}{2}\left(1 - \frac{\phi_m(x) - \phi_{m-1}(x)}{(x_m - x_{m-1})}\right) f_m; \end{aligned} \tag{8}$$

or

$$f^*(x) = \frac{f_0 + f_m}{2} + \frac{1}{2} \sum_{j=0}^{m-1} \frac{\phi_j(x) - \phi_{j+1}(x)}{x_{j+1} - x_j} (f_{j+1} - f_j); \tag{9}$$

or

$$\begin{aligned} f^*(x) =& \frac{1}{2} \sum_{j=1}^{m-1} \left(\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right) \phi_j(x) + \frac{f_0 + f_m}{2} + \\ & \frac{f_1 - f_0}{2(x_1 - x_0)} \phi_0(x) - \frac{f_m - f_{m-1}}{2(x_m - x_{m-1})} \phi_m(x). \end{aligned} \tag{10}$$

Moreover, on $[x_0, x_m]$, we have

$$(f^*(x))' = \frac{1}{2} \sum_{j=0}^{m-1} \frac{\phi_j'(x) - \phi_{j+1}'(x)}{x_{j+1} - x_j} (f_{j+1} - f_j) \tag{11}$$

and

$$(f^*(x))'' = \frac{1}{2} \sum_{j=0}^{m-1} \frac{\phi_j''(x) - \phi_{j+1}''(x)}{x_{j+1} - x_j} (f_{j+1} - f_j). \tag{12}$$

Remark 2.2 We note that formulae (8)–(12) and the linear reproducing property of the quasi-interpolation $f^*(x)$ have no relation to the definition of $\phi_j(x)$, $j = 1, \ldots, m-1$ i.e., (7). In other words, all quasi-interpolation $f^*(x)$ defined by (1), (2), (5) and (6) satisfy (8)–(12) and possesses the linear reproducing property.

Theorem 2.2 ([8–12]) *Denote* $h = \max_{1 \le j \le m} \{x_j - x_{j-1}\}$. $f^*(x)$ *is the univariate multiquadric quasi-interpolation defined by Definition 2.4. For* $c > 0$, *and* $f_x \in C^2(x_0, x_m)$, *we have*

$$\|f^*(x) - f(x)\|_\infty \le K_0 C_h + K_1 h^2 + K_2 ch + K_3 c^2 \log h, \tag{13}$$

where

$$C_h = \min\{c, \frac{c^2}{h}\}, \tag{14}$$

K_0, K_1, K_2, *and* K_3 *are the positive constant independent of* h *and* c.

Remark 2.3 As $c = 0$, $f^*(x)$ changes into $L(x)$, and now $\|f^*(x) - f(x)\|_\infty \le Kh^2$, where K is a constant which is independent of h.

The content above can be found in [8–11] and we iterate them only for the sake of integrality of the paper.

3. Numerical scheme using MQ quasi-interpolation

In this section, we consider the KdV equation, which is a third-order nonlinear equation

$$\frac{\partial u(x,t)}{\partial t} + \varepsilon u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \quad x \in \Omega = [a,b] \subset \mathbb{R}, \ t > 0 \tag{15}$$

with the initial condition

$$u(x,t) = u^0(x), \quad t = 0, \tag{16}$$

and boundary conditions

$$u(x,t) = f(t), \quad x \in \partial\Omega, \ t > 0, \tag{17}$$

$$u_x(b,t) = g(t), \quad t > 0. \tag{18}$$

We show the numerical scheme for solving KdV equation by using the multiquadric(MQ) quasi-interpolation as Chen and Wu did in [9–11].

Discretizing KdV equation

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0, \tag{19}$$

in time with time step τ, we get

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \varepsilon u_j^n (u_x)_j^n + \mu (u_{xxx})_j^n = 0, \tag{20}$$

i.e.,

$$u_j^{n+1} = u_j^n - \tau \cdot (\varepsilon u_j^n (u_x)_j^n + \mu (u_{xxx})_j^n), \tag{21}$$

where u_j^n is the approximation of value $u(x,t)$ at point (x_j, t_n), $x_j = jh$, $t_n = n\tau$. And we use the derivatives of the MQ quasi-interpolation to approximate u_x, where

$$(u_x)_j^n = \frac{1}{2}\sum_{k=0}^{m-1} \frac{\phi'_k(x_j) - \phi'_{k+1}(x_j)}{x_{k+1} - x_k}(u_{k+1}^n - u_k^n). \tag{22}$$

To approximate u_{xxx} more efficiently, we use the following approach instead of using the third derivatives of the MQ quasi-interpolation

$$(u_{xxx})_j^n = \frac{(u_x)_{j+1}^n - 2(u_x)_j^n + (u_x)_{j-1}^n}{h^2}. \tag{23}$$

$\phi_j(x)$, $j = 0, \ldots, m$ is defined in (5) and (7).

We compare the numerical results of the KdV equation by using this scheme with the analytical solutions and the solutions in [20, 21].

4. Numerical examples

In this section, we test our scheme by three examples. In the numerical results, we use the following norms to assess the perfromance of our scheme

$$L_\infty = \max_{1\le i\le N} \mid u_{\text{exact}}(i) - u_{\text{app}}(i) \mid,$$

$$L_2 = \sqrt{\sum_{i=1}^{N}(u_{\text{exact}}(i) - u_{\text{app}}(i))^2},$$

$$RMS = \sqrt{(\sum_{i=1}^{N}(u_{\text{exact}}(i) - u_{\text{app}}(i))^2)/N},$$

where u_{exact} is the exact soltution, u_{app} is the approximate solution of the KdV equation in our scheme and N is the total number of the space joints. Denote our scheme by MQQI. For the sake of the simplification, we set $h_j = h$, then $x_j = jh$, $j = 0, \ldots, m$.

Example 4.1 Propagation of single solitary wave [24]. In this example, we consider the KdV equation (19) with $\varepsilon = 6$ and $\mu = 1$. The initial condition is

$$u^0(x) = \frac{r}{2}\sec h^2(\frac{\sqrt{r}}{2}x - 7), \quad t = 0, r = 0.5 \tag{24}$$

and the exact solution is

$$u(x,t) = \frac{r}{2}\sec h^2(\frac{\sqrt{r}}{2}(x - rt) - 7), \quad r = 0.5 \tag{25}$$

and the boundary functions $f(t)$ and $g(t)$ can be obtained from the exact solution. We consider this example in the domain $0 \le x \le 40$. In our computation, we choose $\tau = 0.001$, $h = 0.2$. Table 1 shows the error with $c = 0.2799$ at $t = 1, 2, 3, 4, 5$. We compare the L_∞-error with the results in [20] in Table 2 and compare the numerical solution with the exact solution in Table 3. Moreover, in Figure 1, we illustrate the profiles of the exact and numerical solution at $t = 5$.

Example 4.2 Propagation of two solitary waves [24]. In this example, we consider the KdV equation (19) with $\varepsilon = 6$ and $\mu = 1$. The initial condition is

$$u^0(x) = 12\Big\{\frac{3 + 4\cosh(2x) + \cosh(4x)}{\{3\cosh(x) + \cosh(3x)\}^2}\Big\} \tag{26}$$

and the exact solution is

$$u(x,t) = 12\Big\{\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\{3\cosh(x - 28t) + \cosh(3x - 36t)\}^2}\Big\}. \tag{27}$$

The boundary functions $f(t)$ and $g(t)$ can be obtained from the exact solution.

t	L_∞-error	L_2-error	RMS-error
1	1.5259×10^{-3}	6.5913×10^{-3}	4.6492×10^{-4}
2	2.8672×10^{-3}	1.2404×10^{-2}	8.7394×10^{-4}
3	4.1428×10^{-3}	1.8221×10^{-2}	1.2852×10^{-3}
4	5.3859×10^{-3}	2.4106×10^{-2}	1.7003×10^{-3}
5	6.8141×10^{-3}	3.0300×10^{-2}	2.1372×10^{-3}

Table 1 The errror between the numerical solution using our scheme and exact solution of Example 4.1 with $c = 0.2799$, $\tau = 0.001$, $h = 0.2$, at $t = 1, 2, 3, 4, 5$

t	MQQI	RBF(MQ)[20]	RBF(IMQ)[20]
1	1.5259×10^{-3}	1.7923×10^{-5}	6.9584×10^{-5}
2	2.8672×10^{-3}	3.0151×10^{-5}	1.9553×10^{-4}
3	4.1428×10^{-3}	3.9839×10^{-5}	3.8286×10^{-3}
4	5.3859×10^{-3}	4.7835×10^{-5}	5.9098×10^{-3}
5	6.8141×10^{-3}	5.4599×10^{-5}	8.3667×10^{-3}

Table 2 The comparison of L_∞-error between the numerical solution using our scheme and the solution in [20] of Example 4.1 with $c = 0.2799$, $\tau = 0.001$, $h = 0.2$, at $t = 1, 2, 3, 4, 5$

x	$t = 1$		$t = 3$		$t = 5$	
	MQQI	Exact	MQQI	Exact	MQQI	Exact
17	0.081817	0.080625	0.045284	0.043573	0.024489	0.022515
18	0.137899	0.137393	0.082636	0.080625	0.045640	0.043573
19	0.204288	0.203886	0.139860	0.137393	0.084426	0.080625
20	0.247960	0.247227	0.206787	0.203886	0.142601	0.137393
21	0.235194	0.235251	0.249011	0.247227	0.208712	0.203886
22	0.176306	0.177627	0.233662	0.235251	0.249883	0.247227
23	0.110928	0.112353	0.173616	0.177627	0.231660	0.235251
24	0.062591	0.063421	0.108716	0.112353	0.171589	0.177627
25	0.033223	0.033545	0.061304	0.063420	0.106221	0.112353

Table 3 The comparison between the numerical solution using our scheme and exact solution of Example 4.1 with $c = 0.2799$, $\tau = 0.001$, $h = 0.2$, at $t = 1, 3, 5$

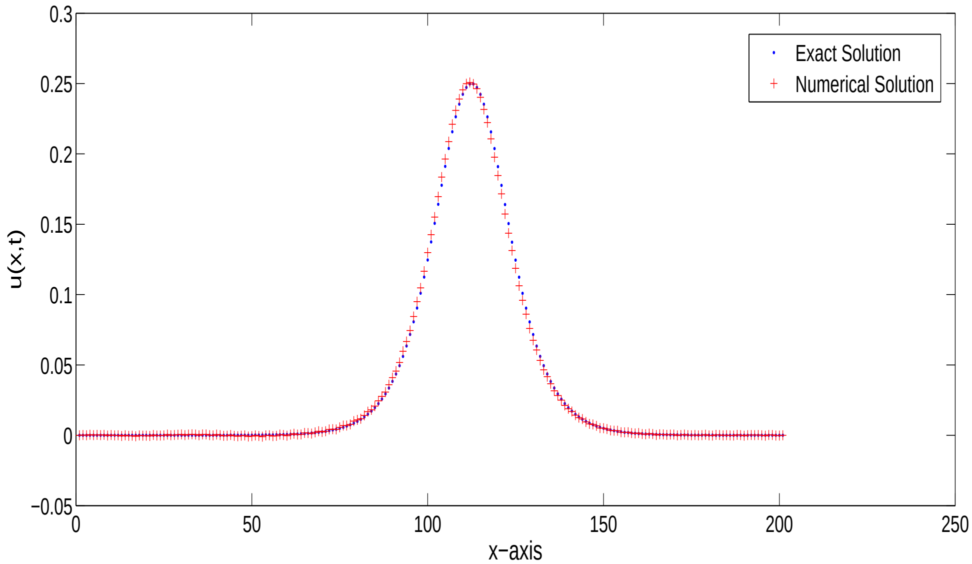

Figure 1 Analytical and estimated fuction of Example 4.1 with $c = 0.2799$, $\tau = 0.001$, $h = 0.2$, at $t = 5$

We consider this example in the domain $-5 \leq x \leq 15$. In our computation, we choose $\tau = 0.00001$, $h = 0.1$. Table 4 shows the error with $c = 0.001$ at $t = 0.01, 0.05, 0.1$. The comparison of the L_∞-error with the results in [20] is given in Table 5 and the comparison with the exact solution is given in Table 6. In Figure 2, the profiles of the exact and numerical solutions at $t = 0.1$ are illustrated.

t	L_∞-error	L_2-error	RMS-error
0.01	7.7405×10^{-3}	2.2328×10^{-2}	1.5749×10^{-3}
0.05	6.3762×10^{-2}	1.7355×10^{-1}	1.2241×10^{-2}
0.1	1.6196×10^{-1}	4.5430×10^{-1}	3.2044×10^{-2}

Table 4 The errror between the numerical solution using our scheme and exact solution of Example 4.2 with $c = 0.001, \tau = 0.00001, h = 0.1$, at $t = 0.01, 0.05, 0.1$

t	MQQI	RBF(MQ)[20]	RBF(IMQ)[20]
0.01	7.7405×10^{-3}	9.2114×10^{-4}	2.2071×10^{-2}
0.05	6.3762×10^{-2}	2.9608×10^{-2}	7.2316×10^{-2}
0.1	1.6196×10^{-1}	1.2806×10^{-2}	1.0121×10^{-1}

Table 5 The comparison of L_∞-error between the numerical solution using our scheme and the solution in [20] of Example 4.2 with $c = 0.001$, $\tau = 0.00001$, $h = 0.1$, at $t = 0.01, 0.05, 0.1$

x	$t=0.01$		$t=0.05$		$t=0.1$	
	MQQI	Exact	MQQI	Exact	MQQI	Exact
-3	0.054358	0.054477	0.038461	0.039507	0.023935	0.026554
-2	0.382756	0.382934	0.279287	0.275315	0.194571	0.188162
-1	2.084206	2.084133	1.380403	1.390946	1.038671	1.045967
0	5.640733	5.638245	2.597785	2.574829	1.990040	2.000572
1	3.186663	3.192964	6.922214	6.881609	1.764999	1.717101
2	0.478633	0.478495	1.191924	1.207024	7.139714	7.171392
3	0.064570	0.064520	0.101458	0.100955	0.452621	0.464299
4	0.008730	0.008723	0.012227	0.012239	0.024443	0.024308
5	0.001181	0.001180	0.001559	0.001630	0.002261	0.002542

Table 6 The comparison between the numerical solution using our scheme and exact solution of Example 4.2 with $c=0.001,\ \tau=0.00001,\ h=0.1$, at $t=0.01, 0.05, 0.1$

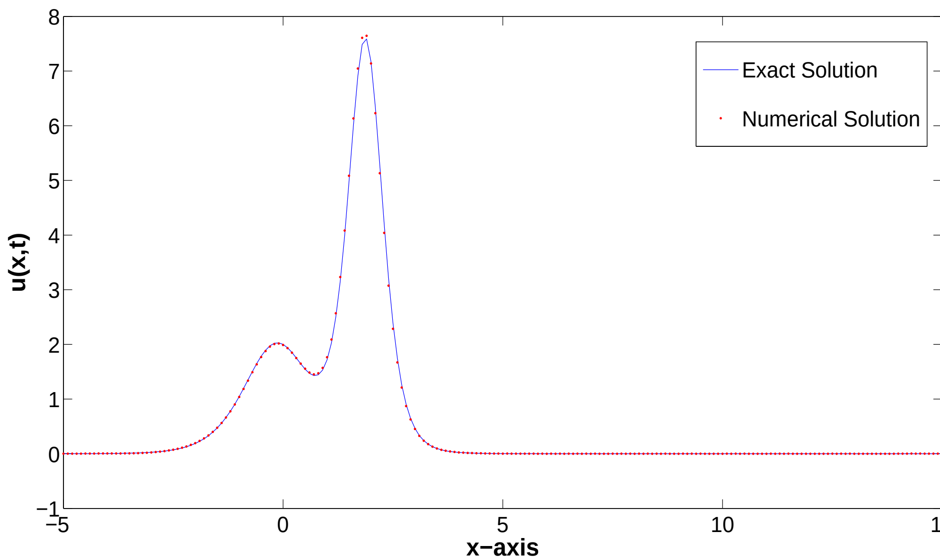

Figure 2 Analytical and estimated function of Example 4.2 with $c=0.001,\ \tau=0.00001,\ h=0.1$, at $t=0.1$

Example 4.3 A special model problem of KdV equation was investigated in [20, 21]. We consider the KdV equation(19) with $\varepsilon=1$ and $\mu=4.84\times10^{-4}$. The initial condition is

$$u^0(x)=u(x,0)=3C_1\sec h^2(A_1x+D_1), 0\le x\le 2 \tag{28}$$

and the boundary condition is

$$u(0,t)=u(2,t)=u_x(2,t)=0,\quad t>0. \tag{29}$$

The exact solution of this problem is taken from [25] and is given by

$$u(x,t)=3C_1\sec h^2(A_1x-B_1t+D_1),\quad 0\le x\le 2 \tag{30}$$

where C_1, D_1 are real constants, $A_1 = \frac{1}{2}\sqrt{\varepsilon C_1/\mu}$ and $B_1 = \varepsilon A_1 C_1$.

In this example, the results of the percentage error are compared with those given in [20, 21] in Table 7. These schemes include RBF(MQ) scheme [20] and HBIM scheme [21] with $\tau = 0.001$, $h = 0.0125$, $C_1 = 0.3$, $D_1 = -6.0$ and the shape parameter $c = 0.0001$.

x	$t = 0.005$			$t = 0.01$	
	MQQI	RBF(MQ)[20]	HBIM[21]	MQQI	HBIM[21]
0.1	0.1082	0.0	3.8033	1.7176	7.7548
0.2	0.1019	0.0003	3.7984	0.2049	7.7418
0.3	0.0597	0.0003	3.7243	0.1310	7.5905
0.4	0.1095	0.0103	2.9326	0.2491	5.9806
0.5	0.0691	0.0060	0.7865	0.1299	1.5010
0.6	0.0821	0.0101	3.2960	0.1626	6.4706
0.7	0.0534	0.0015	3.6331	0.1004	7.1332
0.8	0.0748	0.0007	3.6626	0.1488	7.1911
0.9	0.0767	0.0088	3.6656	0.1533	7.1904
1.0	0.0769	0.0	3.7353	0.1538	7.2016

Table 7 The percentage error using different schemes of Example 4.3 at $t = 0.005, 0.01.$ and choosing $c = 0.0001$, $\tau = 0.001$, $h = 0.0125$ in the scheme

From the tables and figures above, we can say that the results of our scheme are acceptable, although the accuracy is not higher than the scheme in [20]. Besides, our scheme is simple and easy to implement, which means that our scheme is feasible and valid. And we can also find that our results in Example 4.3 are better than the results of the HBIM method in [21].

5. Conclusion

The multiquadric(MQ) quasi-interpolation method is applied to find the numerical solution of the KdV equation, which is a third-order nonlinear equation. From the above tables and figures, we conclude that our scheme is feasible and valid. During the computation, we can find that our scheme is simple and easy to implement and the results have very close relation to the value of the shape parameter c. And in fact, the choice of the shape parameter c is still a pendent question.

The scheme can also be used for non-equidistant grids, although we have used equidistant grids in our numerical experiments. Moreover, we can improve the accuracy by selecting the appropriate shape parameter c and using higher accurate MQ quasi-interpolation.

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