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On Newman-Type Rational Interpolation to $|x|$ at the Adjusted Chebyshev Nodes of the Second Kind

Lai Yi ZHU*, Ying Ying ZHAO
School of Information, People's University of China, Beijing 100872*, P. R. China*

Abstract Recently Brutman and Passow considered Newman-type rational interpolation to $|x|$ induced by arbitrary sets of symmetric nodes in $[-1,1]$ and gave the general estimation of the approximation error. By their methods, one could establish the exact order of approximation for some special nodes. In the present note we consider the sets of interpolation nodes obtained by adjusting the Chebyshev roots of the second kind on the interval $[0,1]$ and then extending this set to $[-1,1]$ in a symmetric way. We show that in this case the exact order of approximation is $O(\frac{1}{n^2})$.

Keywords Newman-type rational interpolation; adjusting the Chebyshev roots of the second kind; exact order of approximation.

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1. Introduction

Let

$$X = \{x_k^{(n)} : k = 1, 2, \ldots, n, 0 < x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)} \le 1\}$$

be a set of n distinct points in $(0, 1]$, and let

$$p_n(x) = \prod_{k=1}^{n} \left(x + x_k^{(n)}\right) \tag{1}$$

(in the sequence, when there is no confusion, the superscript (n) will be omitted).

The Newman-type rational interpolation to $|x|$ (see [3]) at the set of the points

$$\{-x_n, \ldots, -x_2, -x_1, 0, x_1, x_2, \ldots, x_n\} \tag{2}$$

is defined by

$$r_n(x) = r_n(X; x) = x\frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)}. \tag{3}$$

Since $r_n(X; x)$ as well as $|x|$ are even functions, the study of the approximation error

$$e_n(X; x) = |x| - r_n(X; x) \tag{4}$$

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* Corresponding author
E-mail address: zhulaiyi@ruc.edu.cn (L. Y. ZHU)

may be restricted to the interval $[0, 1]$, where it can be represented in the form

$$e_n(X;x) = \frac{2xh_n(X;x)}{1+h_n(X;x)}, \quad 0 \le x \le 1, \tag{5}$$

where

$$h_n(X;x) = \frac{p_n(-x)}{p_n(x)} = \prod_{k=1}^{n} \frac{-x+x_k}{x+x_k}. \tag{6}$$

Bernstein [1] showed that the order of the best uniform approximation of $|x|$ by polynomials is only $O(\frac{1}{n})$. In [6], to demonstrate the rational approximation to $|x|$ is much more favorable in contrast to polynomial approximation to $|x|$, Newman selected

$$x_k = \xi^k, \quad k = 1, 2, \ldots, n, \tag{7}$$

in (2), where

$$\xi = \exp(-n^{-\frac{1}{2}}),$$

and proved that for $n \ge 5$ the following bounds hold

$$\frac{1}{2}\mathrm{e}^{-9\sqrt{n}} \le |e_n(X;x)| \le 3\mathrm{e}^{-\sqrt{n}}, \quad x \in [-1, 1]. \tag{8}$$

There has been a great deal of work on Newman-type rational to $|x|$ since then. Brutman and Passow [3] established general estimates of the approximation error $e_n(X;x)$ for arbitrary set of interpolation points, and obtained

Theorem B *(P1) Let* $S_1 = S_1^{(n)}(X) = \sum_{k=1}^{n} x_k$. *Then*

$$|h_n(X;x)| \le \mathrm{e}^{-xS_1}, \quad x \in [0, 1]. \tag{9}$$

Theorem B *(P2) Let* $A_n = A_n(X) = \sum_{k=1}^{n} x_k^{-1}$. *Then*

$$|e_n(X;x)| \le \frac{1}{A_n}, \quad x \in [-x_1, x_1]. \tag{10}$$

In what follows we denote by c positive constant (different each time, in general) that is absolute or depends on parameters not essential for the argument. If $A(k, n, x, \ldots)$ and $B(k, n, x, \ldots)$ are positive real numbers depending on parameters $q, k, n, x, \ldots$, then the notation

$$A(k, n, x, \ldots) = O(B(k, n, x, \ldots))$$

means that there exists positive real number c independent of $k, n, x, \ldots$, such that

$$A(k, n, x, \ldots) \le cB(k, n, x, \ldots).$$

Brutman [5] studied the special case where the set of nodes is obtained by adjusting the Chebyshev roots $\xi_k^{(n)} = \cos((2k-1)\pi/(2n))$, $k = 1, 2, \ldots, n$, to the interval $[0, 1]$, namely

$$X = \widetilde{T} = \{x_k = \frac{1}{2}(1+\xi_{n-k+1}) = \sin^2\frac{2k-1}{4n}\pi : k = 1, 2, \ldots, n\}$$

and proved that in this case the exact order of approximation is $O(\frac{1}{n^2})$. Zhu and Dong [7] considered the special case where set of interpolation points

$$X = U = \{x_k = \cos\frac{k\pi}{2n+1} : k = 1, 2, \ldots, n\}$$

is that of the zeros contained in $(0,1]$ of the Chebyshev polynomial of the second kind

$$U_{2n}(x)=\frac{\sin[(2n+1)\arccos x]}{\sqrt{1-x^2}},$$

and they proved in this case the exact order of approximation is $O(\frac{1}{n\ln n})$. In present paper we consider the set of nodes obtained by adjusting the Chebyshev roots of the second kind

$$x_k^{(n)}=\cos\frac{k\pi}{n+1},\quad k=1,2,\ldots,n, \tag{11}$$

to the interval $(0,1]$, namely,

$$X=T=\{z_k=\frac{1}{2}(1+x^{(n)}_{n-k+1})=\sin^2\frac{k\pi}{2(n+1)}:k=1,2,\ldots,n\}, \tag{12}$$

and prove that in this case the exact order of approximation is $O(\frac{1}{n^2})$.

2. Results

In order to prove our main results, we firstly estimate S_1 in (9) and A_n in (10) in the case $X=T$.

Lemma 1 *For* $n=1,2,\ldots,$

$$z_k=\sin^2\frac{k\pi}{2(n+1)},\quad k=1,2,\ldots,n,$$

the following estimate holds:

$$S_1=\sum_{k=1}^{n}z_k=\frac{n}{2}. \tag{13}$$

Proof Since

$$\begin{aligned}S_1&=\sum_{k=1}^{n}z_k=\sum_{k=1}^{n}\sin^2\frac{k\pi}{2(n+1)}=\sum_{k=1}^{n}\frac{1-\cos\frac{k\pi}{n+1}}{2}\\&=\frac{n}{2}-\frac{1}{2}\sum_{k=1}^{n}\cos\frac{k\pi}{n+1}=\frac{n}{2}-\frac{1}{2}\frac{\sin\frac{2n+1}{2(n+1)}\pi}{2\sin\frac{\pi}{2(n+1)}}+\frac{1}{4}=\frac{n}{2},\end{aligned}$$

we obtain $S_1=\frac{n}{2}$.

Lemma 2 *For* $n=1,2,\ldots,$

$$z_k=\sin^2\frac{k\pi}{2(n+1)},\quad k=1,2,\ldots,n,$$

the following estimate holds:

$$\frac{4}{\pi^2}(n+1)^2\le A_n=\sum_{k=1}^{n}{z_k}^{-1}\le\frac{\pi^2}{6}(n+1)^2. \tag{14}$$

Proof Using the elementary inequality

$$\frac{2t}{\pi}\le\sin t\le t,\quad t\in[0,\frac{\pi}{2}], \tag{15}$$

we have
$$\Big(\frac{k}{n+1}\Big)^2 \le \sin^2\frac{k\pi}{2(n+1)} \le \Big(\frac{k\pi}{2(n+1)}\Big)^2, \quad k=1,2,\ldots,n.$$
Thus (14) follows immediately from the inequality
$$A_n=\sum_{k=1}^{n}\Big[\sin^2\frac{k\pi}{2(n+1)}\Big]^{-1} \ge \sum_{k=1}^{n}\frac{4(n+1)^2}{k^2\pi^2}=\frac{4(n+1)^2}{\pi^2}\sum_{k=1}^{n}\frac{1}{k^2}\ge\frac{4(n+1)^2}{\pi^2},$$
and the inequality
$$A_n=\sum_{k=1}^{n}\Big[\sin^2\frac{k\pi}{2(n+1)}\Big]^{-1} \le \sum_{k=1}^{n}\frac{(n+1)^2}{k^2}=(n+1)^2\sum_{k=1}^{n}\frac{1}{k^2}\le\frac{\pi^2(n+1)^2}{6}.$$
This completes the proof of Lemma 2. □

It is well-known [2, p37] that the nth Chebyshev polynomial of the second kind $U_n(x)$ has the following three representations
$$U_n(x)=\frac{[\sin(n+1)\arccos x]}{\sqrt{1-x^2}}, \quad x\in[-1,1], \tag{16}$$
$$U_n(x)=\frac{(x+\sqrt{x^2-1})^{n+1}-(x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}, \quad x\in\mathbb{C}, \tag{17}$$
$$U_n(x)=\sum_{k=0}^{n}x^kT_{n-k}(x), \quad x\in\mathbb{C} \tag{18}$$
where $\mathbb{C}$ denotes the set of complex numbers, and
$$T_n(x)=\cos(n\arccos x), \quad x\in[-1,1],$$
$$T_n(x)=\frac{1}{2}\Big[(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n\Big], \quad x\in\mathbb{C}$$
are the nth Chebyshev polynomial, $n=0,1,2,\ldots$. It is easy to see from (1) and (5) that $h_n(T;x)$ can be represented in the form
$$h_n(T;x)=\frac{U_n(1-2x)}{U_n(1+2x)}, \quad x\in[0,1]. \tag{19}$$

Secondly, we need the following estimate.

Lemma 3 *For any $x\in[z_1,\frac{1}{2}]$ and $n=1,2,\ldots$, the following estimate holds:*
$$|h_n(T;x)|\le\frac{1}{2}. \tag{20}$$

Proof By (18),
$$U_n'(x)=\sum_{k=1}^{n}kx^{k-1}T_{n-k}(x)+\sum_{k=0}^{n}x^kT_{n-k}'(x).$$

Since $T_n(x)\ge 0$ and $T_n'(x)=\frac{n}{2}[\frac{(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}}-\frac{(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}]>0$, $|x|\ge 1$. It suffices to verify that $U_n(x)$ is strictly monotone increasing in $x\in[1,\infty)$, and
$$U_n(1)=\lim_{x\to 1}\frac{(x+\sqrt{x^2-1})^{n+1}-(x-\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}$$

$$= \lim_{x\to 1} \frac{(n+1)\left[(x+\sqrt{x^2-1})^{n+1}+(x-\sqrt{x^2-1})^{n+1}\right]/\sqrt{x^2-1}}{2x/\sqrt{x^2-1}}$$
$$= n+1.$$

For any $x \in [z_1, \frac{1}{2}]$ and $n = 1, 2, \ldots$, using (16) yields

$$\begin{aligned} |U_n(1-2x)| &= \frac{|\sin(n+1)\arccos(1-2x)|}{\sqrt{1-(1-2x)^2}} \le \frac{1}{\sqrt{-4x^2+4x}} \le \frac{1}{\sqrt{-4z_1^2+4z_1}} \\ &= \frac{1}{2\sqrt{z_1}\sqrt{1-z_1}} = \frac{1}{\sin\frac{\pi}{n+1}} \le \frac{n+1}{2}. \end{aligned} \tag{21}$$

On the other hand, since $U_n(x)$ is strictly monotone increasing in $x \in [1, \infty)$, and $U_n(1) = n+1$, we can conclude that

$$U_n(1+2x) \ge U_n(1) = n+1. \tag{22}$$

Then it follows from (21) and (22), we can complete the proof of Lemma 3. □

Now we are in position to prove our main results.

Theorem 1 *For any $x \in [-1, 1]$ and $n = 1, 2, \ldots$, there exists a positive number c such that*

$$|e_n(T; x)| \le \frac{c}{n^2}. \tag{23}$$

Proof It suffices to consider $x \in [0, 1]$.

Firstly we consider the case $x \in [0, z_1] = [0, \sin^2\frac{\pi}{2(n+1)}]$. By applying the general estimate (10) and (14), we get

$$|e_n(T; x)| \le \frac{1}{A_n} \le \frac{\pi^2}{4(n+1)^2} \le \frac{\pi^2}{4n^2}, \quad x \in [0, z_1]. \tag{24}$$

Now let us consider the error of approximation in the interval $[z_1, \frac{1}{2}]$.

From (5), (16), (17) and (20), we can get

$$\begin{aligned} |e_n(T;x)| &= \frac{2x|h_n(T;x)|}{|1+h_n(T;x)|} \le \frac{2x|h_n(T;x)|}{1-|h_n(T;x)|} \le 4x|h_n(T;x)| \\ &\le \frac{|\sin(n+1)\arccos(1-2x)|\cdot 8x\sqrt{4x^2+4x}}{\sqrt{-4x^2+4x}\left[\left(1+2x+\sqrt{(1+2x)^2-1}\right)^{n+1}-\left(1+2x-\sqrt{(1+2x)^2-1}\right)^{n+1}\right]} \\ &\le \frac{8x\sqrt{\frac{1+x}{1-x}}}{\left(1+2x+\sqrt{(1+2x)^2-1}\right)^{n+1}-\left(1+2x-\sqrt{(1+2x)^2-1}\right)^{n+1}} \\ &\le \frac{8x\sqrt{\frac{1+1/2}{1-1/2}}}{\left(1+2x+\sqrt{(1+2x)^2-1}\right)^{n+1}-1} \le \frac{8x\sqrt{3}}{1+2n(n+1)x-1} \le \frac{4\sqrt{3}}{n^2}. \end{aligned} \tag{25}$$

Finally, we consider the case $x \in [\frac{1}{2}, 1]$. Using (9) and (13), we get

$$|h_n(T;x)| \le \mathrm{e}^{-\frac{1}{2}S_1} = \mathrm{e}^{-\frac{n}{4}}. \tag{26}$$

Thus for any $x \in [\frac{1}{2}, 1]$ and $n = 1, 2, \dots,$

$$\begin{aligned} |e_n(T;x)| &= \frac{2x|h_n(T;x)|}{|1+h_n(T;x)|} \le \frac{2x|h_n(T;x)|}{1-|h_n(T;x)|} \\ &\le \frac{2\mathrm{e}^{-\frac{n}{4}}}{1-\mathrm{e}^{-\frac{n}{4}}} = \frac{2}{\mathrm{e}^{\frac{n}{4}}-1} \\ &\le \frac{2}{1+\frac{1}{2}(\frac{n}{4})^2-1} = \frac{64}{n^2}. \end{aligned} \tag{27}$$

Comparison of (24), (25), and (27) completes the proof of the theorem. □

Next we show that the estimate (23) is sharp, namely, the following result holds.

Theorem 2 *Let $x^* = \frac{1}{4(n+1)^2}$. Then there exists a positive number c_0 such that*

$$|e_n(T;x^*)| \ge \frac{c_0}{(n+1)^2}. \tag{28}$$

Proof Note first that for $n \ge 1$, $x^* \in [0, z_1]$, and (6) implies $0 < h_n(T;x^*) \le 1$. We can write

$$4(n+1)^2|e_n(T;x^*)| = \frac{|e_n(T;x^*)|}{x^*} = \frac{2h_n(T;x^*)}{1+h_n(T;x^*)} \ge h_n(T;x^*). \tag{29}$$

Let $t_n = \arccos(1-2x^*) = \arccos[1-\frac{1}{2(n+1)^2}]$, which is equivalent to $\sin\frac{t_n}{2} = \frac{1}{2(n+1)}$. By applying the elementary inequality (15), we can get $\frac{t_n}{\pi} \le \sin\frac{t_n}{2} \le \frac{t_n}{2}$, which is equivalent to $\frac{1}{n+1} \le t_n \le \frac{\pi}{2(n+1)}$, thus

$$\sin(n+1)t_n \ge \sin 1. \tag{30}$$

For $n = 1, 2, \dots,$ we can write

$$\begin{aligned} U_n(1+2x^*) =& \frac{\left(1+2x^*+\sqrt{(1+2x^*)^2-1}\right)^{n+1} - \left(1+2x^*-\sqrt{(1+2x^*)^2-1}\right)^{n+1}}{2\sqrt{(1+2x^*)^2-1}} \\ \le& \frac{\left(1+2x^*+\sqrt{(1+2x^*)^2-1}\right)^{n+1}}{2\sqrt{(1+2x^*)^2-1}} \\ =& \frac{\left(1+\frac{1}{2(n+1)^2}+\sqrt{(\frac{1}{(n+1)^2}+\frac{1}{4(n+1)^4}}\right)^{n+1}}{2\sqrt{(1+2x^*)^2-1}} \\ \le& \frac{\left(1+\frac{1}{2(n+1)^2}+\frac{\sqrt{2}}{n+1}\right)^{n+1}}{2\sqrt{(1+2x^*)^2-1}} \le \frac{\left(1+\frac{2}{n+1}\right)^{n+1}}{2\sqrt{(1+2x^*)^2-1}} \le \frac{\mathrm{e}^2}{2\sqrt{4{x^*}^2+4x^*}}. \end{aligned} \tag{31}$$

Applying (30) and (31), we get

$$\begin{aligned} h_n(T;x^*) &= \frac{U_n(1-2x^*)}{U_n(1+2x^*)} \ge \frac{|\sin(n+1)\arccos(1-2x^*)|}{\sqrt{-4{x^*}^2+4x^*}} \frac{2\sqrt{4{x^*}^2+4x^*}}{\mathrm{e}^2} \\ &\ge \frac{2\sin 1}{\mathrm{e}^2}. \end{aligned} \tag{32}$$

Therefore (29) and (32) together yield

$$|e_n(T;x^*)| \ge \frac{\frac{2\sin 1}{e^2}}{4(n+1)^2} = \frac{\sin 1}{2\mathrm{e}^2(n+1)^2}.$$

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