Journal of Mathematical Research & Exposition
Mar., 2011, Vol. 31, No. 2, pp. 215–223
DOI:10.3770/j.issn:1000-341X.2011.02.004
Http://jmre.dlut.edu.cn

Incompleteness and Minimality of Exponential System

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Abstract Necessary and sufficient conditions are obtained for the incompleteness and the minimality of the exponential system $E(\Lambda, M) = \{z^l e^{\lambda_n z} : l = 0, 1, \dots, m_n - 1; n = 1, 2, \dots\}$ in the Banach space $E^2[\sigma]$ consisting of some analytic functions in a half strip. If the incompleteness holds, each function in the closure of the linear span of exponential system $E(\Lambda, M)$ can be extended to an analytic function represented by a Taylor-Dirichlet series. Moreover, by the conformal mapping $\zeta = \phi(z) = e^z$, the similar results hold for the incompleteness and the minimality of the power function system $F(\Lambda, M) = \{(\log \zeta)^l \zeta^{\lambda_n} : l = 0, 1, \dots, m_n - 1; n = 1, 2, \dots\}$ in the Banach space $F^2[\sigma]$ consisting of some analytic functions in a sector.

Keywords incompleteness; minimality; exponential system.

Document code A
MR(2010) Subject Classification 30E05; 41A30
Chinese Library Classification O174.5; O174.52

1. Introduction

Following, e.g., [1] and [2], a system $E = \{e_n : n = 1, 2, \dots\}$ of elements of a Banach space X is called to be (i) incomplete in X if $\overline{\mathrm{span}} E \neq X$; (ii) minimal in X if for all $n = 1, 2, \dots$, $e_n \notin \overline{\mathrm{span}}(E - \{e_n\})$, where span$E$ is the linear span of the system E and $\overline{\mathrm{span}} E$ is the closure of span$E$ in X. The incompleteness of the system E in X is equivalent to the existence of a non-trivial functional f in the dual Banach space X^* of X which annihilates the system E, i.e., $f(e_n) = 0, n = 1, 2, \dots$. The minimality of the system E in X is equivalent to the existence of a system of conjugate functionals $\{f_n : n = 1, 2, \dots\}$ in X^*, i.e., $f_n(e_m) = \delta_{nm}$(Kronneker delta, i.e., $\delta_{nn} = 1$, while $\delta_{nm} = 0$ for $n \neq m$). The system $\{f_n\}$ is also called a biorthogonal system of the system E.

Let $\Lambda = \{\lambda_n : n = 1, 2, \dots\}$ be a sequence of distinct complex numbers in the open right half-plane $\mathbb{C}_0 = \{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, and $M = \{m_n : n = 1, 2, \dots\}$ be a sequence of positive integers. With these sequences Λ and M, we associate the complex exponential system

$$E(\Lambda, M) = \{z^l e^{\lambda_n z} : l = 0, 1, \dots, m_n - 1; n = 1, 2, \dots\}.$$

Received January 17, 2009; Accepted July 19, 2009
Supported by the National Natural Science Foundation of China (Grant No. 11071020) and the Research Foundation for Doctor Program (Grant No. 20100003110004).
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Let $D_{s,\tau}$ be the half strip $\{z \in \mathbb{C} : |\mathrm{Im}z| < s, \mathrm{Re}z < \tau\}$, $\gamma_{s,\tau}$ be a boundary of $D_{s,\tau}$ traced around in the positive direction with respect to $D_{s,\tau}$. When $0 < \sigma < \infty$, let $D_\sigma = D_{\sigma,0}$, $D^*_\sigma = \mathbb{C} \setminus (D_\sigma \cup \gamma_\sigma)$, $\gamma_\sigma = \gamma_{\sigma,0}$. When $1 \le p < \infty$, denote by $E^p[\sigma]$ and $E^p_*[\sigma]$ the sets consisting of all functions f analytic in D_σ and D^*_σ, respectively, such that

$$\sup\{\dot{I}_p(s,\tau,f) : 0 < s < \sigma, \tau < 0\} < \infty \quad \text{and} \quad \sup\{\dot{I}_p(s,\tau,f) : s > \sigma, \tau > 0\} < \infty,$$

respectively. Here, $\dot{I}_p(s,\tau,f) = (\int_{\gamma_{s,\tau}} |f(z)|^p |\mathrm{d}z|)^{\frac{1}{p}}$. By Lemma 5 in [3], $E(\Lambda, M)$ is a subset of $E^2[\sigma]$, and if we define a norm on each of the sets $E^2[\sigma]$ and $E^2_*[\sigma]$ by the equality $||f|| = (\int_{\gamma_\sigma} |f(t)|^2 |\mathrm{d}t|)^{\frac{1}{2}}$, then the sets $E^2[\sigma]$ and $E^2_*[\sigma]$ become Banach spaces.

As in [4], we are interested in the incompleteness and the minimality of $E(\Lambda, M)$ in Banach space $E^2[\sigma]$. Our main conclusions are as follows:

Theorem 1 *Suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\varphi_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_0$, and $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of positive integers, then*

$$E(\Lambda, M) = \{z^l e^{\lambda_n z} : l = 0, 1, \ldots, m_n - 1; n = 1, 2, \ldots\}$$

is incomplete in $E^2[\sigma]$ if and only if

$$\sum_{|\lambda_n| \le 1} \mathrm{Re}\lambda_n < \infty \tag{1}$$

and

$$\limsup_{r \to \infty} (S(r) - \frac{\sigma}{\pi} \log r) < \infty \tag{2}$$

are satisfied, where

$$S(r) = \sum_{1 < |\lambda_n| \le r} m_n (\frac{1}{|\lambda_n|} - \frac{|\lambda_n|}{r^2}) \cos \varphi_n. \tag{3}$$

Remark 1 Theorem 1 was proved by Vinnitskii in [3] when $m_n \equiv 1$.

Theorem 2 *Suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\varphi_n} : n = 1, 2, \ldots\}$ is a sequence of complex numbers in $\mathbb{C}_0$, and $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of positive integers, satisfying*

$$\Theta(\Lambda) = \sup\{|\varphi_n| : n = 1, 2, \ldots\} < \frac{\pi}{2}, \tag{4}$$

$$\delta(\Lambda) = \inf\{|\lambda_{n+1}| - |\lambda_n| : n = 0, 1, 2, \ldots; \lambda_0 = 0\} > 0, \tag{5}$$

and

$$K(M) = \sup\{m_n : n = 1, 2, \ldots\} < \infty. \tag{6}$$

If $S(r) - \frac{\sigma}{\pi} \log r$ is bounded on $(1, \infty)$, then $E(\Lambda, M) = \{z^l e^{\lambda_n z} : l = 0, 1, \ldots, m_n - 1; n = 1, 2, \ldots\}$ is incomplete and minimal in $E^2[\sigma]$, and each function $f \in \overline{\mathrm{span}} E(\Lambda, M)$ can be extended to an analytic function $\tilde{f}(z)$ represented by a Taylor-Dirichlet series

$$\tilde{f}(z) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n - 1} a_{n,k} z^k e^{\lambda_n z}, \quad z \in D(B), \tag{7}$$

where $D(B) = \{z = re^{i\theta} : r \cos(|\pi - \theta| + \Theta(\Lambda)) > B\}$, and B is a positive constant only dependent on Λ, M and σ.

Remark 2 If (4)–(6) hold, $S(r)-\lambda(r)$ is bounded on $(1,\infty)$, here

$$\lambda(r)=\begin{cases}\displaystyle\sum_{|\lambda_n|\le r}\frac{m_n\cos\varphi_n}{|\lambda_n|}\ , & \text{if } r\ge|\lambda_1|;\\ 0\ , & \text{otherwise.}\end{cases}$$

By the conformal mapping $\zeta=\phi(z)=e^z$, each half strip $D_{s,\tau}$ $(0<s<\pi)$ is mapped to the sector $\mathcal{D}_{s,\tau}=\{\zeta=re^{i\theta}:0<r<e^\tau,|\theta|<s<\pi\}=\phi(D_{s,\tau})$, $\kappa_{s,\tau}=\phi(\gamma_{s,\tau})$ is a boundary of $\mathcal{D}_{s,\tau}$ traced around in the positive direction with respect to $\mathcal{D}_{s,\tau}$, $\kappa_\sigma=\kappa_{\sigma,0}$, and $\mathcal{D}_\sigma=\mathcal{D}_{\sigma,0}$. Denote by $F^p[\sigma]$ the linear space of functions F analytic in $\mathcal{D}_\sigma$ such that

$$\sup\{\dot{J}_p(s,\tau,F):0<s<\sigma,\tau<0\}<\infty,$$

where $\dot{J}_p(s,\tau,F)=(\int_{\gamma_{s,\tau}}|J(z)|^p|\mathrm{d}z|)^{\frac{1}{p}}$.

The conformal mapping $\zeta=\phi(z)$ transforms D_σ onto $\mathcal{D}_\sigma$, and

$$\int_{\kappa_{s,\tau}}|F(\zeta)|^p|\mathrm{d}\zeta|=\int_{\gamma_{s,\tau}}|F(\phi(z))|^p|\phi'(z)|\mathrm{d}z,$$

then the mapping $\mathcal{L}:F(\zeta)\longrightarrow f(z)=|F(\phi(z))||\phi'(z)|^{\frac{1}{p}}$ defines an isomorphism between $F^p[\sigma]$ and $E^p[\sigma]$. Define a norm in $F^2[\sigma]$ by the equality $\|F\|=(\int_{\kappa_\sigma}|F(t)|^p|\mathrm{d}t|)^{\frac{1}{2}}$, then $F^2[\sigma]$ is a Banach space.

Suppose that $\Lambda'=\{\lambda'_n=|\lambda'_n|e^{i\varphi'_n}:n=1,2,\ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{-\frac{1}{2}}=\{z\in\mathbb{C}:\mathrm{Re}z>-\frac{1}{2}\}$, then the incompleteness and the minimality of $F(\Lambda',M)=\{(\log\zeta)^l\zeta^{\lambda'_n}:l=0,1,\ldots,m_n-1;n=1,2,\ldots\}$ in $F^2[\sigma]$ are equivalent to the ones of $E(\Lambda,M)=\{z^le^{\lambda_nz}:l=0,1,\ldots,m_n-1;n=1,2,\ldots\}$ in $E^2[\sigma]$, where $\Lambda=\Lambda'+\frac{1}{2}=\{\lambda'_n+\frac{1}{2}:n=1,2,\ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_0$.

Corollary 1 *Suppose that $\Lambda'=\{\lambda'_n=|\lambda'_n|e^{i\varphi'_n}:n=1,2,\ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{-\frac{1}{2}}$ and $M=\{m_n:n=1,2,\ldots\}$ is a sequence of positive integers, then $F(\Lambda',M)=\{(\log\zeta)^l\zeta^{\lambda'_n}:l=0,1,\ldots,m_n-1;n=1,2,\ldots\}$ is incomplete in $F^2[\sigma]$ if and only if Λ' satisfies*

$$\sum_{|\lambda_n|\le 1}m_n\mathrm{Re}\lambda_\mathrm{n}<\infty$$

and

$$\lim_{r\to\infty}(S(r)-\frac{\sigma}{\pi}\log r)<\infty,$$

where $\lambda_n=|\lambda_n|e^{i\varphi_n}=\lambda'_n+\frac{1}{2}$, and $S(r)$ is defined by (3).

Corollary 2 *Suppose that $\Lambda'=\{\lambda'_n=|\lambda'_n|e^{i\varphi'_n}:n=1,2,\ldots\}$ is a sequence of complex numbers in $\mathbb{C}_{-\frac{1}{2}}$ and $M=\{m_n:n=1,2,\ldots\}$ is a sequence of positive integers such that the sequence $\Lambda=\Lambda'+\frac{1}{2}=\{\lambda_n=|\lambda_n|e^{i\varphi_n}=\lambda'_n+\frac{1}{2}:n=1,2,\ldots\}$ and M satisfy (4)–(6). If $S(r)-\frac{\sigma}{\pi}\log r$ is bounded on $(1,\infty)$, then $F(\Lambda',M)=\{(\log\zeta)^l\zeta^{\lambda'_n}:l=0,1,\ldots,m_n-1;n=1,2,\ldots\}$ is incomplete, minimal in $F^2[\sigma]$, and each function $F\in\overline{\mathrm{span}}F(\Lambda',M)$ can be extended to an* *analytic function* $\tilde{F}(\zeta)$ *represented by weighted lacunary power series*

$$\tilde{F}(\zeta)=\sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1}a_{n,k}(\log\zeta)^k\zeta^{\lambda_n'},\quad \zeta\in\mathcal{D}(B),$$

where $\mathcal{D}(B)=\{\zeta\in\mathbb{C}:\cos\Theta(\Lambda)\log|\zeta|+\sin\Theta(\Lambda)|\arg\zeta|+B<0\}$, *and* B *is a positive constant only dependent on* Λ, M *and* σ.

2. Proof of Theorems

Denote by H_σ^p the space consisting of all functions f analytic in $\mathbb{C}_0$ satisfying $\|f\|:=\sup\{(\int_0^\infty|f(re^{i\theta})|^pe^{-p\sigma r|\sin\theta|}\mathrm{d}r)^{\frac{1}{p}}:|\theta|<\frac{\pi}{2}\}<\infty$, and $H(\Lambda,M)$ the class consisting of all functions $f\not\equiv 0$ analytic in $\mathbb{C}_0$ and having zeros of orders m_n at the points λ_n. Hereafter we denote a positive constant by A, not necessarily the same at each occurrence. In order to prove our conclusions, we need the following lemmas.

Lemma 1 *Suppose that* $\Lambda=\{\lambda_n=|\lambda_n|e^{i\varphi_n}:n=1,2,\ldots\}$ *is a sequence of complex numbers in* $\mathbb{C}_0$ *and* $M=\{m_n:n=1,2,\ldots\}$ *is a sequence of positive integers satisfying (4)–(6), then the function*

$$G(z)=\prod_{n=1}^{\infty}\Big(\frac{1-z/\lambda_n}{1+z/\overline{\lambda}_n}\Big)^{m_n}\exp\Big(\frac{m_nz}{\lambda_n}+\frac{m_nz}{\overline{\lambda}_n}\Big)\tag{8}$$

is analytic in the closed right half plane $\overline{\mathbb{C}}_0=\{z\in\mathbb{C}:\mathrm{Re}z\geq 0\}$, *and satisfies the following inequalities*

$$|G(z)|\leq\exp\{2x\lambda(r)+Ax\}\tag{9}$$

for all $z\in\mathbb{C}_0$, *and*

$$|G(z)|\geq\exp\{2x\lambda(r)-Ax\}\tag{10}$$

for all $z\in C(\Lambda,\delta_0)$, *where* $r=|z|$, $4\delta_0=\delta(\Lambda)$ *and* $C(\Lambda,\delta_0)=\{z\in\mathbb{C}_0:|z-\lambda_n|\geq\delta_0,n=1,2,\ldots\}$.

Remark 3 When $\Theta(\Lambda)=0$, $m_n\equiv 1$, $G(z)$ is Fuch's function [5].

Lemma 2 ([3]) *Each continuous linear functional* Φ *on* $E^2[\sigma]$ *is associated with a unique function* $g\in E_*^2[\sigma]$ *such that the value* $\langle\Phi,f\rangle$ *of the functional* Φ *at* $f\in E^2[\sigma]$ *is given by the relation* $\langle\Phi,f\rangle=\int_{\gamma_\sigma}f(t)g(t)\mathrm{d}t$. *In this case, the norm of the functional* Φ *is equivalent to the norm of the function* g *and the space* $(E^2[\sigma])^*$ *(strongly) dual to* $E^2[\sigma]$ *can be realized as* $E_*^2[\sigma]$.

Lemma 3 ([3]) *The equality*

$$f_2(z)=\frac{1}{\sqrt{2\pi}}\int_0^\infty f_1(w)e^{-zw}\mathrm{d}w\tag{11}$$

determines a one-to-one correspondence between the functions $f_1\in H_\sigma^2$ *and* $f_2\in E_*^2[\sigma]$. *The following duality relation is valid*

$$f_1(w)=\frac{1}{\sqrt{2\pi}\,i}\int_{\gamma_\sigma}f_2(z)e^{zw}\mathrm{d}z.\tag{12}$$

Furthermore, $\|f_2\|/A \le \|f_1\| \le 3\|f_2\|$.

Lemma 4 ([3]) *In order that a function* $f \in H_\sigma^2 \cap H(\Lambda, M)$ *exists, it is necessary and sufficient that conditions (1) and (2) are satisfied.*

Lemma 5 ([3]) *If* $f \in E^p[\sigma]$, *then, almost everywhere on* γ_σ, f *has angular limit values belonging to* $L^2[\gamma_\sigma]$ *and, moreover,*

$$\frac{1}{2\pi i}\int_{\gamma_\sigma}\frac{f(t)}{t-z}\mathrm{d}t = \begin{cases} f(z), & z \in D_\sigma; \\ 0, & z \in D_\sigma^*. \end{cases}$$

Proof of Lemma 1 By (4)–(6), $\sum_{n=1}^{\infty} m_n|\lambda_n|^{-2} < \infty$, and the product (8) defines an analytic function in $\overline{\mathbb{C}}_0$, which has zeros of orders m_n at each point λ_n. Let

$$e_n(z) = \Big|\frac{z-\lambda_n}{z+\overline{\lambda}_n}\Big|^2 = 1 - \frac{4x|\lambda_n|\cos\varphi_n}{|z+\overline{\lambda}_n|^2}$$

and

$$E_n(z) = \log\Big|\frac{1-z/\lambda_n}{1+z/\overline{\lambda}_n}\exp\Big(\frac{z}{\lambda_n}+\frac{z}{\overline{\lambda}_n}\Big)\Big| = 2x\frac{\cos\varphi_n}{|\lambda_n|} + \frac{1}{2}\log e_n(z),$$

where $x = \mathrm{Re}z > 0$. When $|\lambda_n| > 8|z|$,

$$l_n(z) = 1 - \frac{|\lambda_n|^2}{|\overline{\lambda}_n + z|^2} \le \frac{288}{49}\frac{|z|}{|\lambda_n|},$$

and

$$1 - e_n(z) = \frac{4x|\lambda_n|\cos\varphi_n}{|z+\overline{\lambda}_n|^2} \le \min\{\frac{4}{7}, \frac{A\sqrt{xr\cos\varphi_n}}{|\lambda_n|}\},$$

so

$$|E_n(z)| = \Big|\frac{2x\cos\varphi_n}{|\lambda_n|}l_n(z) - \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}(1-e_n(z))^k\Big|,$$

hence

$$|E_n(z)| \le \frac{A|x|r\cos\varphi_n}{|\lambda_n|^2}, \quad x = \mathrm{Re}z. \tag{13}$$

By (4)–(6) and $0 \le e_n(z) < 1$,

$$\begin{aligned}\log|G(z)| &\le 2x\sum_{|\lambda_n|\le 8r}\frac{m_n\cos\varphi_n}{|\lambda_n|} + Axr\sum_{|\lambda_n|>8r}\frac{m_n\cos\varphi_n}{|\lambda_n|^2} \\ &\le 2x\lambda(8r) + Axr\sum_{|\lambda_n|>8r}\frac{m_n\cos\varphi_n}{|\lambda_n|^2} \le 2x\lambda(r) + Ax.\end{aligned}$$

Thus inequality (9) holds. In order to prove inequality (10), we note that

$$\log|G(z)| \ge \sum_{|\lambda_n|\le 8r} m_n E_n(z) - \sum_{|\lambda_n|>8r} m_n|E_n(z)| = \Pi_1 - \Pi_2.$$

Inequality (13) yields $\Pi_2 = O(x)$ if $x \ge 0$. Let $n(r) = \sum_{|\lambda_n|\le r} m_n$. Then $n(r) = O(r)$ by (4)–(6). We consider the following two cases for Π_1:

(i) $z \in \{z \in C(\Lambda, \delta_0) : \Theta(\Lambda) + 2\epsilon_1 \le |\theta| < \frac{\pi}{2}\}$;

(ii) $z \in \{z \in C(\Lambda, \delta_0) : |\theta| < \Theta(\Lambda) + 2\epsilon_1\}$, where $z = re^{i\theta}$, and $4\epsilon_1 = \frac{\pi}{2} - \Theta(\Lambda)$.

In case (i), let $\delta_1 = \sin^2 \epsilon_1$. Then

$$|z + \overline{\lambda}_n|^2 \geq 2r|\lambda_n| + 2r|\lambda_n| \cos(|\theta - \varphi_n|) = 4r|\lambda_n|(1 + \delta_1)$$

and

$$0 < 1 - e_n(z) = \frac{4x|\lambda_n| \cos \varphi_n}{|z + \overline{\lambda}_n|^2} \leq \frac{x}{r(1 + \delta_1)}.$$

Since

$$\log(1 - t) \geq -t - \frac{1 + \delta_1}{2\delta_1} t^2 \geq -At, \quad t \in [0, \frac{1}{1 + \delta_1}],$$

by taking $t = 1 - e_n(z)$, then $e_n(z) \geq \exp\{-A\frac{x}{r}\}$. Moreover,

$$\begin{aligned} \Pi_1 &\geq 2x \sum_{|\lambda_n| \leq 8r} \frac{m_n \cos \varphi_n}{|\lambda_n|} - \sum_{|\lambda_n| \leq 8r} \frac{1}{2} m_n \log e_n(z) \\ &\geq 2x\lambda(8r) - \frac{Ax}{r} n(8r) \geq 2x\lambda(r) - Ax. \end{aligned}$$

This implies that inequality (10) holds in this case.

In case (ii), let Λ_k be the set $\{\lambda_n \in \Lambda : \exists\, n, \text{s.t.}\, m_n = k\}$. Then $\Lambda_1, \ldots, \Lambda_{K(M)}$ are disjoint and $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{K(M)}$. Let $\Lambda_k = \{\lambda_{k_n} : n = 1, 2, \ldots\}$, and $n_k(r)$ be the number of $\lambda \leq r$ and $\lambda \in \Lambda_k$. When $|z - \lambda_n| \geq \delta_0$, (4)–(6) and Stirling's formula yield

$$\prod_{\lambda \in \Lambda_k, |\lambda| \leq 8r} |\lambda - z| \geq \delta_0^{N_k} n_k(x)!(N_k - n_k(x))! \geq (\frac{N_k}{A})^{N_k}$$

and

$$\prod_{\lambda \in \Lambda_k, |\lambda| \leq 8r} |\overline{\lambda} + z| \leq (Ar)^{N_k},$$

where $N_k = n_k(8r), k = 1, 2, \ldots, K(M)$. Thus,

$$\begin{aligned} \Pi_1 &\geq \sum_{1 \leq k \leq K(M)} N_k(\log N_k - \log(Ax)) + 2x \sum_{|\lambda_n| \leq 8r} \frac{\cos \varphi_n}{|\lambda_n|} \\ &\geq x\lambda(r) - Ax, \end{aligned}$$

and in the last inequality, we use $N(\log N - \log a) \geq -ae^{-1}$ for $a > 0$. Therefore inequality (10) holds. □

Proof of Theorem 1 According to Lemmas 2 and 3, similarly to the proof of Vinnitskii in [3], the space dual to $E^2[\sigma]$ can be realized in the form H^2_σ. In this case, the value $\langle f_1, f\rangle^*$ of the functional $f_1 \in E^2[\sigma]$ is determined by the equality

$$\langle f_1, f\rangle^* = \int_{\gamma_\sigma} f_2(t) f(t) \mathrm{d}t,$$

where f_2 is defined by (11). In view of (12), we have

$$\langle f_1(z), z^l e^{\lambda_n z}\rangle^* = \int_{\gamma_\sigma} t^l e^{\lambda_n t} f_2(t) \mathrm{d}t = \sqrt{2\pi} i f_1^{(l)}(\lambda_n).$$

Hence, the well-known criterion of completeness implies that system $E(\Lambda, M)$ is incomplete in $E^2[\sigma]$ if and only if there exists a function $f_1 \in H^2_\sigma \cap H(\Lambda, M)$. Therefore, Theorem 1 follows from Lemma 4. □

Proof of Theorem 2 Taking inequality (9) and (10) into account and properly choosing the number M, we can see that the function

$$U(z) = \frac{\exp\{-Mz - \frac{2\sigma}{\pi} z \log z\}}{1+z} G(z)$$

satisfies the following inequalities

$$|U(z)| \le \frac{\exp\{\sigma|y|\}}{|1+z|} \tag{14}$$

for all $z \in \mathbb{C}_0$, and

$$|U(z)| \ge \frac{\exp\{-Ax - \sigma|y|\}}{|1+z|} \tag{15}$$

for all $z \in C(\Lambda, \delta_0)$, where $G(z)$ is defined by (8).

Let $D_n = \{z : |z - \lambda_n| < \delta_0\}$ and $A_{n,j}$ be the coefficients of the principal part of the Laurent series for the function $\frac{1}{U(z)}$ in $D_n - \{\lambda_n\}$, i.e.,

$$\frac{1}{U(z)} = \sum_{j=1}^{m_n} \frac{A_{n,j}}{(z-\lambda_n)^j} + g_n(z), \quad z \in D_n - \{\lambda_n\}, \tag{16}$$

where $g_n(z) \in H(D_n)$. Then

$$A_{n,j} = \frac{1}{2\pi i} \int_{|z-\lambda_n| = \delta_0} \frac{(z-\lambda_n)^{j-1}}{U(z)} \mathrm{d}z.$$

According to inequality (15),

$$\max\{|A_{n,j}| : 1 \le j \le m_n\} \le \exp\{B(|\lambda_n| + 1)\}, \tag{17}$$

where B is a constant only dependent on Λ, M and σ. Let

$$H_{n,k}(z) = U(z) \sum_{l=1}^{m_n - k} \frac{A_{n,k+l}}{k!(z-\lambda_n)^l}, \quad k = 0, 1, \ldots, m_n - 1; \ n = 1, 2, \ldots.$$

By inequalities (14), (17) and Maximum Module Principle, we have

$$|H_{n,k}(z)| \le \frac{A \exp\{\sigma|y|\}}{|1+z| - 2\delta_0} \exp\{B|\lambda_n|\}.$$

Then $H_{n,k}(z) \in H^2_\sigma$, and $||H_{n,k}||_{H^2_\sigma} \le A \exp\{B|\lambda_n|\}$. By Lemma 3, each function

$$h_{n,k}(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty H_{n,k}(t) e^{-zt} \mathrm{d}t$$

belongs to $E^2_*[\sigma]$ and satisfies

$$\|h_{n,k}\|_{E^2_*[\sigma]} \le A \exp\{B|\lambda_n|\}$$

and the duality relation

$$H_{n,k}(z) = \frac{1}{\sqrt{2\pi}\, i} \int_{\gamma_\sigma} h_{n,k}(t) e^{tz} \mathrm{d}t, \quad \mathrm{Re}\, z > 0$$

holds. Next we will prove that

$$H_{n,k}^{(l)}(\lambda_j)=\delta_{nj}\delta_{kl},\quad \text{i.e.,}\quad \frac{1}{\sqrt{2\pi}i}\int_{\gamma_\sigma} t^l e^{\lambda_j t} h_{n,k}(t)\mathrm{d}t=\delta_{nj}\delta_{kl}, \tag{18}$$

where $l=0,1,\ldots,m_j-1$, $k=0,1,\ldots,m_n-1$; $n,j=1,2,\ldots$. It is obvious that if $j\neq n$, then $H_{n,k}^{(l)}(\lambda_j)=0$, $l=0,1,\ldots,m_j-1$. If $j=n$, then by (16), for $z\in D_n$ and $k=0,1,\ldots,m_n-1$, $n=1,2,\ldots,$

$$\begin{aligned}
H_{n,k}(z)&=U(z)\frac{(z-\lambda_n)^k}{k!}\sum_{l=k+1}^{m_n}\frac{A_{n,l}}{(z-\lambda_n)^{l-k}}\\
&=U(z)\frac{(z-\lambda_n)^k}{k!}\left(\frac{1}{U(z)}-\sum_{l=1}^{k}\frac{A_{n,l}}{(z-\lambda_n)^l}-g_n(z)\right)\\
&=\frac{(z-\lambda_n)^k}{k!}+\sum_{l=m_n}^{\infty}B_{n,l}(z-\lambda_n)^l,
\end{aligned}$$

where $B_{n,l}$ are the coefficients of the Taylor expansion of $H_{n,k}(z)$ at λ_n. Thus (18) holds. Define a linear functional $T_{n,k}$ on $E^2[\sigma]$ by

$$T_{n,k}(f)=\frac{1}{\sqrt{2\pi}\,i}\int_{\gamma_\sigma}h_{n,k}(z)f(z)\mathrm{d}z,\quad f(z)\in E^2[\sigma].$$

Then

$$\|T_{n,k}\|\le\frac{1}{\sqrt{2\pi}}\|h_{n,k}\|_{E_*^2[\sigma]}\le A\exp\{B|\lambda_n|\} \tag{19}$$

and

$$T_{n,k}(z^l e^{\lambda_j z})=\frac{1}{\sqrt{2\pi}\,i}\int_{\gamma_\sigma}h_{n,k}(z)z^l e^{\lambda_j z}\mathrm{d}z=H_{n,k}^{(l)}(\lambda_j)=\delta_{nj}\delta_{kl}.$$

Hence $\{T_{n,k}:k=1,2\ldots,m_n;n=1,2,\ldots\}$ is a biorthogonal system of $E(\Lambda,M)$ in $(E^2[\sigma])^*$ and $E(\Lambda,M)$ is minimal in $E^2[\sigma]$.

If $f\in\overline{\operatorname{span}}E(\Lambda,M)$, there exists a sequence of exponential polynomials

$$P_j(z)=\sum_{n=1}^{j}\sum_{k=0}^{m_n-1}a_{n,k}^j z^k e^{\lambda_n z}\in\operatorname{span}E(\Lambda,M)$$

such that

$$\|f-P_j\|_{E^2[\sigma]}\longrightarrow 0,\quad j\longrightarrow\infty. \tag{20}$$

Let $\tilde f(z)$ be defined by (7), where $a_{n,k}=T_{n,k}(f)$, $D(B)=\{z=re^{i\theta}:r\cos(|\pi-\theta|+\Theta(\Lambda))>B\}$. By (19), the function $\tilde f(z)$ is an analytic function in $D(B)$. Since $\frac{1}{t-z}\in L^2[\gamma_\sigma]$, $z\in D_\sigma$, by Lemma 5,

$$|f(z)-P_j(z)|\le\frac{1}{2\pi}\|f-P_j\|_{L^2[\gamma_\sigma]}\|\frac{1}{t-z}\|_{L^2[\gamma_\sigma]}. \tag{21}$$

Note that

$$|a_{nk}-a_{nk}^j|=|T_{n,k}(f)-T_{n,k}(P_j)|\le\|T_{n,k}\|\cdot\|f-P_j\|_{E^2[\sigma]}, \tag{22}$$

so for $z\in D(B)\cap D_\sigma$,

$$|f(z)-\tilde f(z)|\le|f(z)-P_j(z)|+|P_j(z)-\tilde f(z)|$$

$$\leq |f(z) - P_j(z)| + \sum_{n=1}^{j}\sum_{k=0}^{m_n-1} |a_{nk}^j - a_{nk}| r^k e^{\mathrm{Re}(\lambda_k z)} +$$
$$\sum_{k=j+1}^{\infty}\sum_{k=0}^{m_n-1} |a_{nk}| r^k e^{\mathrm{Re}(\lambda_k z)}.$$

Letting $j \longrightarrow \infty$, by (19)–(22), we see that $f(z) = \tilde{f}(z)$ for each $z \in D(B) \cap D_\sigma$. This completes the proof of Theorem 2. $\square$

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