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The Characterization of Parseval Frame Wavelets

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Abstract In this paper, we characterize all generalized low pass filters and MRA Parseval frame wavelets in $L^2(R^n)$ with matrix dilations of the form $(Df)(x)=\sqrt{2}f(Ax)$, where A is an arbitrary expanding $n\times n$ matrix with integer coefficients, such that $|\det A|=2$. We study the pseudo-scaling functions, generalized low pass filters and MRA Parseval frame wavelets and give some important characterizations about them. Furthermore, we give a characterization of the semiorthogonal MRA Parseval frame wavelets and provide several examples to verify our results.

Keywords generalized low pass filter; Pseudo-scaling function; MRA Parseval frame wavelets.

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1. Introduction

Wavelet theory has been studied extensively in both theory and applications since 1980's [1]. One of the basic advantage of wavelets is that an event can be simultaneously described in the frequency domain as well as in the time domain. This feature permits a multiresolution analysis of data with different behavior on different scales. The main advantage of wavelets is their time-frequency localization property. Many signals can be efficiently represented by wavelets.

The classical MRA wavelets are probably the most important class of orthonormal wavelets. Because they guarantee the existence of fast implementation algorithm, many of the famous examples often used in applications belong to this class. However, there are useful "filters", such as $m(\omega)=\frac{1}{2}(1+e^{3i\omega})$, that do not produce orthonormal basis, nevertheless, they do produce systems that have the reconstruction property, as well as many other useful features. It is natural, therefore, to develop a theory involving more general filters that can produce systems having these properties.

A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system [2]. By allowing redundancy in a wavelet system, one has much more freedom in the choice of wavelets. Tight wavelet frames have some desirable features, such as near translation invariant wavelet frame transforms, and it may be easier to recognize

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patterns in a redundant transform. For advantages and applications of tight wavelet frames, the reader is referred to [3–15] and many references therein. Recently, the theory of high dimensional wavelet is widely studied, see, e.g., [19], [20].

In [16], authors discussed wavelet multipliers, scaling function multipliers and low pass filter multipliers in $L^2(R)$. In [17], authors introduced a class of generalized low pass filter that allowed them to define and construct the MRA Parseval frame wavelets. This led them to an associated class of generalized scaling functions that were not necessarily obtained from a multiresolution analysis. Also, they generalized notions of the wavelet multipliers in [16] to the case of wavelet frame and got several properties of the multipliers of Parseval frame wavelets.

In this paper, we characterize all generalized low pass filter and MRA Parseval frame wavelets (PFWs) in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $n \times n$ matrix with integer coefficients, such that $|\det A| = 2$. Firstly, we study some properties of the generalized wavelets, scaling functions and filters in $L^2(R^n)$. Our result is a generalization of the construction of PFWs from generalized low-pass filters that is introduced in [17]. Though we follow [17] as a blueprint, it is well known that the situation in the high dimension is so complex that we have to recur to some special matrices to solve problem. Thus, our approaches are different from original ones. And we borrow some thoughts and technique in [19].

Let us now describe the organization of the material that follows. Section 2 is of a preliminary character: it contains various results on matrices belonging to the class $E_n^{(2)}$ and some facts about a Parseval frame wavelet. In Section 3, we study the pseudo-scaling functions, the generalized low pass filters and the MRA PFWs and give some important characterizations about them. Furthermore, we give a characterization of the semiorthogonal MRA Parseval frame wavelets.

2. Preliminaries

Let us now establish some basic notations.

We denote by $\mathbf{T}^n$ the n-dimensional torus. By $L^p(\mathbf{T}^n)$ we denote the space of all Z^n-periodic functions f (i.e., f is 1-periodic in each variable) such that $\int_{\mathbf{T}^n} |f(x)|^p \mathrm{d}x < +\infty$. The standard unit cube $[-\frac{1}{2}, \frac{1}{2})^n$ will be denoted by C. The subsets of R^n invariant under Z^n translations and the subsets of $\mathbf{T}^n$ are often identified.

We use the Fourier transform in the form

$$\hat{f}(\omega) = \int_{R^n} f(x) e^{-2\pi i \langle x, \omega \rangle} \mathrm{d}x,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in R^n.

For f, $g \in L^2(R^n)$ we denote the function $[f, g](\omega)$ as below:

$$[f, g](\omega) = \sum_{k \in Z^n} f(\omega + k)\overline{g(\omega + k)}.$$

In particular, for $f \in L^2(R^n)$, we will write $\sigma_f(\omega) := \sum_{k \in Z^n} |\hat{f}(\omega + k)|^2$, which is named as the bracket function of f. For $\sigma_f(\omega) = \sum_{k \in Z^n} |\hat{f}(\omega + k)|^2$, we let Ω_f be the Z^n-translation invariant subset of R^n defined, modulo a null set, by $\Omega_f = \operatorname{supp} \sigma_f = \{\omega \in R^n : \hat{f}(\omega + k) \neq 0$, for some $k \in Z^n\}$.

The Lebesgue measure of a set $S \subseteq R^n$ will be denoted by $|S|$. When measurable sets X and Y are equal up to a set of measure zero, we write $X \doteq Y$.

Then we introduce some notations and the existing results about expanding matrices.

Let $E_n^{(2)}$ denote the set of all expanding matrices A such that $|\det A| = 2$. The expanding matrices mean that all eigenvalues have magnitude greater than 1. For $A \in E_n^{(2)}$, we denote by B the transpose of $A : B = A^t$. It is obvious that $B \in E_n^{(2)}$.

In this paper, we will work with two families of unitary operators on $L^2(R^n)$. The first one consists of all translation operators $T_k : L^2(R)^n \to L^2(R^n)$, $k \in Z^n$, defined by $(T_k f)(x) = f(x - k)$. The second one consists of all integer powers of the dilation operator $D_A : L^2(R^n) \to L^2(R^n)$ defined by $(Df)(x) = \sqrt{2} f(Ax)$ with $A \in E_n^{(2)}$.

Let us now fix an arbitrary matrix $A \in E_n^{(2)}$. For a function $\psi \in L^2(R^n)$, we will consider the affine system Ψ defined by

$$\Psi = \{\psi_{j,\ k}(x) | \psi_{j,\ k}(x) = 2^{\frac{j}{2}} \psi(A^j x - k), j \in Z, k \in Z^n\}. \tag{2.1}$$

Let us recall the definition of a Parseval frame and a Parseval frame wavelet.

Definition 2.1 *We say that a countable family $\{f_j\}$, $j \in J$, in a separable Hilbert space H, is a Parseval frame (PF) for H if the equality $\|f\|^2 = \sum_{j \in J} |\langle f,\ f_j \rangle|^2$ is satisfied for all $f \in H$.*

Definition 2.2 *We say that $\psi \in L^2(R^n)$ is a Parseval frame wavelet (briefly: PFW) if the system (2.1) is a Parseval frame for $L^2(R^n)$.*

In the following, we will give some definitions which will be used in this paper. In fact, they are some generalizations of the notations in [16].

Definition 2.3 *A measurable Z^n-periodic function m on R^n is called a generalized filter if it satisfies the equation*

$$|m(\omega)|^2 + |m(\omega + \beta)|^2 = 1, \quad \text{a.e. } \omega, \tag{2.2}$$

where $\alpha \in Z^n / BZ^n$ and $\beta = B^{-1}\alpha$.

We shall denote by $\tilde{F}$ the set of generalized filters and put $\tilde{F}^+ = \{m \in \tilde{F} : m \geq 0\}$. Observe that $m \in \tilde{F} \Rightarrow |m| \in \tilde{F}^+$.

Definition 2.4 *A function $\varphi \in L^2(R^n)$ is called a pseudo-scaling function if there exists a filter $m \in \tilde{F}$ such that*

$$\hat{\varphi}(B\omega) = m(\omega)\hat{\varphi}(\omega), \quad \text{a.e. } \omega. \tag{2.3}$$

Notice that m is not uniquely determined by the pseudo-scaling function φ. Therefore, we shall denote by $\tilde{F}_\varphi$ the set of all $m \in \tilde{F}$ such that m satisfies (2.3) for φ. For example, if $\varphi = 0$, then, $\tilde{F}_\varphi = \tilde{F}$. If φ is a scaling function of an orthonormal MRA wavelet, then $\tilde{F}_\varphi$ is a singleton and its only element is the low-pass filter m associated with φ. Notice that for a pseudo-scaling function φ, the function $|\hat{\varphi}|^{\check{}}$ is also a pseudo-scaling function, and if $m \in \tilde{F}$, then $|m| \in \tilde{F}_{|\hat{\varphi}|^{\check{}}}$.

Suppose that $m \in \tilde{F}^{+}$. Since $0 \le m(\omega) \le 1$, a.e. ω, the function

$$\widehat{\varphi_m} =: \prod_{j=1}^{+\infty} m(B^{-j}\omega)$$

is well defined a.e.. Moreover, we have

$$\widehat{\varphi_m}(B\omega) = m(\omega)\widehat{\varphi_m}(\omega), \quad \text{a.e. } \omega. \tag{2.4}$$

Following [19], the function $\widehat{\varphi_m}$ defined by (2.4) belongs to $L^2(R^n)$ and the function $\widehat{\varphi_m}$ is a pseudo-scaling function such that $m \in \tilde{F}_{\varphi_m}$.

Consequently, if $m \in \tilde{F}$ is an arbitrary generalized filter, then the function $\widehat{\varphi_{|m|}}$ is a pseudo-scaling function and $|m| \in \tilde{F}_{\varphi_{|m|}}$.

Definition 2.5 *For $m \in \tilde{F}^{+}$, define*

$$N_0(m) = \{\omega \in R^n : \lim_{j\to+\infty} \widehat{\varphi_m}(B^{-j}\omega) = 0\}.$$

We say that $m \in \tilde{F}$ is a generalized low-pass filter if $|N_0(|m|)| = 0$. The set of all generalized low-pass filters is denoted by $\tilde{F}_0$.

Then, we will give the definition of MRA PFW.

Definition 2.6 *A PFW ψ is an MRA PFW if there exists a pseudo-scaling function φ and $m \in \tilde{F}_{\varphi}$ and a unimodular function $s \in L^2(T^n)$ such that*

$$\hat{\psi}(B\omega) = e^{2\pi\omega i} s(B\omega)\overline{m(\omega+\beta)}\hat{\varphi}(\omega), \quad \text{a.e. } \omega. \tag{2.5}$$

Let us conclude this introductory section by noting that many of the results that follow can be proved for dilations by expanding integer matrices with arbitrary determinant. Some of these extensions are obtained easily with essentially the same proofs, others require subtler and more involved arguments. But, for the sake of simplicity, we restrict ourselves to the class $E_n^{(2)}$.

3. MRA Parseval frame wavelets

The main purpose of this section is to study the pseudo-scaling functions, the generalized filters and the MRA PFWs in $L^2(R^n)$. We give some important characterizations about them.

In the following we firstly give several lemmas in order to prove our main results.

Lemma 3.1 *Suppose that φ is a pseudo-scaling function and $m \in \tilde{F}_{\varphi}$. If*

$$\lim_{j\to+\infty} |\hat{\varphi}(B^{-j}\omega)| = 1, \quad \text{a.e. } \omega, \tag{3.1}$$

then,

$$|\hat{\varphi}(\omega)| = |\prod_{j=1}^{+\infty} m(B^{-j}\omega)|, \quad \text{a.e. } \omega,$$

and $|N_0(|m|)| = 0$.

Proof By (2.3), we have

$$|\hat{\varphi}(\omega)| = |\prod_{j=1}^{n} m(B^{-j}\omega)||\hat{\varphi}(B^{-j}\omega)|, \quad \text{a.e. } \omega.$$

Using (2.4), we obtain that $|\hat{\varphi}(\omega)| = \widehat{\varphi_{|m|}}$ and $|N_0(|m|)| = 0$ is clearly satisfied. Thus, the function $m \in \tilde{F}$ is a generalized low-pass filter. □

Lemma 3.2 *If $f \in L^1(R^n)$, then, for a.e. $\omega \in R^n$, $\lim_{j\to+\infty} |f(B^j\omega)| = 0$.*

Proof Assuming that $f \in L^1(R^n)$ and applying the monotone convergence theorem, we obtain

$$\begin{aligned}\int_{R^n} \sum_{j\in N} |f(B^j\omega)|\mathrm{d}\omega &= \sum_{j\in N} \int_{R^n} |f(B^j\omega)|\mathrm{d}\omega \\ &= \sum_{j\in N} 2^{-j} \int_{R^n} |f(\xi)\mathrm{d}\xi| = \|f\|_1 < +\infty.\end{aligned}$$

It follows that for a.e. $\omega \in R^n$, $\sum_{j\in N} |f(B^j\omega)|$ is finite. Therefore, for a.e. $\omega \in R^n$, $\lim_{j\to+\infty} |f(B^j\omega)| = 0$. □

Then we recall a result from [18] that characterizes Parseval frame wavelets associated with more general matrix dilations. We state the special case of that theorem appropriate to the discussion in this paper.

Theorem 3.3 ([18, Theorem 6.12]) *Let A be an arbitrary matrix in $E_n^{(2)}$, $B = A^t$ and $\psi \in L^2(R^n)$. Then the system (2.1) is a PFW if and only if both the equality*

$$\sum_{j\in Z} |\hat{\psi}(B^j\omega)|^2 = 1, \quad \text{a.e.}$$

and the equality

$$\sum_{j=0}^{+\infty} \hat{\psi}(B^j\omega)\overline{\hat{\psi}(B^j(\omega + Bk + \alpha))} = 0,$$

$$\text{a.e.}, \forall\, k \in Z^n, \ \alpha \in Z^n/BZ^n$$

are satisfied.

Then, we will give a characterization of the generalized low pass filter.

Lemma 3.4 *Suppose ψ is an MRA PFW and φ is a pseudo-scaling function satisfying (2.5). Then, m defined by (2.5) is a generalized low pass filter.*

Proof Since ψ is an MRA PFW, from Lemma 3.3, Eqs. (2.2) and (2.5), we can obtain

$$\begin{aligned}1 = \sum_{j\in Z} |\hat{\psi}(B^j\omega)|^2 &= \sum_{j\in Z} |m(B^{j-1}\omega + \beta)|^2 |\hat{\varphi}(B^{j-1}\omega)|^2 \\ &= \lim_{n\to+\infty} \sum_{j=-n}^{n} |m(B^{j-1}\omega + \beta)|^2 |\hat{\varphi}(B^{j-1}\omega)|^2\end{aligned}$$

$$= \lim_{n\to+\infty} \sum_{j=-n}^{n} [1 - |m(B^{j-1}\omega)|^2] |\hat{\varphi}(B^{j-1}\omega)|^2$$
$$= \lim_{n\to+\infty} \{|\hat{\varphi}(B^{-n-1}\omega)|^2 - |\hat{\varphi}(B^{n}\omega)|^2\}.$$

Since $\varphi \in L^2(R^n)$, Lemma 3.2 implies $\lim_{n\to+\infty} |\hat{\varphi}(B^n\omega)|^2 = 0$ for a.e. ω. This shows that for a.e. ω, $\lim_{n\to+\infty} |\hat{\varphi}(B^{-n}\omega)| = 1$, thus, by Lemma 3.1, m is a generalized low pass filter. □

We have described the MRA PFWs. Let us specify this. In [17, 18], authors introduced the definition of the semiorthogonal wavelet. Recall that this term means that the spaces $W_j = \mathrm{span}\{\psi(B^j \cdot -k) : k \in Z\}$ are orthogonal to each other as j ranges through Z. In the following, we shall show that we have the following characterization of the semiorthogonal MRA PFWs.

Theorem 3.5 *Suppose that ψ is an MRA Parseval frame wavelet (MRA PFW), then, ψ is a semiorthogonal MRA PFW if and only if ψ is an MRA PFW whose translates form a Parseval frame for $W_0 := V_1 \cap V_0^{\perp}$.*

Proof Sufficiency. Suppose that ψ is an MRA PFW whose translates form a Parseval frame for $W_0 := V_1 \cap V_0^{\perp}$. Then, the construction of ψ as a function in W_0, a space orthogonal to V_0, makes it clear that ψ is a semiorthogonal MRA PFW.

Necessity. We assume that ψ is an MRA PFW which is semiorthogonal. We claim that the sequence of subspaces $V_j = \bigoplus_{l=-\infty}^{j-1} W_l$, together with the function φ in equality (2.5), form a Parseval frame MRA. That is, the translates, $\varphi(\cdot - n), n \in Z$, form the desired Parseval frame for V_0. From equations (2.2), (2.3) and (2.5), we easily obtain

$$|\hat{\varphi}(\omega)|^2 = \sum_{j=1}^{+\infty} |\hat{\psi}(B^j\omega)|^2. \tag{3.2}$$

We claim that

$$\sigma_{\varphi}(\omega) = \sum_{k\in Z^n} |\hat{\varphi}(\omega + k)|^2 = \sum_{j=1}^{+\infty} \sum_{k\in Z^n} |\hat{\psi}(B^j(\omega + k))|^2 = \chi_U(\omega) \tag{3.3}$$

for some Z^n-periodic set U. This would then tell us that $\{\varphi(\cdot - n)\}_{n\in Z}$ is a Parseval frame for $S \equiv \overline{\mathrm{span}\{\varphi_n\}_{n\in Z}}$. We also claim that

$$S = V_0 = \bigoplus_{l=-\infty}^{-1} W_l. \tag{3.4}$$

Once these claims are established, since the semiorthogonality of the system $\{\psi_{j,k}\}$ implies that $\{\psi(\cdot - n)\}_{n\in Z}$ is a Parseval frame for the subspace W_0, this would establish Theorem 3.5.

In order to establish (3.3), we fix ω and let

$$\Psi_j(\omega) = \{\hat{\psi}(B^j(\omega + k)),\ k \in Z^n\}, \quad j \geq 1.$$

From Eq. (3.1) in [20], we can obtain

$$\hat{\psi}(B^r\omega) = \sum_{j=1}^{+\infty} \hat{\psi}(B^j\omega) \sum_{k\in Z^n} \hat{\psi}(B^r(\omega + k))\overline{\hat{\psi}(B^j(\omega + k))}. \tag{3.5}$$

According to the fact that $\sigma_\psi(\omega) \le 1$, we see that $\Psi_j(\omega) \in \ell^2(Z^n)$ for a.e. $\omega \in R$. But, from (3.5), we see that

$$\Psi_j(\omega) = \sum_{k=1}^{+\infty} \langle \Psi_k(\omega),\ \Psi_j(\omega)\rangle_{\ell^2(Z^n)} \Psi_k(\omega) \tag{3.6}$$

for $j \ge 1$.

Let

$$D_\psi(\omega) = \sum_{j=1}^{+\infty} \sum_{k\in Z^n} |\hat{\psi}(B^j(\omega+k))|^2.$$

By the definition of $D_\psi(\omega)$ and Eq. (3.6), we have

$$D_\psi(\omega) = \sum_{j=1}^{+\infty} |\Psi_j(\omega)|^2_{\ell^2(Z^n)}.$$

Because [7, Lemma 3.7, p.359] is valid in R^n with $A \in E_n^{(2)}$, we conclude that

$$D_\psi(\omega) = \dim \Gamma_\psi(\omega) \ \text{ for a.e. } \ \omega \in R, \tag{3.7}$$

where $\Gamma_\psi(\omega) = \overline{\operatorname{span}\{\Psi_j(\omega) :\ j \ge 1\}}$; this is a well-defined subspace of $\ell^2(Z^n)$.

In a conclusion, (3.7) implies that $D_\psi(\omega)$ is integer-valued a.e..

If we apply (2.5), $\hat{\psi}(B\eta) = e^{2\pi\eta i}\overline{m(\eta+\beta)}\hat{\varphi}(\eta)$, when $\eta = B^l(\omega+k), k \in Z^n$, we obtain

$$\nu_1 = \{\hat{\psi}(B^j(\omega+k))\}_{k\in Z^n} = e^{2\pi\omega i}\overline{m(\eta+\beta)}\{\hat{\varphi}(B^j(\omega+k))\}_{k\in Z^n}$$

and, for $l \ge 2$,

$$\begin{aligned}\nu_l &= \{\hat{\psi}(B^l(\omega+k))\}_{k\in Z^n} \\ &= \{e^{2\pi B^{l-1}\omega i}\overline{m(B^{l-1}\omega+\beta)\prod_{i=0}^{l-2} m(B^k\omega)}\}\{\hat{\varphi}(B^j(\omega+k))\}_{k\in Z^n}.\end{aligned}$$

This shows that ν_l is a multiple of the same vector $\{\hat{\varphi}(B^j(\omega+k))\}_{k\in Z^n}$ for all $l \ge 1$. Consequently, the dimension of $\overline{\operatorname{span}\{\nu_j(\omega) :\ j \ge 1\}}$ is either 1 or 0. Hence, $D_\psi(\omega) = \chi_U(\omega)$, where U is a Z^n-periodic set. This proves (3.3).

We only need to show

$$S = V_0 = \bigoplus_{l=-\infty}^{-1} W_l = \overline{\operatorname{span}\{\varphi_n\}_{n\in Z}} \equiv S. \tag{3.8}$$

It is obvious that $V_0 = \bigoplus_{l=-\infty}^{-1} W_l \subset S$. If we show that $S \perp W_j$ for all $j \ge 0$, we would then have the desired equality (3.8). Toward this end, we observe that it suffices to show $S \perp W_0$.

Similarly to the proof in [17], we can get

$$\sum_{k\in Z^n} \hat{\psi}(B\omega+k)\overline{\hat{\varphi}(B\omega+k)} = 0,$$

and, consequently,

$$\langle \psi, \varphi(\cdot - n)\rangle = \int_{R^n} \hat{\psi}(\omega)\overline{\hat{\varphi}(\omega)}e^{2\pi i n\omega}\mathrm{d}\omega$$

$$= \int_{R^n} \hat{\psi}(B\omega)\overline{\hat{\varphi}(B\omega)}e^{2B\pi in\omega}\mathrm{d}\omega$$
$$= 2\sum_{k\in Z^n}\int_{[0,1)^n+k} \hat{\psi}(B\omega)\overline{\hat{\varphi}(B\omega)}e^{2B\pi in\omega\omega}\mathrm{d}\omega$$
$$= 2\int_{[0,1)^n} (\sum_{k\in Z^n} \hat{\psi}(B\omega+k)\overline{\hat{\varphi}(B\omega+k)})e^{2B\pi in\omega}\mathrm{d}\omega = 0.$$

This implies $S\perp W_0$ and the proof of Theorem 3.5 is completed. $\Box$

Next, we provide several examples to verify our results.

Example 3.6 Let A be the quincunx matrix $Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in E_n^{(2)}$. Then we get $B = Q^t = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in E_n^{(2)}$. Furthermore, we have $B^{-1}C \subseteq C$, where C is the standard unit square in R^2. It is well known that the family $\{B^j(C\backslash B^{-1}C) : j \in Z\}$ is a partition of R^2. We define the function ψ by $\hat{\psi}(\omega) = \chi_{BC\backslash C}$, so we obtain a PFW from [19]. Because its dimensional function $D_\psi(\omega) = 1$, a.e., $\psi(x)$ is a PFW associated to FMRA by Theorem 3.3 (see [20]). Thus, there exist a pseudo-scaling function φ and $m \in \tilde{F}_\varphi$ such that the function $\psi(x)$ satisfies Eq.(2.5). From Theorem 3.5, we know that ψ is a semiorthogonal MRA Parseval frame wavelet. In fact, $\psi(x)$ is an orthonormal wavelet associated to MRA, which serves as an analog of the Shannon wavelet $\hat{\psi}(\omega) = \chi_{2I\backslash I}$, where I denotes the unit interval $[-\frac{1}{2}, \frac{1}{2}]$.

Example 3.7 Let A, B and C be defined as in Example 3.6. Then, we have $B^jC \subseteq B^{j+1}C$, $\forall j \in N$, where C is the standard unit square in R^2.

Let us define $\hat{\varphi}(\omega) = \begin{cases} \frac{1}{2}, & \omega \in B^{-1}C\backslash B^{-2}C \\ 1, & \omega \in B^{-2}C \\ 0, & \omega \bar{\in} B^{-1}C \end{cases}$ and $m(\omega) = \begin{cases} 0, & \omega \in B^{-1}C\backslash B^{-2}C \\ \frac{1}{2}, & \omega \in B^{-2}C\backslash B^{-3}C \\ 1, & \omega \in B^{-3}C \end{cases}$.

Now we extend $m(\omega)$ to C such that the equality $|m(\omega)|^2 + |m(\omega+\beta)|^2 = 1$ is satisfied for all $\omega \in C$, where we take $\beta = (\frac{1}{2}, \frac{1}{2})$, and extend $m(\omega)$ to R^2 by Z^2-periodicity.

From the definitions of the functions φ and m, we easily deduce that φ is a pseudo-scaling function and m is a generalized low pass filter.

Finally, we define
$$\hat{\psi}(B\omega) = e^{2\pi i\omega_{j(A)}}\overline{m(\omega+\beta)}\hat{\varphi}(\omega).$$

Therefore we get an MRA PFW ψ.

However, we know this PFW is not a PFW associated to FMRA [20], which does not permit that $\psi's$ translates form a Parseval frame for $W_0 := V_1 \cap V_0^\perp$.

From Theorem 3.5, we know that ψ is not a semiorthogonal MRA Parseval frame wavelet.

4. Conclusion

In this paper, we characterize all generalized low pass filter and MRA Parseval frame wavelets in $L^2(R^n)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where A is an arbitrary expanding $n \times n$ matrix with integer coefficients, such that $|\det A| = 2$. We study some properties of generalized wavelets, scaling functions and filters in $L^2(R^n)$. Our result is a generalization of the construction of PFWs from generalized low-pass filters that was introduced in [17]. However, our approaches are different from original ones.

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