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Strong Convergence Theorems for a Family of Quasi-ϕ-Asymptotically Nonexpansive Mappings

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Abstract The purpose of this article is to propose a modified hybrid projection algorithm and prove a strong convergence theorem for a family of quasi-ϕ-asymptotically nonexpansive mappings. Its results hold in reflexive, strictly convex, smooth Banach spaces with the property(K). The results of this paper improve and extend recent some relative results.

Keywords quasi-ϕ-asymptotically nonexpansive mapping; hybrid algorithm; generalized projection; strong convergence theorem.

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1. Introduction

Let X be a Banach space and X^* its dual space. Let C be a nonempty subset of X. Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \tag{1.1}$$

for all $x, y \in C$ and all $n \ge 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that, if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, host of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings (see [1–3] and the references therein).

It is well known that, in an infinite-dimensional Hilbert space, the normal Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann's iteration algorithm, and the so called hybrid projection iteration method is such a modification.

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The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [4] in 1968. For 40 years, HPIA has received rapid developments. For details, the readers are referred to papers [5–13] and the references therein.

In 2003, Nakajo and Takahashi [6] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is a closed convex subset of H, and P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

In 2006, Kim and Xu [7] extended the result of Nakajo and Takahashi [6] from nonexpansive mappings to asymptotically nonexpansive mappings. They proposed the following modification of the Mann iteration method for asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is a bounded closed convex subset and

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\operatorname{diam} C)^2 \to 0 \ \text{ as } n \to \infty.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$.

They also proposed the following modification of the Mann iteration method for asymptotically nonexpansive semigroup $\Im$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) \dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s) x_n \mathrm{d}s, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \overline{\theta}_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a bounded closed convex subset,

$$\overline{\theta}_n = (1-\alpha_n)[(\frac{1}{t_n}\int_0^{t_n} L(s)\mathrm{d}s)^2 - 1](\operatorname{diam} C)^2 \to 0 \ \text{ as } n \to \infty,$$

and $L : (0, \infty) \to [0, \infty)$ is a nonincreasing in s and bounded measurable function such that,

$L(s) \geq 1$ for all $s > 0$, $L(s) \to 1$ as $s \to \infty$, and for each $s > 0$,

$$\|T(s)x - T(s)y\| \leq L(s)\|x - y\|, \quad x, y \in C.$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(\Im)}(x_0)$, where $F(\Im)$ denotes the common fixed point set of $\Im$.

In 2005, Matsushita and Takahashi [8] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.5}$$

They proved the following convergence theorem.

Theorem MT *Let E be a uniformly convex and uniformly smooth Banach space, C be a nonempty closed convex subset of E, T be a relatively nonexpansive mapping from C into itself, and $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.5), where J is the normalized duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

Recently, Zhou, Gao and Tan [12] extended the result of Kim and Xu [7] from asymptotically nonexpansive mappings or asymptotically nonexpansive semigroup to a family of quasi-ϕ-asymptotically nonexpansive mappings and the result of Matsushita and Takahashi [8] from relatively nonexpansive mappings to a family of quasi-ϕ-asymptotically nonexpansive mappings. They proposed the following hybrid iteration method with generalized projection for a family of closed and quasi-ϕ-asymptotically nonexpansive mappings in a Banach space E:

$$\begin{cases} x_0 \in C \ \text{ chosen arbitrarily,} \\ y_{n,i} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_i^n x_n), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \zeta_{n,i}\}, \\ C_n = \bigcap\limits_{i \in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.6}$$

They proved the following convergence theorem.

Theorem ZGT *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_n\}$ be a real sequence in $[0, 1)$ such that*

$\limsup_{n\to\infty}\alpha_n<1$. *Suppose that* $\{x_n\}$ *is given by (1.6), then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* $\zeta_{n,i}=(1-\alpha_n)(k_{n,i}-1)M$, $M\ge\phi(z,x_n)$ *for all* $z\in F, x_n\in C$ *and* Π_F *is the generalized projection from* C *onto* F.

At this point, we put forth the following two questions:

Question 1 Can Theorem ZGT be extended to more general reflexive, strictly convex, smooth Banach spaces with the property(K)?

Question 2 Can the algorithm in Theorem ZGT be replaced by another simpler one?

The purpose of this article is to solve above questions by introducing a new and simple hybrid projection iteration algorithm and by proving a strong convergence theorem for a family of closed and quasi-$\phi-$asymptotically nonexpansive mappings which are asymptotically regular on C by using new analysis techniques in the setting of reflexive, strictly convex, smooth Banach spaces with the property(K). The results of this paper improve and extend the results of Nakajo and Takahashi [6], Kim and Xu [7], Matsushita and Takahashi [8], Zhou, Gao, Tan [12] and others.

2. Preliminaries

Let X be a Banach space and X^* its dual space. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx=\{f\in X^*:\langle x,f\rangle=\|x\|^2=\|f\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if X^* is uniformly convex, then J is uniformly continuous on bounded subsets of X.

It is also very well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C: H\to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [14] introduced a generalized projection operator Π_C in a Banach space X which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that X is a real smooth Banach space. Let us consider the functional defined as in [8] by

$$\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2 \quad \text{for} \quad x,y\in X. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x,y)=\|x-y\|^2$, $x,y\in H$.

The generalized projection $\Pi_C: X\to C$ is a map that assigns to an arbitrary point $x\in X$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x=\bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x},x)=\min_{y\in C}\phi(y,x), \tag{2.2}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see [14, 15]). In Hilbert spaces, $\Pi_C=P_C$. It is obvious from the definition of function ϕ that

$$(\|y\|-\|x\|)^2\le\phi(y,x)\le(\|y\|+\|x\|)^2 \quad \text{for all} \quad x,y\in X. \tag{2.3}$$

Remark 2.1 If X is a reflexive strictly convex and smooth Banach space, then for $x, y \in X$, $\phi(x, y) = 0$ if and only if $x = y$. It suffices to show that if $\phi(x, y) = 0$, then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of J, we have $Jx = Jy$. Since X is strictly convex, J is strictly monotone, and hence, $x = y$. One may consult [15] for the details.

Let C be a closed convex subset of X, and T a mapping from C into itself. We use $F(T)$ to denote the fixed point set of T. A point p in C is said to be asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T will be denoted by $\widetilde{F(T)}$.

A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

A mapping T from C into itself is said to be quasi-ϕ-nonexpansive if $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

A mapping T from C into itself is said to be quasi-ϕ-asymptotically nonexpansive [9] if there exist some real sequence $\{k_n\}$ with $k_n \geq 1$ and $k_n \to 1$ and $F(T) \neq \emptyset$ such that $\phi(p, T^n x) \leq k_n\phi(p, x)$ for all $n \geq 1$, $x \in C$ and $p \in F(T)$.

Remark 2.2 The class of quasi-ϕ-asymptotically nonexpansive mappings contains properly the class of quasi-ϕ-nonexpansive mappings as a subclass and the class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings.

Remark 2.3 An asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may be not true.

A mapping $T : C \to C$ is said to be asymptotically regular on C if for any bounded subset $\tilde{C}$ of C, there holds the following equality:

$$\lim_{n\to\infty} \sup\{\|T^{n+1}x - T^n x\| : x \in \tilde{C}\} = 0.$$

We present some examples which are closed and quasi-ϕ-asymptotically nonexpansive in Zhou, Gao and Tan [12].

Example 2.1 Let E be a real line. We define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \dfrac{x}{2}\sin\dfrac{1}{x}, & \text{if } \ x \neq 0, \\ 0, & \text{if } \ x = 0. \end{cases}$$

Then T is continuous quasi-nonexpansive and hence it is closed and quasi-asymptotically nonexpansive with the constant sequence $\{1\}$ but not asymptotically nonexpansive.

Example 2.2 Let X be a uniformly smooth and strictly convex Banach space and $A \subset X \times X^*$ is a maximal monotone mapping such that $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed and quasi-ϕ-asymptotically nonexpansive mapping from X onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.3 Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space X onto a nonempty closed convex subset C of X. Then, Π_C is a closed and quasi-ϕ-asymptotically nonexpansive mapping from X onto C with $F(\Pi_C) = C$.

Recall that a Banach space X has the Kadeč-Klee property (property(K) for brevity) if for any sequence $\{x_n\} \subset X$ and $x \in X$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning property(K) the reader is referred to [16] and references cited there.

Now we are in a position to prove the main results of this paper.

3. Main results

Theorem 3.1 *Let X be a reflexive, strictly convex, smooth Banach space such that X and X^* have the property(K). Assume C is a nonempty closed convex subset of X. Let $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-asymptotically nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Assume that every T_i $(i \in I)$ is asymptotically regular on C. Let $\{\alpha_{n,i}\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_{n,i} < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \bigcap\limits_{i\in I} C_{1,i}, x_1 = \Pi_{C_1}(x_0), \\ y_{n,i} = J^{-1}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J(T_i^n x_n)),\ n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \zeta_{n,i}\}, \\ C_{n+1} = \bigcap\limits_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0,\ n \geq 0, \end{cases} \tag{3.1}$$

where

$$\zeta_{n,i} = (1-\alpha_{n,i})(k_{n,i} - 1) \sup_{z\in A} \phi(z, x_n),$$

and

$$A = \{y \in F : \|y - p_1\| \leq 1\}, \quad p_1 = \Pi_F x_0.$$

Then $\{x_n\}$ converges strongly to $p_1 = \Pi_F x_0$.

We remark that the sets $\{k_{n,i}\}, \{\alpha_{n,i}\}$ and $\{\zeta_{n,i}\}$ in Theorem 3.1 are nets when I is an infinite uncountable set; otherwise, they are all sequences of real numbers.

Proof We split the proof into six steps.

Step 1. Show that $\Pi_F x_0$ is well defined for every $x_0 \in C$.

To this end, we prove first that $F(T_i)$ is closed and convex for each $i \in I$. Let $\{p_n\}$ be a sequence in $F(T_i)$ with $p_n \to p$ as $n \to \infty$. We prove that $p \in F(T_i)$. From the definition of T_i, one has $\phi(p_n, T_i p) \leq k_{1,i}\phi(p_n, p)$, where $1 \leq k_{1,i} < \infty$. This implies that $\phi(p_n, T_i p) \to 0$ as $n \to \infty$. Noticing that

$$\phi(p_n, T_i p) = \|p_n\|^2 - 2\langle p_n, J(T_i p)\rangle + \|T_i p\|^2. \tag{3.2}$$

By taking limit on the both sides of (3.2), we have

$$\lim_{n\to\infty} \phi(p_n, T_i p) = \|p\|^2 - 2\langle p, J(T_i p)\rangle + \|T_i p\|^2 = \phi(p, T_i p).$$

Hence $\phi(p, T_i p) = 0$, which implies that $p = T_i p$. Hence $F(T_i)$ is closed, for all $i \in I$. We next show that $F(T_i)$ is convex. To this end, for arbitrary $p, q \in F(T_i), t \in (0,1)$, by setting $w = tp + (1-t)q$, it suffices to show that $T_i w = w$. Indeed, in view of definition of $\phi(x,y)$, we have

$$\begin{aligned}\phi(w, T_i^n w) &= \|w\|^2 - 2\langle w, JT_i^n w\rangle + \|T_i^n w\|^2 \\ &= \|w\|^2 - 2t\langle p, JT_i^n w\rangle - 2(1-t)\langle q, JT_i^n w\rangle + \|T_i^n w\|^2 \\ &= \|w\|^2 + t\phi(p, T_i^n w) + (1-t)\phi(q, T_i^n w) - t\|p\|^2 - (1-t)\|q\|^2 \\ &\le \|w\|^2 + k_{n,i} t\phi(p, w) + k_{n,i}(1-t)\phi(q, w) - t\|p\|^2 - (1-t)\|q\|^2 \\ &= (k_{n,i} - 1)(t\|p\|^2 + (1-t)\|q\|^2 - \|w\|^2).\end{aligned}$$

It follows from $k_{n,i} \to 1$ that $\phi(w, T_i^n w) \to 0$ as $n \to \infty$. Note that $0 \le (\|w\| - \|T_i^n w\|)^2 \le \phi(w, T_i^n w)$. Hence $\|T_i^n w\| \to \|w\|$ and consequently $\|J(T_i^n w)\| \to \|Jw\|$. This implies that $\{J(T_i^n w)\}$ is bounded. Since X is reflexive, X^* is also reflexive. So we can assume that

$$J(T_i^n w) \to f_0 \in X^* \tag{3.3}$$

weakly. On the other hand, in view of the reflexivity of X, one has $J(X) = X^*$, which means that for $f_0 \in X^*$, there exists $x \in X$, such that $J(x) = f_0$. Noting that

$$\begin{aligned}\phi(w, T_i^n w) &= \|w\|^2 - 2\langle w, J(T_i^n w)\rangle + \|T_i^n w\|^2 \\ &= \|w\|^2 - 2\langle w, J(T_i^n w)\rangle + \|J(T_i^n w)\|^2\end{aligned}$$

and using weakly lower semi-continuity of $\|\cdot\|^2$ and (3.3), we have

$$\begin{aligned}\liminf_{n\to\infty} \phi(w, T_i^n w) &= \liminf_{n\to\infty}(\|w\|^2 - 2\langle w, J(T_i^n w)\rangle + \|J(T_i^n w)\|^2) \\ &\ge \|w\|^2 - 2\langle w, f_0\rangle + \|f_0\|^2 \\ &= \|w\|^2 - 2\langle w, Jx\rangle + \|Jx\|^2 \\ &= \phi(w, x).\end{aligned}$$

From $\phi(w, T_i^n w) \to 0$ as $n \to \infty$, we have $\phi(w, x) = 0$ and consequently $w = x$, which implies that $f_0 = Jw$. Hence

$$J(T_i^n w) \to Jw \in X^*$$

weakly. Since $\|J(T_i^n w)\| \to \|Jw\|$ and X^* has the property(K), we have

$$\|J(T_i^n w) - Jw\| \to 0. \tag{3.4}$$

Noting that $J^{-1} : X^* \to X$ is demi-continuous, we have

$$T_i^n w \to w$$

weakly. Since $\|T_i^n w\| \to \|w\|$, by using the property(K) of X, we have

$$\|T_i^n w - w\| \to 0. \tag{3.5}$$

Since T_i is asymptotically regular, we have $T_i(T_i^n w) = T_i^{n+1} w \to w$ as $n \to \infty$. Since T_i is closed, we see that $w = T_i w$. Hence $F(T_i)$ is closed and convex for each $i \in I$ and consequently

$F=\bigcap_{i\in I}F(T_i)$ is closed and convex. By our assumption that $F=\bigcap_{i\in I}F(T_i)\neq\emptyset$, we have $\Pi_F x_0$ is well defined for every $x_0\in C$.

Step 2. Show that C_n is closed and convex for all $n\geq 1$.

It suffices to show that for each $i\in I$, $C_{n,i}$ is closed and convex for every $n\geq 1$. This can be proved by induction on n. In fact, for $n=1$, $C_{1,i}=C$ is closed and convex. Assume that $C_{n,i}$ is closed and convex for some $n\geq 1$. For $z\in C_{n+1,i}\subset C_{n,i}$, one obtains that

$$\phi(z,y_{n,i})\leq\phi(z,x_n)+\zeta_{n,i}$$

is equivalent to

$$2\langle z,Jx_n-Jy_{n,i}\rangle\leq\|x_n\|^2-\|y_{n,i}\|^2+\zeta_{n,i}.$$

It is easy to see that $C_{n+1,i}$ is closed and convex. Then, for all $n\geq 1$, $C_{n,i}$ is closed and convex. Consequently, $C_n=\bigcap_{i\in I}C_{n,i}$ is closed and convex for all $n\geq 1$.

In addition, it is obvious that

$$A=\{y\in F:\|y-p_1\|\leq 1\}$$

is bounded closed convex subset of X, where $p_1=\Pi_F x_0$.

Step 3. Show that $A\subset C_n$ for all $n\geq 1$.

It suffices to show that for each $i\in I$, $A\subset C_{n,i}$. $A\subset F\subset C_{1,i}=C$ is obvious. Assume that $A\subset C_{n,i}$ is closed and convex for some $n\geq 1$. For any $z\in A\subset F$, we have $z\in C_{n,i}$. From the definition of quasi-ϕ-asymptotically nonexpansive mappings, one has

$$\begin{aligned}
\phi(z,y_{n,i})&=\phi(z,J^{-1}(\alpha_{n,i}Jx_n+(1-\alpha_{n,i})J(T_i^nx_n)))\\
&=\|z\|^2-2\langle z,\alpha_{n,i}Jx_n+(1-\alpha_{n,i})J(T_i^nx_n)\rangle+\|J^{-1}(\alpha_{n,i}Jx_n+(1-\alpha_{n,i})J(T_i^nx_n))\|^2\\
&\leq\|z\|^2-2\langle z,\alpha_{n,i}Jx_n+(1-\alpha_{n,i})J(T_i^nx_n)\rangle+\alpha_{n,i}\|x_n\|^2+(1-\alpha_{n,i})\|T_i^nx_n\|^2\\
&=\alpha_{n,i}\phi(z,x_n)+(1-\alpha_{n,i})\phi(z,T_i^nx_n)\\
&\leq\alpha_{n,i}\phi(z,x_n)+(1-\alpha_{n,i})k_{n,i}\phi(z,x_n)\\
&=\phi(z,x_n)+(1-\alpha_{n,i})(k_{n,i}-1)\phi(z,x_n)\\
&\leq\phi(z,x_n)+(1-\alpha_{n,i})(k_{n,i}-1)(\sup_{z\in A}\phi(z,x_n))\\
&=\phi(z,x_n)+\zeta_{n,i},
\end{aligned}$$

which implies that $z\in C_{n+1,i}$ and consequently $z\in C_{n,i}$ for all $n\geq 1$ and $i\in I$. Therefore, $A\subset\bigcap_{i\in I}C_{n,i}=C_n$ and consequently $A\subset\bigcap_{n=1}^{\infty}C_n=D$. So $D\neq\emptyset$.

Step 4. Show that $\|x_n-p_0\|\to 0$, where $p_0=\Pi_D x_0$.

From Steps 2 and 3, we obtain that D is a nonempty, closed and convex subset of C. Hence $\Pi_D x_0$ is well defined for every $x_0\in C$. From the construction of C_n, we know that

$$C\supset C_1\supset C_2\supset\cdots.$$

Let $p_0=\Pi_D x_0$. Since $x_n=\Pi_{C_n}x_0$, we have

$$\phi(x_1,x_0)\leq\phi(x_2,x_0)\leq\cdots\leq\phi(p_0,x_0).$$

By the reflexivity of X, we can assume that $x_n \to g_1 \in X$ weakly as $n \to \infty$ (passing to a subsequence if necessary). Since $C_j \subset C_n$, for $j \geq n$, we have $x_j \in C_n$ for $j \geq n$. Since C_n is closed and convex, by the Mazur Theorem, $g_1 \in C_n$ for any $n \geq 1$. Hence $g_1 \in D$. Moreover, using weakly lower semi-continuity of $\phi(\cdot, x_0)$, we obtain that

$$\phi(p_0, x_0) \leq \phi(g_1, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(p_0, x_0),$$

which implies that $g_1 = p_0$ and $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(p_0, x_0)$, i.e.,

$$\lim_{n\to\infty}(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2) = \|p_0\|^2 - 2\langle p_0, Jx_0\rangle + \|x_0\|^2,$$

which shows that $\lim_{n\to\infty} \|x_n\| = \|p_0\|$. By the property(K) of X, we have

$$\|x_n - p_0\| \to 0 \quad \text{as} \quad n \to \infty,$$

where $p_0 = \Pi_D x_0$.

Step 5. Show that $p_0 = T_i p_0$, for all $i \in I$.

Since $x_{n+1} \in C_{n+1} = \bigcap_{i\in I} C_{n+1,i}$ for all $n \geq 0$ and $i \in I$, we have

$$0 \leq \phi(x_{n+1}, y_{n,i}) \leq \phi(x_{n+1}, x_n) + \zeta_{n,i}. \tag{3.6}$$

It follows from $\|x_n - p_0\| \to 0$ that

$$\phi(x_{n+1}, x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Since $k_{n,i} \to 1$ as $n \to \infty$, we have

$$\zeta_{n,i} = (1 - \alpha_{n,i})(k_{n,i} - 1)(\sup_{z\in A} \phi(z, x_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.8}$$

From (3.6), (3.7) and (3.8), we have

$$\phi(x_{n+1}, y_{n,i}) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Note that $0 \leq (\|x_{n+1}\| - \|y_{n,i}\|)^2 \leq \phi(x_{n+1}, y_{n,i})$. Hence $\|y_{n,i}\| \to \|p_0\|$ and consequently $\|J(y_{n,i})\| \to \|Jp_0\|$. This implies that $\{J(y_{n,i})\}$ is bounded. Since X is reflexive, X^* is also reflexive. So we can assume that

$$J(y_{n,i}) \to g_0 \in X^* \tag{3.10}$$

weakly. On the other hand, in view of the reflexivity of X, one has $J(X) = X^*$, which means that for $g_0 \in X^*$, there exists $y \in X$, such that $J(y) = g_0$. Noting that

$$\begin{aligned}\phi(x_{n+1}, y_{n,i}) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J(y_{n,i})\rangle + \|y_{n,i}\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, J(y_{n,i})\rangle + \|J(y_{n,i})\|^2\end{aligned}$$

and using weakly lower semi-continuity of $\|\cdot\|^2$ and (3.10), we have

$$\begin{aligned}\liminf_{n\to\infty} \phi(x_{n+1}, y_{n,i}) &= \liminf_{n\to\infty}(\|x_{n+1}\|^2 - 2\langle x_{n+1}, J(y_{n,i})\rangle + \|J(y_{n,i})\|^2) \\ &\geq \|p_0\|^2 - 2\langle p_0, g_0\rangle + \|g_0\|^2 \\ &= \|p_0\|^2 - 2\langle p_0, Jy\rangle + \|Jy\|^2 \\ &= \phi(p_0, y).\end{aligned}$$

From (3.9), we have $\phi(p_0, y) = 0$ and consequently $p_0 = y$, which implies that $g_0 = Jp_0$. Hence

$$J(y_{n,i}) \to Jp_0 \in X^*$$

weakly. Since $\|J(y_{n,i})\| \to \|Jp_0\|$ and X^* has the property(K), we have

$$\|J(y_{n,i}) - Jp_0\| \to 0. \tag{3.11}$$

On the other hand, since $\|x_n - p_0\| \to 0$, noting that $J : X \to X^*$ is demi-continuous, we have

$$Jx_n \to Jp_0 \in X^*$$

weakly. Notice that

$$|\|Jx_n\| - \|Jp_0\|| = |\|x_n\| - \|p_0\|| \leq \|x_n - p_0\| \to 0,$$

which implies that $\|Jx_n\| \to \|Jp_0\|$. By using the property(K) of X^*, we have

$$\|Jx_n - Jp_0\| \to 0. \tag{3.12}$$

From (3.1), (3.11), (3.12) and $\liminf_{n\to\infty} \alpha_{n,i} < 1$, we have

$$\|J(T_i^n x_n) - Jp_0\| \to 0.$$

Since $J^{-1} : X^* \to X$ is demi-continuous, we have

$$T_i^n x_n \to p_0$$

weakly in X. Moreover,

$$|\|T_i^n x_n\| - \|p_0\|| = |\|J(T_i^n x_n)\| - \|Jp_0\|| \leq \|J(T_i^n x_n) - Jp_0\| \to 0,$$

which implies that $\|T_i^n x_n\| \to \|p_0\|$. By the property(K) of X, we have

$$T_i^n x_n \to p_0.$$

By using the asymptotic regularity of T_i, we have

$$T_i^{n+1} x_n \to p_0.$$

Hence $T_i(T_i^n x_n) \to p_0$. From the closeness property of T_i, we have

$$T_i p_0 = p_0,$$

which implies that $p_0 \in F = \bigcap_{i\in I} F(T_i)$.

Step 6. Show that $p_0 = p_1 = \Pi_F x_0$.

Suppose $p_0 \neq p_1$. From $p_0 \in F$, we have

$$\phi(p_0, x_0) > \phi(\Pi_F x_0, x_0) = \phi(p_1, x_0).$$

From $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(p_0, x_0)$, there must exist a positive integer N such that

$$\phi(x_n, x_0) > \phi(p_1, x_0) \tag{3.13}$$

whenever $n > N$. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(y, x_0), \quad \forall y \in C_n. \tag{3.14}$$

From (3.13) and (3.14), we can obtain $p_1 \overline{\in} C_n$, for $n > N$. From Step 3, we have $p_1 \overline{\in} A$. This is a contradiction. This completes the proof. □

Remark 3.1 Our Theorem 3.1 improves and extends Theorem ZGT in several respects:

(i) It extends from uniform convex and uniform smooth Banach spaces to reflexive, strictly convex, smooth Banach spaces with the property(K). In our Theorem 3.1 the hypotheses on X are weaker than usual assumptions of uniform convexity and uniform smoothness. For example, any strictly convex, reflexive and smooth Musielak-Orlicz space satisfies our assumptions [16] while, in general, these spaces need not to be uniform convex or uniform smooth.

(ii) It relaxes the restriction on $\{\alpha_{n,i}\}$ from $\limsup_{n\to\infty} \alpha_{n,i} < 1$ to $\liminf_{n\to\infty} \alpha_{n,i} < 1$ and removes the condition of boundedness of subset C.

(iii) Our algorithm is simpler than the one used in Theorem ZGT.

Remark 3.2 Theorem 3.1 presents some affirmative answers to Questions 1 and 2.

From Theorem 3.1, we deduce the following corollary immediately.

Corollary 3.1 *Let X be a reflexive, strictly convex and smooth Banach space such that X and X^* have the property(K), C be a nonempty closed convex subset of X, and $\{T_i\}_{i\in I} : C \to C$ be a family of closed and quasi-ϕ-nonexpansive mappings such that $F = \bigcap_{i\in I} F(T_i) \neq \emptyset$. Let $\{\alpha_{n,i}\}$ be a real sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_{n,i} < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ C_{1,i} = C,\ C_1 = \bigcap\limits_{i\in I} C_{1,i},\ x_1 = \Pi_{C_1}(x_0), \\ y_{n,i} = J^{-1}(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J(T_i^n x_n)), \quad n \geq 1, \\ C_{n+1,i} = \{z \in C_{n,i} : \phi(z, y_{n,i}) \leq \phi(z, x_n)\}, \\ C_{n+1} = \bigcap\limits_{i\in I} C_{n+1,i}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0. \end{cases} \tag{3.2}$$

Then $\{x_n\}$ converges strongly to $p_0 = \Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Remark 3.3 In Theorem 3.1 and Corollary 3.1, if one takes $I = \{1, 2, \ldots, N\}$, $I = \{1, 2, \ldots\}$ and $I = R^+$, respectively, then one can obtain strong convergence theorems for a finite, an infinite countable and an infinite uncountable families of quasi-ϕ-asymptotically nonexpansive mappings, respectively. The results of this paper improve and extend the recent results of [6–13].

References

[1] GOEBEL K, KIRK W A. *A fixed point theorem for asymptotically nonexpansive mappings* [J]. Proc. Amer. Math. Soc., 1972, **35**: 271–174.

[2] SCHU J. *Iterative construction of fixed points of asymptotically nonexpansive mappings* [J]. J. Math. Anal. Appl., 1991, **158**(2): 407–413.

[3] ZHOU Haiyun, CHO Y J, KANG S M. *A new iterative algorithm for approximating common fixed points*

for asymptotically nonexpansive mappings [J]. Fixed Point Theory Appl. 2007, Art. ID 64874, 10 pp.

[4] HAUGAZEAU Y. *Sur les Inéquations Variationnelles et la Minimisation de Fonctionnelles Convexes* [M]. Thése, Université de Paris, Paris, France, 1968.

[5] BAUSCHKE H H, COMBETTES P L. *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces* [J]. Math. Oper. Res., 2001, **26**(2): 248–264.

[6] NAKAJO K, TAKAHASHI W. *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups* [J]. J. Math. Anal. Appl., 2003, **279**(2): 372–379.

[7] KIM T H, XU Hongkun. *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups* [J]. Nonlinear Anal., 2006, **64**(5): 1140–1152.

[8] MATSUSHITA S Y, TAKAHASHI W. *A strong convergence theorem for relatively nonexpansive mappings in a Banach space* [J]. J. Approx. Theory, 2005, **134**(2): 257–266.

[9] SU Yongfu, QIN Xiaolong. *Strong convergence of modified Ishikawa iterations for nonlinear mappings* [J]. Proc. Indian Acad. Sci. Math. Sci., 2007, **117**(1): 97–107.

[10] SU Yongfu, WANG Dongxing, SHANG Meijuan. *Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings* [J]. Fixed Point Theory Appl., 2008, Art. ID 284613, 8 pp.

[11] HE Huimin, CHEN Rudong. *Strong convergence theorems of the CQ method for nonexpansive semigroups* [J]. Fixed Point Theory Appl., 2007, Art. ID 59735, 8 pp.

[12] ZHOU Haiyun, GAO Gailiang, TAN Bin. *Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings* [J]. J. Appl. Math. Comput., 2010, **32**(2): 453–464.

[13] SU Yongfu, GAO Junyu, ZHOU Haiyun. *Monotone CQ algorithm of fixed points for weak relatively nonexpansive mappings and applications* [J]. J. Math. Res. Exposition, 2008, **28**(4): 957–967.

[14] ALBER YA I. *Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications* [M]. Dekker, New York, 1996.

[15] TAKAHASHI W. *Nonlinear Functional Analysis* [M]. Yokohama Publishers, Yokohama, 2000. (in Japanese)

[16] HUDZIK H, KOWALEWSKI W, LEWICKI G. *Approximative compactness and full rotundity in Musielak-Orlicz spaces and Lorentz-Orlicz spaces* [J]. Zeitschrift fuer Analysis und ihre Anwendungen, 2006, **25**: 163–192.