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Exponential Polynomial Approximation with Unrestricted Upper Density

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Abstract We take a new approach to obtaining necessary and sufficient conditions for the incompleteness of exponential polynomials in L^p_α, where L^p_α is the weighted Banach space of complex continuous functions f defined on the real axis $\mathbb{R}$ satisfying $(\int_{-\infty}^{+\infty}|f(t)|^pe^{-\alpha(t)}\mathrm{d}t)^{1/p}$, $1<p<\infty$, and $\alpha(t)$ is a nonnegative continuous function defined on the real axis $\mathbb{R}$. In this paper, the upper density of the sequence which forms the exponential polynomials is not required to be finite. In the study of weighted polynomial approximation, consideration of the case is new.

Keywords incompleteness; Banach space; exponential system.

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1. Introduction

Let $\alpha(t)$ be a nonnegative continuous function defined on $\mathbb{R}$, henceforth called a weight, satisfying

$$\lim_{|t|\to\infty}|t|^{-1}\alpha(t)=\infty. \tag{1}$$

Given a weight $\alpha(t)$, the weighted Banach space L^p_α consists of complex continuous functions f defined on the real axis $\mathbb{R}$, satisfying $\|f\|_{p,\alpha}<\infty$ where

$$\|f\|_{p,\alpha}=(\int_{-\infty}^{+\infty}|f(t)|^pe^{-\alpha(t)}\mathrm{d}t)^{1/p},\quad 1<p<\infty,$$

for $f\in L^p_\alpha$. Denote by $\mathbf{M}(\Lambda)$ the set of exponential polynomials which are finite linear combinations of exponential system $\{e^{\lambda t}:\lambda\in\Lambda\}$ where $\Lambda=\{\lambda_n:n=1,2,\ldots\}$ is a sequence of complex numbers in $\mathbb{C}$. Our condition (1) guarantees that $\mathbf{M}(\Lambda)$ is a subspace of L^p_α.

The problem of completeness of $\mathbf{M}(\Lambda)$ in L^p_α in the norm $\|\cdot\|_{p,\alpha}$ is the so-called exponential polynomial approximation, which is similar to the classical Bernstein problem on polynomial approximation in [5]. And the similar problem is also considered in [1–4, 11, 12]. Motivated by Bernstein problem and by using a generalization of Malliavin's uniqueness theorem in [8]

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about Watson's problem, Deng [2] obtained a necessary and sufficient condition for $\mathbf{M}(\Lambda)$ to be complete in L^p_α. The result is described below.

Theorem A ([2]) *Let $\alpha(t)$ be a nonnegative convex function defined on $\mathbb{R}$ satisfying (1). Let $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of positive real numbers satisfying*

$$\delta(\Lambda) = \inf\{\lambda_{n+1} - \lambda_n : n = 1, 2, \ldots\} > 0. \tag{2}$$

A necessary and sufficient condition for $\mathbf{M}(\Lambda)$ to be complete in L^p_α is

$$\int_1^{+\infty} \frac{\alpha(\lambda(r) - a)}{1 + t^2} \mathrm{d}t = \infty, \tag{3}$$

for each real number a, where $\lambda(r) = 2\sum_{\lambda_n \le r} \frac{1}{\lambda_n}$, if $r \ge \lambda_1$; $\lambda(r) = 0$, otherwise.

We want to point out that in the study of weighted polynomial approximation, the sequence Λ is often required to have a finite upper density, in other words, the relation $\limsup_{t\to\infty} \frac{n_\Lambda(t)}{t} < \infty$, is often required [1–4, 11]. In [12], by applying Khabibullin's uniqueness theorems on entire functions, the author investigated the weighted polynomial approximation problem where the sequence Λ does not have a finite upper density, in other words, the relation $\limsup_{t\to\infty} \frac{n_\Lambda(t)}{t} = \infty$, is allowed.

The main technical tool for proving Theorem A is the Proposition 1 in [2] which is a generalization of Malliavin's uniqueness theorem in [8] about Watson's problem. We observe it cannot be applied to the situation where Λ does not have a finite upper density, Λ is not separated or $\alpha(t)$ is not necessarily convex. That is why we take the new approach from [12] to dealing with the situation. Luckily, taking Khabibullin's uniqueness theorems of entire functions from [6] and [7] as the basic ingredients, we could investigate the weighted exponential polynomial approximation problem where Λ does not have a finite upper density, Λ is not separated or $\alpha(t)$ is not necessarily convex. It is convenient to introduce some terminology before we present our results.

With a sequence of numbers $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$, $\lambda_n \in \mathbb{C}$, we associate the averaged counting function [6]

$$N_\Lambda(r) = \int_0^r \frac{n_\Lambda(t)}{t} \mathrm{d}t, \quad n_\Lambda(t) = \sum_{|\lambda_n| \le t} 1.$$

Let us state the main results of our paper. Let A denote positive constants, which may be different at each occurrence.

Theorem 1 *Let $\alpha(t)$ be a nonnegative even continuous function defined on $\mathbb{R}$, satisfying (1). If $\mathbf{M}(\Lambda)$ is incomplete in L^p_α, then for any constant $A > 0$, $r = |z| > 0$,*

$$N_\Lambda(r) \le \frac{1}{p} \alpha^*(r + A), \tag{4}$$

where $\alpha^*(r) = \sup\{rt - \alpha(t) : t \in \mathbb{R}\}$.

Theorem 2 *Let $\alpha(t)$ be a nonnegative continuous function defined on $\mathbb{R}$, satisfying*

$$A_1 t^2 \le \alpha(t) \le A_2 t^2, \quad 0 < A_1 \le A_2. \tag{5}$$

If

$$N_\Lambda(r) \le A\alpha^*(Ar) \tag{6}$$

for some constant $A > 0$ and

$$A^3 < \frac{A_1}{54A_2}, \tag{7}$$

then $\mathbf{M}(\Lambda)$ is incomplete in L^p_α.

Theorem 3 *Let $\alpha(t) = ct^2$ for $c > 0$. We have*

(i) If $\limsup_{r\to\infty} \frac{N_\Lambda(r)}{r^2} < \frac{1}{16\pi c}$, then $\mathbf{M}(\Lambda)$ is incomplete in L^p_α;

(ii) If $\limsup_{r\to\infty} \frac{N_\Lambda(r)}{r^2} > \frac{1}{4pc}$, then $\mathbf{M}(\Lambda)$ is complete in L^p_α.

2. Proof of theorems

In order to prove the theorems, we need the results from [6] and [7].

Lemma 1 ([6]) *Let k(r) be a positive monotonic nondecreasing unbounded function. Let $\Lambda = \{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of complex numbers in $\mathbb{C}$. Then there is an entire function f(z) which satisfies $f(\Lambda) = 0$ and*

$$|f(z)| \le \exp(A_f k(A_f|z|)),$$

where A_f is a constant which depends on f, if and only if there is a constant A such that (6) holds for Λ and $k(r)$ instead of α^*.

Remark 1 Lemma 1 is Khabibullin's generalization of uniqueness theorems from [9] and [10]. From Khabibullin's proof, we see that the hypothesis that $k(r)$ is continuous can be omitted. And from the proof of Theorem A in [6], one has instead the weaker estimate

$$|f(z)| \le \exp(3Ak(3A|z|)),$$

in case $N_\Lambda(r) \le Ak(Ar)$.

Lemma 2 ([7]) *If $\rho \ge \frac{1}{2}$ and*

$$\limsup_{r\to+\infty} \frac{N_\Lambda(r)}{r^\rho} < \sigma,$$

then there exists an entire function f which is not equivalently equal to 0 such that $f(\Lambda)$=0 and

$$\limsup_{r\to+\infty} \frac{\log|f(z)|}{|z|^\rho} < \pi\rho\sigma.$$

Proof of Theorem 1 If the space $\mathbf{M}(\Lambda)$ is incomplete in L^p_α, then there exists a bounded linear functional T such that $\|T\|_{p,\alpha} = 1$ and $T(e^{\lambda t}) = 0$ for $\lambda \in \Lambda$. So by the Riesz representation theorem, there exists a function $g(t) \in L^q_\alpha$ satisfying $\|g\|_{q,\alpha} = 1$ for $h \in L^p_\alpha$,

$$T(h) = \int_{-\infty}^{+\infty} h(t)g(t)e^{-\alpha(t)}\mathrm{d}t,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$f(z) = \int_{-\infty}^{+\infty} e^{\frac{tz}{p}} g(t)e^{-\alpha(t)}\mathrm{d}t,$$

then for arbitrary $A > 0$, obviously $e^{-A|t|} \in L^p_\alpha$. By Hölder's inequality and the fact that $\alpha(t)$ is an even continuous function, we have

$$|f(z)| \leq \|g\|_{q,\alpha} e^{\frac{1}{p}\alpha^*(|z|+A)} \int_{-\infty}^{+\infty} e^{-pA|t|}\mathrm{d}t$$

holds for all $z \in \mathbb{C}$. We know that $f(z)$ is an entire function from Fubini's theorem and Morera's theorem. Thus by Jensen's formula, we could obtain (4).

Proof of Theorem 2 If there exist real constants A, A_1, A_2 such that (5) and (6) hold, by a weaker form of Lemma 1 there exists an entire function $g(z)$ satisfying $g(\Lambda) = 0$ and

$$|g(z)| \leq e^{3A\alpha^*(3A|z|)} \tag{8}$$

holds for all $z \in \mathbb{C}$. Define

$$g_1(z) = g(z)e^{B_1 z^2},$$

where B_1 is a positive constant, satisfying

$$\frac{1}{8A_2} > B_1 > \frac{27A^3}{4A_1}. \tag{9}$$

From $\alpha^*(|z|) = \sup\{|z|t - \alpha(t) : t \in \mathbb{R}\}$, (5) and (8), by direct calculation, we have

$$|g_1(z)| \leq e^{(\frac{27A^3}{4A_1}+B_1)x^2-(B_1-\frac{27A^3}{4A_1})y^2}. \tag{10}$$

Let

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_1(1+iy)e^{-(1+iy)t}\mathrm{d}y.$$

We see the above definition is reasonable from (9) and (10), and $h_0(t)$ is continuous on $(-\infty, +\infty)$. By Cauchy's formula

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_1(x+iy)e^{-(x+iy)t}\mathrm{d}y, \quad \forall x \in \mathbb{R}. \tag{11}$$

From (10), we have

$$|h_0(t)| \leq B_2 e^{B_3 x^2 - xt},$$

where $B_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-(B_1-\frac{27A^3}{4A_1})y^2}\mathrm{d}y$ and $B_3 = \frac{27A^3}{4A_1} + B_1$. Thus

$$|h_0(t)| \leq B_2 e^{\inf\{B_3x^2 - xt : x \in \mathbb{R}\}},$$

and by direct calculation

$$|h_0(t)| \leq B_2 e^{-\frac{t^2}{4B_3}}. \tag{12}$$

From (12) we know that $h_0(t)$ is in L^1. Take the inverse Fourier transform in (11), we obtain

$$g_1(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)e^{zt}\mathrm{d}t.$$

Therefore from (5), (9), (10) and (12), if (7) holds, by properly choosing B_1, we obtain the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)h(t)\mathrm{d}t, \quad h \in L^p_\alpha$$

satisfying $T(e^{\lambda t}) = 0$ for $\lambda \in \Lambda$ and $\|T\| = \|h_0 e^{\alpha}\|_{q,\alpha} > 0$.

Proof of Theorem 3

Proof of (i) In this case, we have $\alpha^*(r) = \frac{r^2}{4c}$. And we have

$$\limsup_{r\to+\infty} \frac{N_\Lambda(r)}{r^2} < \sigma < \frac{1}{16\pi c},$$

for some $\sigma > 0$. It follows from Lemma 2 for $\rho = 2$ that there exists an entire function $g(z)$ satisfying $g(\Lambda) = 0$ and

$$\limsup_{|z|\to+\infty} \frac{\log|g(z)|}{|z|^2} < 2\pi\sigma. \tag{13}$$

Thus $|g(z)| \le Ae^{2\pi\sigma r^2}$ for some $A > 0$ and all $z \in \mathbb{C}$. Define

$$g_1(z) = g(z)e^{B_1 z^2},$$

where B_1 is a positive constant, satisfying

$$2\pi\sigma < B_1 < 2\pi\sigma + \delta, \quad 0 < \delta < \frac{1 - 16\pi\sigma c}{4c}. \tag{14}$$

Thus, we have

$$|g_1(z)| \le Ae^{(2\pi\sigma + B_1)x^2 - (B_1 - 2\pi\sigma)y^2}. \tag{15}$$

Let

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_1(1 + iy)e^{-(1+iy)t}\mathrm{d}y.$$

From (15), we see the above definition is reasonable, and $h_0(t)$ is continuous on $(-\infty, +\infty)$. By Cauchy's formula

$$h_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_1(x + iy)e^{-(x+iy)t}\mathrm{d}y, \quad \forall x \in \mathbb{R}. \tag{16}$$

From (15), we have

$$|h_0(t)| \le B_2 e^{B_3 x^2 - xt},$$

where $B_2 = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{+\infty} e^{-(B_1 - 2\pi\sigma)y^2}\mathrm{d}y$ and $B_3 = 2\pi\sigma + B_1$. Thus

$$|h_0(t)| \le B_2 e^{\inf\{B_3 x^2 - xt : x \in \mathbb{R}\}},$$

and by direct calculation

$$|h_0(t)| \le B_2 e^{-\frac{t^2}{4B_3}}. \tag{17}$$

From (17) we know that $h_0(t)$ is in L^1. Taking the inverse Fourier transform in (16), we obtain

$$g_1(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)e^{zt}\mathrm{d}t.$$

Therefore from (14) and (17), if (i) holds, then by properly choosing B_1, we obtain the bounded linear functional

$$T(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h_0(t)h(t)\mathrm{d}t, \quad h \in L^p_\alpha$$

satisfying $T(e^{\lambda t}) = 0$ for $\lambda \in \Lambda$ and $\|T\| = \|h_0 e^{\alpha}\|_{q,\alpha} > 0$.

Proof of (ii) If the space $\mathbf{M}(\Lambda)$ is incomplete in L^p_α, then there exists a bounded linear functional T such that $\|T\|_{p,\alpha}=1$ and $T(e^{\lambda t})=0$ for $\lambda\in\Lambda$. So by the Riesz representation theorem, there exists a function $g(t)\in L^q_\alpha$ satisfying $\|g\|_{q,\alpha}=1$ for $h\in L^p_\alpha$,

$$T(h)=\int_{-\infty}^{+\infty}h(t)g(t)e^{-\alpha(t)}\mathrm{d}t,$$

where $\frac{1}{p}+\frac{1}{q}=1$. Define

$$f(z)=\int_{-\infty}^{+\infty}e^{\frac{tz}{p}}g(t)e^{-\alpha(t)}\mathrm{d}t,$$

then for some $A>0$, obviously $e^{-A|t|}\in L^p_\alpha$. By Hölder's inequality and the fact that $\alpha(t)$ is an even continuous function, we have

$$|f(z)|\le\|g\|_{q,\alpha}e^{\frac{1}{p}\alpha^*(|z|+A)}\int_{-\infty}^{+\infty}e^{-pA|t|}\mathrm{d}t$$

holds for all $z\in\mathbb{C}$. We know that $f(z)$ is an entire function from Fubini's theorem and Morera's theorem. Thus by Jensen's formula, we could obtain $N_\Lambda(r)\le\frac{(r+A)^2}{4pc}$ which is a contradiction.

An open problem In Theorem 3, we consider the completeness of $\mathbf{M}(\Lambda)$ in C_α where $\alpha(t)=ct^2$ for some $c>0$ and $\limsup_{r\to\infty}\frac{N_\Lambda(r)}{r^2}<\frac{1}{16\pi c}$ or $\limsup_{r\to\infty}\frac{N_\Lambda(r)}{r^2}>\frac{1}{4pc}$. If $\limsup_{r\to\infty}\frac{N_\Lambda(r)}{r^2}\in[\frac{1}{16\pi c},\frac{1}{4pc}]$, our method could not be applied. The completeness problem of $\mathbf{M}(\Lambda)$ remains open in this situation.

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