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Adjacent Vertex Distinguishing Incidence Coloring of the Cartesian Product of Some Graphs

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Abstract An adjacent vertex distinguishing incidence coloring of graph G is an incidence coloring of G such that no pair of adjacent vertices meets the same set of colors. We obtain the adjacent vertex distinguishing incidence chromatic number of the Cartesian product of a path and a path, a path and a wheel, a path and a fan, and a path and a star.

Keywords Cartesian product; incidence coloring; adjacent vertex distinguishing incidence coloring; adjacent vertex distinguishing incidence chromatic number.

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1. Introduction

All graphs in this paper are simple, connected and undirected. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of a graph G, respectively. And we denote the maximum degree of G by $\Delta(G)$. The undefined terminology can be found in [1].

Let G be a graph of order n. For any vertex $u \in V(G)$, $N(u)$ denotes the set of all vertices adjacent to vertex u. Obviously, $d(u)$ is equal to $|N(u)|$.

Let $I(G) = \{(v,e) \in V(G) \times E(G) \mid v \text{ is incident with } e\}$ be the set of incidences of G. Two incidences (v,e) and (w,f) are said to be adjacent if one of the following holds: (1) $v = w$; (2) $e = f$; (3) the edge vw equals e or f. $I_v = \{(v,vu) \mid u \in N(v)\}$ and $A_v = \{(u,uv) \mid u \in N(v)\}$ are called the close-incidence set and far-incidence set of v, respectively.

Definition 1.1 ([2]) *An incidence coloring of G is a mapping σ from $I(G)$ to a color set C such that any two adjacent incidences have different images. If σ: $I(G) \to C$ is an incidence coloring of G and $|C| = k$, k is a positive integer, then we say that G is k-incidence colorable and σ is a k-incidence coloring of G; The minimum value of k such that G is k-incidence colorable is called the incidence chromatic number of G, and denoted by $\chi_i(G)$.*

Definition 1.2 ([3]) *Let $Q_u = I_u \cup A_u$ and let σ be a k-incidence coloring of a graph G*

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with color set C. $S(u)$ denotes the set of colors assigned to Q_u (with respect to σ). If for any $uv \in E(G)$, $S(u) \neq S(v)$, then σ is called a k-adjacent vertex distinguishing incidence coloring of G. And $\chi_{ai}(G)$=min$\{k \mid$ there exists a k-adjacent vertex distinguishing incident coloring of $G\}$ is called the adjacent vertex distinguishing incidence chromatic number of G. An adjacent vertex distinguishing incidence coloring f of G using k colors is denoted by k-AVDIC. And let $\overline{S}(u) = C \setminus S(u)$.

Definition 1.3 ([3]) *The Cartesian product of simple graphs G and H is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent to (u', v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.*

Adjacent strong edge coloring, adjacent vertex distinguishing total coloring and vertex distinguishing edge coloring of the Cartesian product of some special graphs were studied in [4–6], respectively. In this paper, we will investigate adjacent vertex distinguishing incidence coloring of the Cartesian product of some graphs.

The following lemma will be used.

Lemma 1.1 ([3]) *Let C_n be a cycle of order n, where n is at least 3. Then*

$$\chi_{ai}(C_n) = \begin{cases} 4, & \text{if } n \geq 3 \text{ and } n \neq 5; \\ 5, & \text{if } n = 5. \end{cases}$$

Lemma 1.2 ([3]) *Let G be a graph of order at least 3. If there exist the adjacent vertices of maximum degree in G, then $\chi_{ai}(G) \geq \Delta(G) + 2$.*

Lemma 1.3 ([3]) *For any graph G, $\chi_{ai}(G) \geq \chi_i(G) \geq \Delta(G) + 1$.*

2. Main results

Theorem 2.1 *Let P_n and P_m be two paths of order n and m, respectively, where $n \geq m \geq 2$. Then*

$$\chi_{ai}(P_n \times P_m) = \begin{cases} 4, & \text{if } m = n = 2; \\ 6, & \text{if } m \geq 3 \text{ and } n \geq 4; \\ 5, & \text{otherwise.} \end{cases}$$

Proof Let $\{u_1, u_2, \ldots, u_n\}$ be vertex set of P_n and $\{u'_1, u'_2, \ldots, u'_m\}$ be vertex set of P_m. Let $v_{ij} = (u_i, u'_j)$. We consider the following four cases separately.

Case 1 Suppose $m = n = 2$. Obviously, $P_2 \times P_2 \cong C_4$. By Lemma 1.1, $\chi_{ai}(C_4) = 4$.

Case 2 Suppose $m \geq 3$ and $n \geq 4$. Obviously, there exist the adjacent vertices of maximum degree in $P_n \times P_m$. By Lemma 1.2, $\chi_{ai}(P_n \times P_m) \geq \Delta(P_n \times P_m) + 2 = 6$. We now only need to give a 6-AVDIC of $P_n \times P_m$.

We construct a mapping f from $I(P_n \times P_m)$ to $\{0, 1, 2, 3, 4, 5\}$ as follows:

$$f(A_{v_{ij}}) = (2i + j - 3) \pmod 6, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

It is easy to see that f is an incidence coloring of $P_n \times P_m$. For convenience, we use the matrix $B=(b_{ij})_{n\times m}$ to describe the incidence coloring f of $P_n \times P_m$, where $b_{ij}=f(A_{v_{ij}})$ denotes the color which is received by the far-incidence set of vertex v_{ij}, and all b_{ij} are taken modulo 6.

$$B=\begin{pmatrix} 0 & 1 & 2 & \cdots & m-2 & m-1 \\ 2 & 3 & 4 & \cdots & m & m+1 \\ 4 & 5 & 6 & \cdots & m+2 & m+3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2n-4 & 2n-3 & 2n-2 & \cdots & 2n+m-6 & 2n+m-5 \\ 2n-2 & 2n-1 & 2n & \cdots & 2n+m-4 & 2n+m-3 \end{pmatrix}$$

From the matrix $B=(b_{ij})_{n\times m}$, it is clear that f is adjacent vertex distinguishing. Hence f is a 6-AVDIC of $P_n \times P_m$ and thus $\chi_{ai}(P_n \times P_m)=6$.

Case 3 Suppose $m=n=3$. It is clear that $\chi_{ai}(P_3\times P_3)\geq \Delta(P_3\times P_3)+1=5$. We now only need to give a 5-AVDIC of $P_3 \times P_3$.

We construct a mapping f from $I(P_3\times P_3)$ to $\{0,1,2,3,4\}$ as follows:

$$f(A_{v_{ij}})=(2i+j-3) \pmod 5,\quad i,j=1,2,3.$$

Similarly, we use the matrix $B'=(b'_{ij})_{3\times 3}$ to describe the incidence coloring f of $P_3\times P_3$. The matrix B' is similar to B in case 1, however, all b'_{ij} are taken modulo 5.

It is easy to see that f is a 5-AVDIC of $P_3\times P_3$ and thus $\chi_{ai}(P_3\times P_3)=5$.

Case 4 Suppose $m=2$ and $n\geq 3$. It is clear that $\chi_{ai}(P_n\times P_2)\geq \Delta(P_n\times P_2)+2=5$. We now only need to give a 5-AVDIC of $P_n\times P_2$.

In the same way as in Case 3, we construct a mapping f from $I(P_n\times P_2)$ to $\{0,1,2,3,4\}$ as follows:

$$f(A_{v_{ij}})=(2i+j-3) \pmod 5,\quad i=1,2,\ldots,n,\ j=1,2.$$

Similarly, we use the matrix $B''=(b''_{ij})_{n\times 2}$ to describe the incidence coloring f of $P_n\times P_2$. The matrix $B^{''}$ is similar to B in Case 1, however, all b''_{ij} are taken modulo 5.

It is easy to see that f is a 5-AVDIC of $P_n\times P_2$ and thus $\chi_{ai}(P_n\times P_2)=5$.

The proof of this theorem is completed. □

Theorem 2.2 *Let P_n be a path of order $n\geq 2$, and let G be a star S_m, a wheel W_m or a fan F_m of order $m+1$, where m is at least 5. Then*

$$\chi_{ai}(P_n\times G)=\begin{cases} m+3, & \text{if } n=2,3; \\ m+4, & \text{if } n\geq 4. \end{cases}$$

Proof Let $\{u_1,u_2,\ldots,u_n\}$ be vertex set of P_n and let $\{u'_0,u'_1,\ldots,u'_m\}$ be vertex set of G, where u'_0 is a vertex with degree m. Let $v_{ij}=(u_i,u'_j)$. We consider the following three cases separately.

Case 1 Suppose $n=2$. Then there exist the adjacent vertices of maximum degree in $P_2\times G$. By Lemma 1.2, $\chi_{ai}(P_2\times G)\geq \Delta(P_2\times G)+2=m+3$. We now only need to give an $(m+3)$-AVDIC of $P_2\times G$.

Let $C = \{a_1, a_2, a_3\} \cup \{0, 1, 2, \ldots, m-1\}$ be the set of colors such that $|C| = m+3$. We construct a mapping f from $I(P_2 \times G)$ to C as follows:

$$f(A_{v_{10}}) = a_1,\ f(A_{v_{1j}}) = j-1, \quad j = 1, 2, \ldots, m.$$

$$f(A_{v_{20}}) = a_2,\ f(A_{v_{2j}}) = j+1 \pmod m,\ f(A_{v_{2m}}) = a_3, \quad j = 1, 2, \ldots, m-1.$$

For convenience, we use the matrix $B = (b_{ij})_{2\times(m+1)}$ to describe the incidence coloring f of $P_2 \times G$,

$$B = \begin{pmatrix} a_1 & 0 & 1 & \cdots & m-3 & m-2 & m-1 \\ a_2 & 2 & 3 & \cdots & m-1 & 0 & a_3 \end{pmatrix}$$

where $b_{ij} = f(A_{v_{ij}})$ denotes the color which is received by the far-incidence set of vertex v_{ij}.

From the matrix $B = (b_{ij})_{2\times(m+1)}$, obviously, f is adjacent vertex distinguishing. Hence f is an $(m+3)$-AVDIC of $P_2 \times G$ and thus $\chi_{ai}(P_2 \times G) = m+3$.

Case 2 Suppose $n = 3$. By Lemma 1.3, $\chi_{ai}(P_3 \times G) \geq \Delta(P_3 \times G) + 1 = m+3$. We now only need to give an $(m+3)$-AVDIC of $P_3 \times G$.

Let $C = \{a_1, a_2, a_3\} \cup \{0, 1, 2, \ldots, m-1\}$ be a color set such that $|C| = m+3$. We construct a mapping f from $I(P_3 \times G)$ to C as follows:

$$f(A_{v_{i0}}) = a_i,\ f(A_{v_{ij}}) = (2i+j-3) \pmod m, \quad i = 1, 2, 3,\ j = 1, 2, \ldots, m.$$

It is easy to see that f is an incidence coloring of $P_3 \times G$. Similarly, we use the matrix $B = (b_{ij})_{3\times(m+1)}$ to describe the incidence coloring of $P_3 \times G$,

$$B = \begin{pmatrix} a_1 & 0 & 1 & \cdots & m-5 & m-4 & m-3 & m-2 & m-1 \\ a_2 & 2 & 3 & \cdots & m-3 & m-2 & m-1 & 0 & 1 \\ a_3 & 4 & 5 & \cdots & m-1 & 0 & 1 & 2 & 3 \end{pmatrix}$$

where $b_{ij} = f(A_{v_{ij}})$ denotes the color which is received by the far-incidence set of vertex v_{ij}.

From the matrix $B = (b_{ij})_{3\times(m+1)}$, obviously, $\overline{S}(v_{01}) = \{a_3\}$, $\overline{S}(v_{02}) = \emptyset$, $\overline{S}(v_{03}) = \{a_1\}$, and for any $j = 1, 2, \ldots, m$, $a_2 \notin S(v_{1j})$, $a_2 \in S(v_{2j})$, $a_3 \notin S(v_{2j})$, $a_3 \in S(v_{3j})$. Hence $S(v_{ij}) \neq S(v_{i+1,j})$ for any copy $P_3 \times \{u'_j\}$ of P_3, where $j = 0, 1, 2, \ldots, m$, $i = 1, 2$. On the other hand, for any copy $\{u_i\} \times G$ of G, $S(v_{i0}) \neq S(v_{ij})$, and $(j+1) \pmod m \in S(v_{1j})$, $(j+2) \pmod m \notin S(v_{1j})$, $(j+3) \pmod m \in S(v_{2j})$, $(j+4) \pmod m \notin S(v_{2j})$, $(j+1) \pmod m \in S(v_{3j})$, $(j+2) \pmod m \notin S(v_{3j})$, where $j = 1, 2, \ldots, m$. Hence for any pair of adjacent vertices v_{ij} and v_{ik} in $\{u_i\} \times G$, $S(v_{ij}) \neq S(v_{ik})$, where $i = 1, 2, 3$. Consequently f is adjacent vertex distinguishing and thus $\chi_{ai}(P_3 \times G) = m+3$.

Case 3 Suppose $n \geq 4$. Obviously, there exist the adjacent vertices of maximum degree in $P_n \times G$. By Lemma 1.2, $\chi_{ai}(P_n \times G) \geq \Delta(P_n \times G) + 2 = m+4$. We now only need to give an $(m+4)$-AVDIC of $P_n \times G$.

Let $C = \{a_0, a_1, a_2, a_3\} \cup \{0, 1, 2, \ldots, m-1\}$ be the set of colors such that $|C| = m+4$. We now construct a mapping f from $I(P_n \times G)$ to C as follows: for any $i = 1, 2, \ldots, n$ and

$j = 0, 1, \ldots, m$, let

$$f(A_{v_{i0}}) = a_{i-1},\ f(A_{v_{ij}}) = (2i + j - 3) \pmod m,$$

where the suffix of a_{i-1} is taken modulo 4.

It is easy to see that f is an incidence coloring of $P_n \times G$. Similarly, we use the matrix $B = (b_{ij})_{n\times(m+1)}$ to describe the incidence coloring of $P_n \times G$,

$$B = \begin{pmatrix} a_0 & 0 & 1 & \cdots & m-2 & m-1 \\ a_1 & 2 & 3 & \cdots & m & m+1 \\ a_2 & 4 & 5 & \cdots & m+2 & m+3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & 2n-2 & 2n-1 & \cdots & 2n+m-4 & 2n+m-3 \end{pmatrix}$$

where $b_{ij} = f(A_{v_{ij}})$ denotes the color which is received by the far-incidence set of vertex v_{ij} (here $b_{i0} = a_{i-1}$), and the suffix of a_{i-1} is taken modulo 4, b_{ij} are taken modulo m, where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

From the matrix $B = (b_{ij})_{n\times(m+1)}$, it is clear that $S(v_{ij}) \neq S(v_{i+1,j})$ for any copy $P_n \times \{u_j'\}$ of P_n, where $j = 0, 1, 2, \ldots, m$, $i = 1, 2, \ldots, n-1$. On the other hand, for any copy $\{u_i\} \times G$ of G, $S(v_{i0}) \neq S(v_{ij})$, and $(2i+j-1) \pmod m \in S(v_{ij})$, $(2i+j) \pmod m \notin S(v_{ij})$, $i = 1, 2, \ldots, n-1$, $(2n + j - 5) \pmod m \in S(v_{nj})$, $(2n + j - 4) \pmod m \notin S(v_{nj})$, where $j = 1, 2, \ldots, m$. Hence for any pair of adjacent vertices v_{ij} and v_{ik} in $\{u_i\} \times G$, $S(v_{ij}) \neq S(v_{ik})$, where $i = 1, 2, \ldots, n$. Consequently, f is adjacent vertex distinguishing and thus $\chi_{ai}(P_n \times G) = m + 4$.

The proof of this theorem is completed. □

References

[1] BONDY J A, MURTY U S R. *Graph Theory with Applications* [M]. American Elsevier Publishing Co., Inc., New York, 1976.

[2] CHEN Dongling, LIU Xikui. *The incidence chromatic number and the incidence coloring conjecture of graphs* [J]. J. Math. Econom., 1998, **15**(3): 47–51.

[3] Wang Yaqin. *The incidence coloring and adjacent vertex distingushing incidence coloring of graphs* [D]. Shandong University of Science and Technology, Shandong, 2007.

[4] Tian Shuangliang, Zhang Zhongfu. *A note of adjacent strong edge chromatic number of product graphs* [J]. J. Lanzhou Railw. Univ. Nat. Sci., 2005, **24**(3): 136–137.

[5] TIAN Shuangliang, CHEN Ping. *Adjacent vertex-distinguishing total coloring of two classes of product graphs* [J]. J. Math. Res. Exposition, 2007, **27**(4): 733–737. (in Chinese)

[6] TIAN Shuangliang, CHEN Ping. *On the vertex distinguishing edge coloring of some product graph* [J]. J. Shandong Univ. Nat. Sci. Ed., 2006, **41**(4): 53–56.