Journal of Mathematical Research & Exposition
May, 2011, Vol. 31, No. 3, pp. 393–401
DOI:10.3770/j.issn:1000-341X.2011.03.002
Http://jmre.dlut.edu.cn

On the Optimal Controller for LTV Measurement Feedback Control Problem

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Abstract In this paper, we consider the measurement feedback control problem for discrete linear time-varying systems within the framework of nest algebra consisting of causal and bounded linear operators. Based on the inner-outer factorization of operators, we reduce the control problem to a distance from a certain operator to a special subspace of a nest algebra and show the existence of the optimal LTV controller in two different ways: one via the characteristic of the subspace in question directly, the other via the duality theory. The latter also gives a new formula for computing the optimal cost.

Keywords LTV systems; nest algebra; control; optimal controller; duality.

Document code A
MR(2010) Subject Classification 93D99
Chinese Library Classification O231.9

1. Introduction

Analysis of control problems for linear time-varying (LTV) systems has received more and more considerable attention in recent years (take [1–4] for example), because one can often recover the results of the time-invariant problems as special cases of the more general time-varying problem [5, 6]. Here, the measurement feedback control, the problem which is at the heart of control theory [7], is considered in the time-varying setting. Many methods have been developed to solve this problem in other different cases such as [7–10] and references therein, but only a few papers concern with the time-varying case. The description of the measurement feedback control problem was first introduced in [11]. In [12], Feintuch and Markus used inner-outer factorization in respect of the nest algebra of lower triangular bounded linear operators to construct isometric and co-isometric operators and transferred this time-varying linear control problem into a 4-block problem of the type discussed in [13].

In this paper, we mainly concern with the existence of the LTV optimal controller for the time-varying measurement feedback control problem. On the basis of the factorization given by Feintuch and Markus, we reduce this control problem to a distance from a specific operator to a special subspace of a nest algebra and show the existence of the optimal controller in two

Received October 25, 2010; Accepted April 22, 2011
Supported by the National Natural Science Foundation of China (Grant No. 10971020).
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different ways. The first one is via the weak closeness of the operator subspace in question directly. An analogous method has ever been used in [3] and [14] to guarantee the attainment of the solution for the Nehari formula. The second one is via the Banach duality theory, a technique which was used by Zames and Owen in [15, 16] to characterize the solutions of some H^∞ control problems and generalized by Djouadi to the TV control problems defined in the context of nest algebra [1, 4, 17]. Our approach used here is inspired by the work mentioned above. We give the structure of a preannihilator for the special subspace consisting of 2×2 operator matrices we need so that the duality theory can be used to solve the TV measurement feedback control problem. Moreover, a new formula for computing the optimal cost is obtained.

The paper is organized as follows. In Section 2, we give some basic definitions and notations. The TV measurement feedback control problem is formulated in Section 3. In Section 4, the existence of the optimal controller is obtained in two ways. A new formula for the optimal performance is also given . In Section 5, we give the conclusion.

2. Preliminaries

We first recall some basic concepts and standard notations from [1, 13, 18] used in what follows.

If X, Y are Banach spaces, $\mathcal{B}(X, Y)$ denotes the space of all the bounded linear operators from X to Y. $A \in \mathcal{B}(X, Y)$ is endowed with the induced operator norm

$$\|A\| := \sup_{x \in X, \|x\| \leq 1} \|Ax\|.$$

A^* means the adjoint of operator A.

Definition 2.1 *A nest is a chain $\mathcal{N}$ of closed subspaces of a Hilbert space $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under intersection and closed span.*

Definition 2.2 *The nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$.*

In this paper, the Hilbert sequence space

$$\ell^2 = \{x = (x_0, x_1, x_2, \ldots) : x_i \in \mathbb{C}^n, \sum_{i=0}^{\infty} \|x_i\|^2 < +\infty\}$$

is chosen to be our input-output space. The corresponding nest is

$$\mathcal{N} = \{Q_n \ell^2 = (I - P_n)\ell^2 : n = -1, 0, 1, \ldots\},$$

where P_n $(n \geq 0)$ is the standard truncation projection on ℓ^2 defined as follows:

$$P_n(x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) = (x_0, x_1, \ldots, x_n, 0, \ldots).$$

Let $P_{-1} = 0$ and $P_\infty = I$.

Definition 2.3 *An operator $A \in \mathcal{B}(\ell^2, \ell^2)$ is called causal if it satisfies*

$$P_n A = P_n A P_n, \quad \forall n \geq 0.$$

All the causal bounded linear operators on ℓ^2 are denoted by $\mathcal{B}_c(\ell^2, \ell^2)$ (see [1]). Obviously, $\mathcal{B}_c(\ell^2, \ell^2)$ is the required nest algebra $\mathcal{T}(\mathcal{N})$ in this paper.

Here we also need some notions and results about Banach dual space [4, 19–21].

Definition 2.4 *The collection of all bounded linear functionals on Banach space X, denoted by X^*, is called the dual space of X. X_* is said to be the predual space of X if*

$$(X_*)^* \simeq X,$$

where $\simeq$ denotes isometric isomorphism between Banach spaces.

Definition 2.5 *For a subspace M of X, the annihilator of M in X^* is defined by*

$$M^{\perp} := \{\phi \in X^* : \phi(x) = 0, \forall x \in M\}.$$

A subspace ${}^{\perp}M$ of X_* is called the preannihilator of M if $({}^{\perp}M)^{\perp} \simeq M$.

When a predual and preannihilator exist, we have the following standard result of Banach space duality theory [1, 19].

Proposition 2.1 *X is a Banach space with its predual space X_*. ${}^{\perp}M \subset X_*$ is the preannihilator of the subspace $M \subset X$. x is an element of X. Then we have*

$$\operatorname{dist}(x, M) = \inf_{m \in M} \|x - m\| = \min_{m \in M} \|x - m\| = \sup_{y \in {}^{\perp}M, \|y\| \leq 1} |\langle x, y \rangle|,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product.

The last proposition will be crucial for us to solve the shortest distance problem raised in the LTV measurement feedback control.

3. Problem formulation

Consider the feedback arrangement shown in Figure 1,

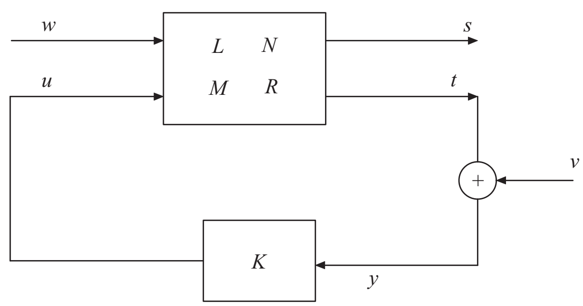

Figure 1 LTV measurement feedback control configuration

where L, M, N, R are known operators in $\mathcal{B}_c(\ell^2, \ell^2)$ satisfying the equation

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} L & N \\ M & R \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

All the signals are from ℓ^2 (that is, they are of finite energy), where w can be considered as the process noise, v the measurement noise that corrupts the output signal t, u the control output used to influence the dynamic behavior of the plant and s the regulated output.

The original LTV measurement feedback control problem stated in [11, 12] is to find

$$\gamma_{opt} = \inf\{\|T_K\| : \ K \text{ being stabilizing LTV controller}\}$$

with

$$T_K = \begin{bmatrix} L + NK(I-RK)^{-1}M & NK(I-RK)^{-1} \\ K(I-RK)^{-1}M & K(I-RK)^{-1} \end{bmatrix},$$

the transfer matrix from $\begin{bmatrix} w \\ v \end{bmatrix}$ to $\begin{bmatrix} s \\ u \end{bmatrix}$. By using the Youla parametrization theorem in [13], we can relate the Youla parameter $Q := K(I-RK)^{-1}$ to the controller K uniquely. Then the above problem is equivalent to finding

$$\gamma_{opt} = \inf\{\|T_Q\| : Q \in \mathcal{B}_\rfloor(\ell^\in, \ell^\in)\},$$

where

$$T_Q = \begin{bmatrix} L + NQM & NQ \\ QM & Q \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N \\ I \end{bmatrix} Q \begin{bmatrix} M & I \end{bmatrix}.$$

The following lemmas state the special inner-outer factorizations first used by Feintuch (see Lemmas 1 and 5, Theorems 2 and 5 in [12]). They are also of importance for us to reduce the TV control problem to a special distance formula.

Lemma 3.1 *For $M \in \mathcal{B}_c(\ell^2, \ell^2)$, there exist $B, W_1, W_2 \in \mathcal{B}_c(\ell^2, \ell^2)$ with the following properties:*

(1) $W_1W_1^* + W_2W_2^* = I$;

(2) For each n, $B^*P_n\ell^2 = \operatorname{Ran} B^* \cap P_n\ell^2$;

(3) If R_{B^*} is the orthogonal projection on $\operatorname{Ran} B^*$, then for each n, $R_{B^*}P_n = P_nR_{B^*}$;

(4) $\begin{bmatrix} M & I \end{bmatrix} = B \begin{bmatrix} W_1 & W_2 \end{bmatrix}$;

(5) B is invertible in $\mathcal{B}_c(\ell^2, \ell^2)$.

The dual result to Lemma 3.1 is the following.

Lemma 3.2 *For any $N \in \mathcal{B}_c(\ell^2, \ell^2)$, there exist $A, V_1, V_2 \in \mathcal{B}_c(\ell^2, \ell^2)$ satisfying:*

(1) $V_1^*V_1 + V_2^*V_2 = I$;

(2) $AQ_n\ell^2 = \operatorname{Ran} A \cap Q_n\ell^2$, $\forall n$;

(3) $R_AQ_n = Q_nR_A$, *for each n;*

(4) $\begin{bmatrix} N \\ I \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} A$;

(5) A is invertible in $\mathcal{B}_c(\ell^2, \ell^2)$.

Thus,

$$T_Q = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} N \\ I \end{bmatrix} Q \begin{bmatrix} M & I \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} AQB \begin{bmatrix} W_1 & W_2 \end{bmatrix}.$$

The invertibility of operators A, B guarantees the bijection of the map

$$\mathcal{B}_c(\ell^2, \ell^2) \longrightarrow A\mathcal{B}_c(\ell^2, \ell^2)B.$$

By absorbing the invertible operators into the free parameter Q, we can get

$$\begin{aligned}\gamma_{opt} &= \inf\{\|T_Q\| : Q \in \mathcal{B}_\rfloor(\ell^\in, \ell^\in)\} \\ &= \inf\{\| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q \begin{bmatrix} W_1 & W_2 \end{bmatrix} \| : Q \in \mathcal{B}_c(\ell^2, \ell^2)\}.\end{aligned}$$

We transfer the measurement feedback control problem to a distance between the operator $\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$ and a special subspace $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ of the nest algebra $\mathcal{B}_c(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$ determined by the projection $\begin{bmatrix} Q_n & 0 \\ 0 & Q_n \end{bmatrix}$, that is,

$$\gamma_{opt} = \mathrm{dist}(\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}).$$

In next section, we study the distance minimization problem and the existence of the solutions in the context of the operator algebra.

4. The existence of the LTV optimal controller

This section is divided into two parts with each part showing an approach to obtain the existence of the optimal solution for the distance formula.

4.1 The weak closeness of the special subspace

In this subsection, we characterize the property of the subspace appearing in the distance formula in Section 3. It is the property that ensures the attainment of an optimal controller.

We begin with a fundamental lemma from [18].

Lemma 4.1 *$\mathcal{T}(\mathcal{N})$ is a weak operator closed subalgebra of $\mathcal{B}(\mathcal{H})$.*

The following lemma states the required property. The main idea of the proof is from [14].

Lemma 4.2 *The subspace $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ is a weak operator closed subalgebra of $\mathcal{B}(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$.*

Proof Suppose $\{X_\alpha\}$ is a net in $\mathcal{B}_c(\ell^2, \ell^2)$ such that $\{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_\alpha \begin{bmatrix} W_1 & W_2 \end{bmatrix}\}$ converges weakly to X, then we aim at showing that the limit X belongs to $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Obviously, we can get the fact that $X_\alpha \to \begin{bmatrix} V_1^* & V_1^* \end{bmatrix} X \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix}$ weakly from the definition and the property of $\begin{bmatrix} W_1 & W_2 \end{bmatrix}$, $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ shown in Lemmas 3.1 and 3.2. By Lemma 4.1, there is a Y in $\mathcal{B}_c(\ell^2, \ell^2)$ such that

$$Y = \begin{bmatrix} V_1^* & V_1^* \end{bmatrix} X \begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix}.$$

An other argument as above shows $\{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_\alpha \begin{bmatrix} W_1 & W_2 \end{bmatrix}\} \to \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Y \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ weakly.

Therefore, we get the desired result that

$$X = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Y \begin{bmatrix} W_1 & W_2 \end{bmatrix}$$

is in $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. We now apply this lemma to show that the distance defined in Section 3 can be achieved.

Theorem 4.1 *There exists a Q_0 in $\mathcal{B}_c(\ell^2, \ell^2)$ such that*

$$\begin{aligned} \gamma_{opt} &= \inf\{\| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q \begin{bmatrix} W_1 & W_2 \end{bmatrix} \| : Q \in \mathcal{B}_c(\ell^2, \ell^2)\} \\ &= \| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q_0 \begin{bmatrix} W_1 & W_2 \end{bmatrix} \|. \end{aligned}$$

Proof As shown in Section 3,

$$\gamma_{opt} = \mathrm{dist}(\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}).$$

Then there exists a sequence $\{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_n \begin{bmatrix} W_1 & W_2 \end{bmatrix}\}$ in $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ such that

$$\lim_{n\to\infty} \| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_n \begin{bmatrix} W_1 & W_2 \end{bmatrix} \| = \gamma_{opt}.$$

It follows that $\{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_n \begin{bmatrix} W_1 & W_2 \end{bmatrix}\}$ is bounded and therefore weakly compact. Thus, there exists a subsequence $\{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} X_{n_k} \begin{bmatrix} W_1 & W_2 \end{bmatrix}\}$ converging weakly to $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q_0 \begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ for some Q_0 in $\mathcal{B}_c(\ell^2, \ell^2)$. Noticing that the weak operator topology closure of $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ equals the strong operator topology closure of $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ (see Corollary 8.2 in [22]), together with the definition of γ_{opt}

and induced operator norm, we can get

$$\gamma_{opt} = \| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q_0 \begin{bmatrix} W_1 & W_2 \end{bmatrix} \|,$$

showing the attaintment of the infimum.

Remark 4.1 Let $\tilde{Q}_0 = A^{-1} Q_0 B^{-1}$, where A, B, Q_0 are defined in Lemmas 3.1, 3.2 and Theorem 4.1, respectively. Since the Youla parameter Q and the controller K determine each other uniquely under the relationship $Q = K(I - RK)^{-1}$, we can define

$$K_0 = (I + \tilde{Q}_0 R)^{-1} \tilde{Q}_0 = (A + Q_0 B^{-1} R)^{-1} Q_0 B^{-1}.$$

It is obvious that K_0 is one of the optimal LTV controllers for the TV control problem discussed.

4.2 Duality theory for the TV control problem

For the shortest distance problem we deal with, we endow the notations appearing in Proposition 2.1 with

$$X = \mathcal{B}(\ell^2 \times \ell^2, \ell^2 \times \ell^2), \quad x = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \in X,$$

$$M = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \mathcal{B}_c(\ell^2, \ell^2) \begin{bmatrix} W_1 & W_2 \end{bmatrix} \subset X.$$

It is crucial for us to establish the existence of a predual and determine the form of the preannihilator of subspace M.

Let $\mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$ be the space of trace class or Schatten 1-class operators acting on the Hilbert space $\ell^2 \times \ell^2$. We can then identify $\mathcal{B}(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$ with the dual space of $\mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$ under the bounded linear functionals defined as follows: for $A \in \mathcal{B}(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$,

$$\phi_A(T) = \mathrm{tr}(A^* T), \ \forall T \in \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2).$$

So, we have $X_* = \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$.

The preannihilator of $\mathcal{B}_c(\ell^2, \ell^2)$ is given by [17]

$$\mathcal{S} := \{T \in \mathcal{C}_1(\ell^2, \ell^2) : (I - Q_{n+1}) T Q_n = 0, \text{ for all } n\}.$$

The following lemma describes the form of the preannihilator of M.

Lemma 4.3 *Let $R_1 \in \mathcal{B}(\ell^2, \ell^2 \times \ell^2)$ be an isometry and $R_2 \in \mathcal{B}(\ell^2 \times \ell^2, \ell^2)$ be a co-isometry, i.e., $R_1^* R_1 = I$, $R_2 R_2^* = I$. Then*

$$\begin{aligned} {}^{\perp}(R_1 \mathcal{B}_c(\ell^2, \ell^2) R_2) &= (I - R_1 R_1^*) \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2) + \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)(I - R_2^* R_2) + R_1 \mathcal{S} R_2 \\ &:= \mathcal{S}_0 \subset \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2). \end{aligned}$$

Proof Due to the definition of the preannihilator, it suffices to show $\mathcal{S}_0^{\perp} = R_1 \mathcal{B}_c(\ell^2, \ell^2) R_2$. For any $B \in \mathcal{S}_0^{\perp}$,

$$\begin{aligned} &\phi_B((I - R_1 R_1^*) T_1 + T_2 (I - R_2^* R_2) + R_1 S R_2) \\ &= \mathrm{tr}(B^* (I - R_1 R_1^*) T_1) + \mathrm{tr}(B^* T_2 (I - R_2^* R_2)) + \mathrm{tr}(B^* R_1 S R_2) = 0, \end{aligned}$$

$\forall T_1, T_2 \in \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$, $\forall S \in \mathcal{S}$. Because of the relations $\mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)^{\perp} = 0$ and ${}^{\perp}\mathcal{B}_c(\ell^2, \ell^2) = \mathcal{S}$, we have

$$B^*(I - R_1 R_1^*) = 0, \ (I - R_2^* R_2)B^* = 0, \ R_2 B^* R_1 = A^* \in \mathcal{B}_c^*(\ell^2, \ell^2),$$

which implies

$$B^* = R_2^* R_2 B^* R_1 R_1^* = R_2^* A^* R_1^*.$$

By taking adjoints, we have

$$B = R_1 A R_2 \in R_1 \mathcal{B}_c(\ell^2, \ell^2) R_2.$$

On the other hand, if $B_1 = R_1 A_1 R_2 \in R_1 \mathcal{B}_c(\ell^2, \ell^2) R_2$ for some $A_1 \in \mathcal{B}_c(\ell^2, \ell^2)$,

$$\begin{aligned}
&\phi_{B_1}((I - R_1 R_1^*)T_1 + T_2(I - R_2^* R_2) + R_1 S R_2) \\
&\quad = \operatorname{tr}(R_2^* A_1^* R_1^* (I - R_1 R_1^*)T_1) + \operatorname{tr}((I - R_2^* R_2) R_2^* A_1^* R_1^* T_2) + \operatorname{tr}(R_2 R_2^* A_1^* R_1^* R_1 S) \\
&\quad = \phi_{A_1}(S) = 0,
\end{aligned}$$

for any $T_1, T_2 \in \mathcal{C}_1(\ell^2 \times \ell^2, \ell^2 \times \ell^2)$, $S \in \mathcal{S}$. Therefore, $B_1 \in \mathcal{S}_0^{\perp}$. The proof is completed. □

In our case, $R_1 = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, R_2 = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, the preannihilator of M can be computed directly from Lemma 4.3 denoted by $\mathcal{S}_0$ as well.

As an application of Proposition 2.1, we give the following theorem to solve the optimal LTV control problem and show the existence of the optimal $Q_0 \in \mathcal{B}_c(\ell^2, \ell^2)$.

Theorem 4.2 *For the TV measurement feedback control problem,*

$$\begin{aligned}
\gamma_{opt} &= \min_{Q \in \mathcal{B}_\rceil(\ell^\in, \ell^\in)} \left\| \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} Q \begin{bmatrix} W_1 & W_2 \end{bmatrix} \right\| \\
&= \sup_{Y \in \mathcal{S}_0, \|Y\| \le 1} \left| \left\langle \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}, Y \right\rangle \right| \\
&= \sup_{Y \in \mathcal{S}_0, \|Y\| \le 1} \left| \operatorname{tr}\left(\begin{pmatrix} L^* & 0 \\ 0 & 0 \end{pmatrix} Y \right) \right|,
\end{aligned}$$

and there exists at least one optimal $Q_0 \in \mathcal{B}_c(\ell^2, \ell^2)$ achieving the minimization in the above formula.

The proof of the theorem follows directly by taking the specific parameters in Proposition 2.1.

Remark 4.2 An optimal controller with the similar form as shown in Remark 4.1 can also be constructed by the Youla parameter Q_0 in Theorem 4.2.

5. Conclusions

In this paper, we study the LTV measurement feedback control problem in the context of nest algebras. Two methods are proposed to solve the existence of the LTV optimal controller, where duality theory is a more systemic approach. Moreover, the results about duality theory given here can also be used to solve other TV control problems of the similar form, such as the model-matching problem proposed in [13].

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