Journal of Mathematical Research & Exposition
May, 2011, Vol. 31, No. 3, pp. 411–419
DOI:10.3770/j.issn:1000-341X.2011.03.004
Http://jmre.dlut.edu.cn

Lower Semicontinuity of the Solution Sets to Parametric Generalized Vector Equilibrium Problems

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Abstract In this paper, by a scalarization method, the lower semicontinuity of the solution mappings to two kinds of parametric generalized vector equilibrium problems involving set-valued mappings is established under new assumptions which are weaker than the C-strict monotonicity. These results extend the corresponding ones. Some examples are given to illustrate our results.

Keywords parametric generalized vector equilibrium problems; lower semicontinuity; scalarization.

Document code A
MR(2010) Subject Classification 90C46; 90C31
Chinese Library Classification O224

1. Introduction

It is well known that the stability analysis of solution mappings for vector equilibrium problems is an important topic in vector optimization theory. Recently, the semicontinuity, especially the lower semicontinuity of the solution mappings for parametric vector equilibrium problems with the parameter perturbed in the space of parameters has been intensively studied in the literature, such as [1, 2, 5, 6, 12–14].

Among those papers, the scalarization technique plays an important role in dealing with the lower semicontinuity of solution mappings to parametric vector variational inequalities and parametric vector equilibrium problems. In [7], by a scalarization method, Cheng and Zhu first obtained a result on the lower semicontinuity of solution mappings to a finite-dimensional parametric weak vector variational inequality. In [12], by virtue of a density result and scalarization technique, Gong and Yao discussed the lower semicontinuity of efficient solutions for parametric vector equilibrium problems, which are called generalized systems in their papers. By using the ideas of [7], Gong [13] studied the upper and lower semicontinuity of the solution set mappings to a parametric weak vector equilibrium problems. In [6], by using a new proof which is different from the ones of [12, 13], Chen et al. discussed the lower semicontinuity and continuity of solution mappings to a parametric generalized vector equilibrium problems involving set-valued mappings. In [18], Li et al. obtained the sufficient conditions for the lower semicontinuity and

Received December 20, 2009; Accepted November 20, 2010
Supported by the Fundamental Research Funds for the Central Universities (Grant No. CDJXS10100008).
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continuity of solution mappings to a parametric generalized weak vector equilibrium problem with set-valued mappings. Note that, in [6, 7, 12, 13, 18], f-efficient solution set (see Definitions 2.1 and 2.2) is confined to be a singleton because of the assumption of C-strict monotonicity. In [17], Li and Fang established the lower semicontinuity of the weak efficient solution mappings and globally efficient solution mappings to parametric vector equilibrium problems by a scalarization method under an assumption which is weaker than the assumption of C-strict monotonicity. Under their assumption (see condition (iii) of Lemma 3.1), the f-efficient solution set is a general set, which improved the foregoing results.

Motivated by [6, 17, 18], this paper aims to investigate the lower semicontinuity of the solution set mappings to two kinds of parametric generalized vector equilibrium problems involving set-valued mappings by using the ideas of [17]. These models are different from the ones considered in [17]. Under our assumptions, the f-efficient solution set may be a set, but not a singleton. We will give some examples to illustrate that our results extend the corresponding ones in [6, 17, 18].

The rest of the paper is organized as follows. In Section 2, we introduce two parametric generalized vector equilibrium problems, and recall some notions. In Sections 3 and 4, we discuss the lower continuity of solution mappings to parametric generalized vector equilibrium problem and parametric generalized weak vector equilibrium problem, respectively. In Section 5, we will give a conclusion.

2. Preliminaries

Throughout this paper, let X, Y and Z be Banach spaces. Assume that C is a pointed closed convex cone in Y with nonempty interior $\operatorname{int} C$. Let Y^* be the topological dual space of Y and $C^* := \{f \in Y^* | f(y) \geq 0, \forall y \in C\}$ be the dual cone of C.

Let A be a nonempty subset of X and $F : A \times A \to 2^Y \backslash \{\emptyset\}$ be a set-valued mapping. We consider the following generalized vector equilibrium problem (GVEP)

$$\text{Find } x \in A \text{ such that } F(x, y) \subset Y \backslash -\operatorname{int} C, \quad \forall y \in A,$$

and generalized weak vector equilibrium problem (GWVEP)

$$\text{Find } x \in A \text{ such that } F(x, y) \cap (Y \backslash -\operatorname{int} C) \neq \emptyset, \quad \forall y \in A.$$

When the subset A and the function F are perturbed by a parameter μ which varies over a subset Λ of Z, we consider the following parametric generalized vector equilibrium problem

$$\text{(PGVEP)} \quad \text{Find } x \in A(\mu) \text{ such that } F(x, y, \mu) \subset Y \backslash -\operatorname{int} C, \quad \forall y \in A(\mu),$$

and parametric generalized weak vector equilibrium problem

$$\text{(PGWVEP)} \quad \text{Find } x \in A(\mu) \text{ such that } F(x, y, \mu) \cap (Y \backslash -\operatorname{int} C) \neq \emptyset, \quad \forall y \in A(\mu),$$

where $A : \Lambda \to 2^X \backslash \{\emptyset\}$ is a set-valued mapping, B is a nonempty subset of X, $F : B \times B \times \Lambda \subset X \times X \times Z \to 2^Y \backslash \{\emptyset\}$ is a set-valued mapping with $A(\Lambda) = \cup_{\mu \in \Lambda} A(\mu) \subset B$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the solution set of (PGVEP), i.e.,

$$S(\mu) = \{x \in A(\mu) | F(x, y, \mu) \subset Y \backslash - \operatorname{int} C, \quad \forall y \in A(\mu)\},$$

and $S_w(\mu)$ denote the solution set of (PGWVEP), i.e.,

$$S_w(\mu) = \{x \in A(\mu) | F(x, y, \mu) \cap (Y \backslash - \operatorname{int} C) \neq \emptyset, \quad \forall y \in A(\mu)\}.$$

Throughout this paper, we always assume $S(\mu) \neq \emptyset$ and $S_w(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. This paper aims to investigate the lower semicontinuity of the solution set maps $S(\cdot)$ and $S_w(\cdot)$.

Now we recall some basic definitions and the properties.

Definition 2.1 ([6]) *Let $f \in C^* \backslash \{0\}$. A vector $x \in A(\mu)$ is called an f-efficient solution to the (PGVEP) if*

$$\inf_{z \in F(x,y,\mu)} f(z) \geq 0, \quad \forall y \in A(\mu).$$

The set of the f-efficient solution to the (PGVEP) is denoted by $S_f(\mu)$.

Now we give the scalarization results for $S(\mu)$ and $S_w(\mu)$.

Definition 2.2 ([18]) *Let $f \in C^* \backslash \{0\}$. A vector $x \in A(\mu)$ is called an f-efficient solution to the (PGWVEP) if*

$$\exists z \in F(x, y, \mu), \text{ s.t. } f(z) \geq 0, \quad \forall y \in A(\mu).$$

The set of the f-efficient solution to the (PGWVEP) is denoted by $\bar{S}_f(\mu)$.

Lemma 2.1 ([6]) *For each $\mu \in \Lambda$, if for each $x \in A(\mu)$, $F(x, A(\mu), \mu) + C$ is a convex set, then $S(\mu) = \bigcup_{f \in C^* \backslash \{0\}} S_f(\mu)$.*

Lemma 2.2 ([18]) *For each $\mu \in \Lambda$, if for each $x \in S_w(\mu)$ and $y \in A(\mu)$, there exists a selection $z(y)$ of $F(x, y, \mu) \backslash - \operatorname{int} C$ (i.e., $z(y) \in F(x, y, \mu) \backslash - \operatorname{int} C$), such that $\bigcup_{y \in A(\mu)} z(y) + C$ is a convex set, then $S_w(\mu) = \bigcup_{f \in C^* \backslash \{0\}} \bar{S}_f(\mu)$.*

Let $F : \Lambda \to 2^X$ be a set-valued mapping, and given $\bar{\lambda} \in \Lambda$.The notion $B(\bar{\lambda}, \delta)$ denotes the open ball with center $\bar{\lambda} \in \Lambda$ and radius $\delta > 0$.

Definition 2.3 ([15])

(i) F is called lower semicontinuous (l.s.c) at $\bar{\lambda}$ if for any open set V satisfying $V \cap F(\bar{\lambda}) \neq \emptyset$, there exists $\delta > 0$ such that for every $\lambda \in B(\bar{\lambda}, \delta)$, $V \cap F(\lambda) \neq \emptyset$.

(ii) F is called upper semicontinuous (u.s.c) at $\bar{\lambda}$ if for any open set V satisfying $F(\bar{\lambda}) \subset V$, there exists $\delta > 0$ such that for every $\lambda \in B(\bar{\lambda}, \delta)$, $F(\lambda) \subset V$.

We say F is l.s.c (resp.u.s.c) on Λ, if it is l.s.c (resp.u.s.c) at each $\lambda \in \Lambda$. F is said to be continuous on Λ if it is both l.s.c and u.s.c on Λ.

Proposition 2.1 ([3, 8])

(i) F is l.s.c at $\bar{\lambda}$ if and only if for any sequence $\lambda_n \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and any $\bar{x} \in F(\bar{\lambda})$, there exists $x_n \in F(\lambda_n)$ such that $x_n \to \bar{x}$.

(ii) If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c at $\bar{\lambda}$ if and only if for any sequence $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \to \bar{\lambda}$ and for any $x_n \in F(\lambda_n)$, there exist $\bar{x} \in F(\bar{\lambda})$ and a subsequence $\{x_{n_k}\}$ of x_n, such that $x_{n_k} \to \bar{x}$.

The following lemma plays an important role in the proof of lower semicontinuity of the solution set mappings $S(\mu)$ and $S_w(\mu)$.

Lemma 2.3 ([4]) *The union $\Gamma = \bigcup_{i \in I} \Gamma_i$ of a family of l.s.c set-valued mappings Γ_i from a topological space X into a topological space Y is also an l.s.c set-valued mapping from X into Y, where I is an index set.*

3. Lower semicontinuity of solution map for (PGVEP)

In this section, we establish the lower semicontinuity of the solution set mapping for (PGVEP). The notion $B(0, d(x, y))$ denotes the open ball with center 0 and radius $d(x, y) > 0$, $d(x, y) = \|x - y\|$.

Lemma 3.1 *Let $f \in C^* \backslash \{0\}$. Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ is continuous with compact convex values on Λ;

(ii) F is u.s.c with nonempty compact on $B \times B \times \Lambda$;

(iii) For each $\mu \in \Lambda$, $x \in A(\mu) \backslash S_f(\mu)$, there exists $y \in S_f(\mu)$ such that

$$F(x, y, \mu) + F(y, x, \mu) + B(0, d(x, y)) \subset -C.$$

Then, $S_f(\cdot)$ is l.s.c on Λ.

Proof Suppose to the contrary that there exists $\mu_0 \in \Lambda$ such that $S_f(\cdot)$ is not l.s.c at μ_0. Then, there exist a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in S_f(\mu_0)$ such that for any $x_n \in S_f(\mu_n)$, $x_n \not\to x_0$.

From $x_0 \in S_f(\mu_0)$, we have $x_0 \in A(\mu_0)$. Since $A(\cdot)$ is l.s.c at μ_0, there exists a sequence $\bar{x}_n \in A(\mu_n)$ such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A(\mu_n) \backslash S_f(\mu_n)$. Then, by (iii), there exists $y_n \in S_f(\mu_n)$ such that

$$F(\bar{x}_n, y_n, \mu_n) + F(y_n, \bar{x}_n, \mu_n) + B(0, d(\bar{x}_n, y_n)) \subset -C. \tag{1}$$

Since $y_n \in S_f(\mu_n)$ implies $y_n \in A(\mu_n)$, it follows from the upper semicontinuity and compactness of $A(\cdot)$ at μ_0 that there exist $y_0 \in A(\mu_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$. Particularly, for (1), we have

$$F(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k}) + F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k}) + B(0, d(\bar{x}_{n_k}, y_{n_k})) \subset -C. \tag{2}$$

Then there exist $z^1_{n_k} \in F(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k})$ and $z^2_{n_k} \in F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k})$ such that

$$z^1_{n_k} + z^2_{n_k} + B(0, d(\bar{x}_{n_k}, y_{n_k})) \subset -C. \tag{3}$$

Since $F(\cdot, \cdot, \cdot)$ is u.s.c with compact values, there exist $z_{0_1} \in F(x_0, y_0, \mu_0)$ and $z_{0_2} \in F(y_0, x_0, \mu_0)$ such that $z^1_{n_k} \to z_{0_1}, z^2_{n_k} \to z_{0_2}$. By (3), we have

$$z_{0_1} + z_{0_2} + B(0, d(x_0, y_0)) \subset -C. \tag{4}$$

If $x_0 \neq y_0$, by (4), we have $z_{0_1} + z_{0_2} \in -\operatorname{int} C$. Thus,

$$f(z_{0_1} + z_{0_2}) < 0. \tag{5}$$

Noting that $x_0 \in S_f(\mu_0)$ and $y_0 \in A(\mu_0)$, we get $\inf_{z \in F(x_0, y_0, \mu_0)} f(z) \geq 0$. Particularly, we have

$$f(z_{0_1}) \geq 0. \tag{6}$$

On the other hand, since $y_{n_k} \in S_f(\mu_{n_k})$ and $\bar{x}_{n_k} \in A(\mu_{n_k})$, we have $\inf_{z \in F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k})} f(z) \geq 0$. Also, we have $f(z^2_{n_k}) \geq 0$. It follows from the continuity of f that we have

$$f(z_{0_2}) \geq 0. \tag{7}$$

By (6), (7) and the linearity of f, we get

$$f(z_{0_1} + z_{0_2}) \geq 0, \tag{8}$$

which contradicts (5). Therefore, we have $y_0 = x_0$. This is impossible by the contradiction assumption. Thus, our result holds and the proof is completed. □

Remark 3.1 (i) When F is a vector-valued mapping, the (PGVEP) reduces to $(\text{VEP})_\mu$ considered in [17]. Then, Lemma 3.1 reduces to Lemma 3.1 in [17].

(ii) In [5], Chen et al. obtained the continuity of the f-efficient solution set of (PGVEP) by virtue of C-strict monotonicity. But this condition is so strict that the f-efficient solution set is confined to be a singleton. In our paper, by using the ideas in [17], we introduce the assumption (iii) of Lemma 3.1, which abates the condition of C-strict monotonicity. In the case, the f-efficient solution set of (PGVEP) may be a set, but not a singleton. We also obtain the lower semicontinuity of the f-efficient solution set. Now we give the following example to show that the f-efficient solution set is a set.

Example 3.1 Let $X = R$, $Y = R^2$, $C = R^2_+$, $\Lambda = [1, 2]$, $A(\mu) = [-1, 1]$. For each $\mu \in \Lambda$ and $x \in A(\mu)$, $\forall y \in A(\mu)$, let

$$F(x, y, \mu) = \{(a, b) \in R^2 | (a, b) = (1 - t)(-1, \mu x) + t(-2, 2\mu x), t \in [0, 1]\}.$$

For any given $\mu \in \Lambda$, let $f((x, y)) = \frac{1}{\mu} y \in C^* \backslash \{0\}$. It follows from a direct computation that $S_f(\mu) = [0, 1]$. It is obvious that the set of f-efficient solution to (PGVEP) is not a singleton but a general set. The assumption (iii) in Lemma 3.1 can be checked as follows: For any $x \in A(\mu) \backslash S_f(\mu) = [-1, 0)$, there exists $y = 0 \in S_f(\mu) = [0, 1]$ such that

$$\begin{aligned} F(x, y, \mu) + F(y, x, \mu) + B(0, d(x, y)) &= (t + 1)(-2, \mu x) + B(0, d(x, y)) \\ &= (t + 1)(-2, \mu x) + B(0, d(x, 0)) \subset -C. \end{aligned}$$

However, the assumption of C-strict monotonicity in Lemma 3.2 of [6] is violated. Indeed, for $\forall x \in A(\mu) \backslash S_f(\mu) = [-1, 0)$, there exists $y = -x \in S_f(\mu) = [0, 1]$ such that

$$F(x, y, \mu) + F(y, x, \mu) = (t + 1)(-2, 0) \in -\partial C \backslash \{0\}$$

where ∂C is the boundary of C. Obviously, $F(x,y,\mu)+F(y,x,\mu)\notin -\mathrm{int}\,C$, which implies that $F(\cdot,\cdot,\mu)$ is not C-strictly monotone on $A(\mu)\times A(\mu)$.

Theorem 3.1 *For each $f\in C^*\backslash\{0\}$, suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ is continuous with compact convex values on Λ;

(ii) F is u.s.c with nonempty compact on $B\times B\times\Lambda$;

(iii) For each $\mu\in\Lambda$ and $x\in A(\mu)$, $F(x,\cdot,\mu)$ is C-convexlike on $A(\mu)$, i.e., for any $x_1,x_2\in A(\mu)$ and any $\rho\in[0,1]$, there exists $x_3\in A(\mu)$ such that $\rho F(x,x_1,\mu)+(1-\rho)F(x,x_2,\mu)\subset F(x,x_3,\mu)+C$;

(iv) For each $\mu\in\Lambda$, $x\in A(\mu)\backslash S_f(\mu)$, there exists $y\in S_f(\mu)$ such that

$$F(x,y,\mu)+F(y,x,\mu)+B(0,d(x,y))\subset -C.$$

Then, $S(\cdot)$ is l.s.c on Λ.

Proof For each $\mu\in\Lambda$ and $x\in A(\mu)$, since $F(x,\cdot,\mu)$ is C-convexlike on $A(\mu)$, $F(x,A(\mu),\mu)+C$ is a convex set. Then, it follows from Lemma 2.1 that for each $\mu\in\Lambda$,

$$S(\mu)=\bigcup_{f\in C^*\backslash\{0\}} S_f(\mu).$$

By Lemma 3.1, for each $f\in C^*\backslash\{0\}$, $S_f(\cdot)$ is l.s.c on Λ. Thus, in view of Lemma 2.3, we have $S(\cdot)$ is l.s.c on Λ. The proof is completed. □

Now, we give an example to illustrate that our result is different from that of [6].

Example 3.2 Let $X=R$, $Y=R^2$, $C=R^2_+$, $\Lambda=[1,2]$, $A(\mu)=[-1,0]$ and $F(x,y,\mu)=\{(a,b)\in R^2|(a,b)=(1-t)(-1,\mu x)+t(-2,2\mu x),t\in[0,1]\}$.

For any $f\in C^*\backslash\{0\}$, it follows from a direct computation that if $S_f(\mu)\neq\emptyset$, $0\in S_f(\mu)$. It is clear that conditions (i)–(iii) of Theorem 3.1 are satisfied. For any $x\in A(\mu)\backslash S_f(\mu)$, there exists $y=0\in S_f(\mu)$ such that

$$\begin{aligned}F(x,y,\mu)+F(y,x,\mu)+B(0,d(x,y))&=(t+1)(-2,\mu x)+B(0,d(x,y))\\&=(t+1)(-2,\mu x)+B(0,d(x,0))\subset -C.\end{aligned}$$

Thus, the condition (iv) of Theorem 3.1 is also satisfied. By Theorem 3.1, $S(\cdot)$ is lower semicontinuous on Λ.

However, for $\bar{x}=\bar{y}=0$,

$$F(\bar{x},\bar{y},\mu)+F(\bar{y},\bar{x},\mu)=(-2,0)\in -\partial C\backslash\{0\},$$

where ∂C is the boundary of C. Obviously, $F(\bar{x},\bar{y},\mu)+F(\bar{y},\bar{x},\mu)\not\subset -intC$, i.e., $F(\cdot,\cdot,\mu)$ is not C-strictly monotone on $A(\mu)\times A(\mu)$. Thus, Theorem 3.1 of [6] is not applicable.

4. Lower semicontinuity of solution map for (PGWVEP)

In this section, we establish the lower semicontinuity of the solution set mapping for (PGWVEP).

Lemma 4.1 *Let $f \in C^*\backslash\{0\}$. Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ is continuous with compact convex values on Λ;

(ii) F is continuous with nonempty compact on $B \times B \times \Lambda$;

(iii) For each $\mu \in \Lambda$, $x \in A(\mu)\backslash\bar{S}_f(\mu)$, there exists $y \in \bar{S}_f(\mu)$ such that

$$F(x, y, \mu) + F(y, x, \mu) + B(0, d(x, y)) \subset -C.$$

Then, $\bar{S}_f(\cdot)$ is l.s.c on Λ.

Proof Suppose to the contrary that there exists $\mu_0 \in \Lambda$ such that $\bar{S}_f(\cdot)$ is not l.s.c at μ_0. Then, there exist a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ and $x_0 \in \bar{S}_f(\mu_0)$ such that for any $x_n \in \bar{S}_f(\mu_n)$, $x_n \not\to x_0$.

From $x_0 \in \bar{S}_f(\mu_0)$, we have $x_0 \in A(\mu_0)$. Since $A(\cdot)$ is l.s.c at μ_0, there exists a sequence $\bar{x}_n \in A(\mu_n)$ such that $\bar{x}_n \to x_0$. Obviously, $\bar{x}_n \in A(\mu_n)\backslash\bar{S}_f(\mu_n)$. Then, by (iii), there exists $y_n \in \bar{S}_f(\mu_n)$ such that

$$F(\bar{x}_n, y_n, \mu_n) + F(y_n, \bar{x}_n, \mu_n) + B(0, d(\bar{x}_n, y_n)) \subset -C. \tag{9}$$

Since $y_n \in \bar{S}_f(\mu_n)$ implies $y_n \in A(\mu_n)$, it follows from the upper semicontinuity and compactness of $A(\cdot)$ at μ_0 that there exist $y_0 \in A(\mu_0)$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to y_0$. Particularly, for (9), we have

$$F(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k}) + F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k}) + B(0, d(\bar{x}_{n_k}, y_{n_k})) \subset -C. \tag{10}$$

Since $x_0 \in \bar{S}_f(\mu_0)$, for $y_0 \in A(\mu_0)$, there exists $z_{0_1} \in F(x_0, y_0, \mu_0)$ such that

$$f(z_{0_1}) \geq 0. \tag{11}$$

By the lower semicontinuity of $F(\cdot,\cdot,\cdot)$ at (x_0, y_0, μ_0), there exists $z^1_{n_k} \in F(\bar{x}_{n_k}, y_{n_k}, \mu_{n_k})$ such that $z^1_{n_k} \to z_{0_1}$. On the other hand, from $y_{n_k} \in \bar{S}_f(\mu_{n_k})$ and $\bar{x}_{n_k} \in A(\mu_{n_k})$, there exists $z^2_{n_k} \in F(y_{n_k}, \bar{x}_{n_k}, \mu_{n_k})$ such that

$$f(z^2_{n_k}) \geq 0. \tag{12}$$

Since $F(\cdot,\cdot,\cdot)$ is u.s.c at (y_0, x_0, μ_0) with compact values, there exists $z_{0_2} \in F(y_0, x_0, \mu_0)$ such that $z^2_{n_k} \to z_{0_2}$. It follows from the continuity of f and (12) that we get

$$f(z_{0_2}) \geq 0. \tag{12}$$

By (11), (13) and the linearity of f

$$f(z_{0_1} + z_{0_2}) \geq 0. \tag{14}$$

From (10), we can obtain $z^1_{n_k} + z^2_{n_k} + B(0, d(\bar{x}_{n_k}, y_{n_k})) \subset -C$. Taking $n_k \to \infty$, we get $z_{0_1} + z_{0_2} + B(0, d(x_0, y_0)) \subset -C$. If $x_0 \neq y_0$, we have $z_{0_1} + z_{0_2} \in -\operatorname{int} C$.

Thus,

$$f(z_{0_1} + z_{0_2}) < 0, \tag{15}$$

which contradicts (14). Therefore, we have $x_0 = y_0$. This is impossible by the contradiction assumption. Thus, our result holds and the proof is completed. □

Remark 4.1 In [18], Li et al. obtained the continuity of the f-efficient solution set of (PG-WVEP) by virtue of C-strict monotonicity. But this condition is so strict that the f-efficient solution set is confined to be a singleton. In our paper, we introduce the assumption (iii) of Lemma 4.1, which abates the condition of C-strict monotonicity. In the case, the f-efficient solution set of (PGWVEP) may be a set, but not a singleton. Note that $S_f(\mu) \subset \bar{S}_f(\mu)$. Thus, Example 3.1 shows that $\bar{S}_f(\mu)$ may not be a singleton.

Theorem 4.1 *For each $f \in C^*\backslash\{0\}$, suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ is continuous with compact convex values on Λ;

(ii) F is continuous with nonempty compact on $B \times B \times \Lambda$;

(iii) For each $\mu \in \Lambda$, $x \in A(\mu)\backslash\bar{S}_f(\mu)$, there exists $y \in \bar{S}_f(\mu)$ such that

$$F(x, y, \mu) + F(y, x, \mu) + B(0, d(x, y)) \subset -C;$$

(iv) For each $\mu \in \Lambda$, if for each $x \in S_w(\mu)$ and $y \in A(\mu)$, there exists a selection $z(y)$ of $F(x, y, \mu)\backslash - \operatorname{int} C$ (i.e. $z(y) \in F(x, y, \mu)\backslash - \operatorname{int} C$), such that $\bigcup_{y\in A(\mu)} z(y) + C$ is a convex set, then $S_w(\cdot)$ is l.s.c on Λ.

Proof By virtue of the condition (iv) and Lemma 2.2, for each $\mu \in \Lambda$,

$$S_w(\mu) = \bigcup_{f\in C^*\backslash\{0\}} \bar{S}_f(\mu).$$

By Lemma 4.1, for each $f \in C^*\backslash\{0\}$, $\bar{S}_f(\cdot)$ is l.s.c on Λ. Therefore, in view of Lemma 2.3, we have $S_w(\cdot)$ is l.s.c on Λ. The proof is completed. □

Remark 4.2 The condition (iii) of Theorem 4.1 is weaker than C-strict monotonicity for a set-valued map. Thus, Theorem 4.1 is different from Theorem 3.7 of [18]. Now we give an example to illustrate it.

Example 4.1 Let $X = R$, $Y = R^2$, $C = R^2_+$, $\Lambda = [1, 2]$, $A(\mu) = [-1, 0]$ and $F(x, y, \mu) = \{(a, b) \in R^2|(a, b) = (1 - t)(-1, \mu x) + t(-3, 3\mu x), t \in [0, 1]\}$.

It is similar to Example 3.2. For any $x \in A(\mu)\backslash S_f(\mu)$, there exists $y = 0 \in S_f(\mu)$ such that

$$\begin{aligned} F(x, y, \mu) + F(y, x, \mu) + B(0, d(x, y)) &= (2t + 1)(-1, \mu x) + B(0, d(x, y)) \\ &= (2t + 1)(-1, \mu x) + B(0, d(x, 0)) \subset -C. \end{aligned}$$

Thus, the conditions of Theorem 4.1 are all satisfied. By Theorem 4.1, $S(\cdot)$ is lower semicontinuous on Λ.

However, for $\bar{x} = \bar{y} = 0$,

$$F(\bar{x}, \bar{y}, \mu) + F(\bar{y}, \bar{x}, \mu) = (-1, 0) \in -\partial C\backslash\{0\},$$

where ∂C is the boundary of C. Obviously, $F(\bar{x}, \bar{y}, \mu) + F(\bar{y}, \bar{x}, \mu) \not\subset -intC$, i.e., $F(\cdot, \cdot, \mu)$ is not C-strictly monotone on $A(\mu) \times A(\mu)$. Thus, Theorem 3.7 of [18] is not applicable.

5. Conclusion

In our paper, we investigate the lower semicontinuity of the solution set mappings of two kinds of parametric multivalued vector quasiequilibrium problems involving set-valued mappings under new assumptions. Our results extend the corresponding ones in [6, 17, 18] since our assumptions are weaker than C-strict monotonicity and the f-efficient solution set is a general set but a singleton. Some examples are also given to illustrate this.

References

[1] ANH L Q, KHANH P Q. *Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems* [J]. J. Math. Anal. Appl., 2004, **294**(2): 699–711.

[2] ANH L Q, KHANH P Q. *On the stability of the solution sets of general multivalued vector quasiequilibrium problems* [J]. J. Optim. Theory Appl., 2007, **135**(2): 271–284.

[3] AUBIN J P, EKELAND I. *Applied Nonlinear Analysis* [M]. Wiley, New York, 1984.

[4] BERGE C. *Topological Spaces* [M]. Oliver and Boyd, London, 1963.

[5] CHEN Chunrong, LI Shengjie. *On the solution continuity of parametric generalized systems* [J]. Pac. J. Optim., 2010, **6**(1): 141–151.

[6] CHEN Chunrong, LI Shengjie, TEO K L. *Solution semicontinuity of parametric generalized vector equilibrium problems* [J]. J. Global Optim., 2009, **45**(2): 309–318.

[7] CHENG Yonghong, ZHU Daoli. *Global stability results for the weak vector variational inequality* [J]. J. Global Optim., 2005, **32**(4): 543–550.

[8] FERRO F. *A minimax theorem for vector-valued functions* [J]. J. Optim. Theory Appl., 1989, **60**(1): 19–31.

[9] GONG Xunhua. *Efficiency and Henig efficiency for vector equilibrium problems* [J]. J. Optim. Theory Appl., 2001, **108**(1): 139–154.

[10] GONG Xunhua. *Connectedness of the solution sets and scalarization for vector equilibrium problems* [J]. J. Optim. Theory Appl., 2007, **133**(2): 151–161.

[11] GONG Xunhua, YAO J C. *Connectedness of the set of efficient solutions for generalized systems* [J]. J. Optim. Theory Appl., 2008, **138**(2): 189–196.

[12] GONG Xunhua, YAO J C. *Lower semicontinuity of the set of the efficient solutions for generalized systems* [J]. J. Optim. Theory Appl., 2008, **138**(2):197–205.

[13] GONG Xunhua. *Continuity of the solution set to parametric weak vector equilibrium problems* [J]. J. Optim. Theory Appl., 2008, **139**(1): 35–46.

[14] HUANG Nanjing, LI Jun, THOMPSON H B. *Stability for parametric implict vector equilibrium problems* [J]. Math. Comput. Modelling, 2006, **43**(11-12): 1267–1274.

[15] KIEN B T. *On the lower semicontinuity of optimial solution sets* [J]. Optimization, 2005, **54**(2): 123–130.

[16] KIMURA K, YAO J C. *Sensitivity analysis of solution mappings of parametric vector quasi-equilibrium problems* [J]. J. Global Optim., 2008, **41**(2): 187–202.

[17] LI Shengjie, FANG Zhimiao. *Lower semicontinuity of the solution mappings to a parametric generalized Ky Fan inequality* [J]. J. Optim. Theory Appl., 2010, **147**(3): 507–515.

[18] LI Shengjie, LIU Huimin, CHEN Chunrong. *Lower semicontinuity of parametric generalized weak vector equilibrium problems* [J]. Bull. Austral. Math. Soc., 2010, **81**(1): 85–95.