

Remarks on Representations of Finite Groups over an Arbitrary Field of Characteristic Zero

Jin Ke HAI*, Zheng Xing LI

College of Mathematics, Qingdao University, Shandong 266071, P. R. China

Abstract Let G be a finite group and K a field of characteristic zero. It is well-known that if K is a splitting field for G , then G is abelian if and only if any irreducible representation of G has degree 1. In this paper, we generalize this result to the case that K is an arbitrary field of characteristic zero (that is, K need not be a splitting field for G), and we also obtain the orthogonality relations of irreducible K -characters of G in this case. Our results generalize some well-known theorems.

Keywords Γ_K -action; Γ_K -classes; orthogonality relations.

Document code A

MR(2010) Subject Classification 20C15; 20C20

Chinese Library Classification O152.6

1. Introduction

Let G be a finite group and K an arbitrary field of characteristic zero. Let e be the exponent of G , i.e., the least common multiple of the orders of the elements of G , and L be the field generated over K by the e th roots of unity. It is clear that the extension L/K is Galois and that the Galois group $\text{Gal}(L/K)$, which is the group of all K -automorphisms of L , is isomorphic to a subgroup Γ_K of the multiplicative group $(\mathbb{Z}/e\mathbb{Z})^*$ of invertible elements of $\mathbb{Z}/e\mathbb{Z}$ (see [1]). Let ω be an e th root of unity. For $\sigma \in \text{Gal}(L/K)$, there exists a unique element $t \in \Gamma_K$ such that $\sigma(\omega) = \omega^t$, so we write $\sigma = \sigma_t$.

For any $t \in \Gamma_K$, an action on G , denoted by t also, is defined as follows $t : G \rightarrow G, x \mapsto x^t$. This is well defined as $(t, |G|) = 1$. By sending $t \in \Gamma_K$ to the permutation $x \mapsto x^t$, we map Γ_K to a permutation group on the underlying set of G . It is easy to see that Γ_K is independent of the choice of ω , but dependent on e (see [2]).

Definition 1.1 ([3]) *Two elements $s, s' \in G$ are said to be Γ_K -conjugate if there exists a $t \in \Gamma_K$ such that s' and s^t are conjugate in G . Γ_K -conjugate is an equivalence relationship, and its classes are called the Γ_K -classes of G .*

Received April 17, 2009; Accepted September 15, 2009

Supported by the National Natural Science Foundation of China (Grant No. 10771132) and the Natural Science Foundation of Shandong Province (Grant No. Y2008A03).

* Corresponding author

E-mail address: haijinke2002@yahoo.com.cn (J. K. HAI)

It is an obvious but important fact that the Γ_K -action on G commutes with the G -conjugate action on G , i.e., $\Gamma_K \times G$ acts on G with $x^{(t,y)} = (x^t)^y = (x^y)^t$ for any $(t, y) \in \Gamma_K \times G$.

We shall use the following notations:

For $x \in G$, set $N_G(x) = \{y \in G | (x^t)^y = x \text{ for some } t \in \Gamma_K\}$;

$N_{\Gamma_K}(x) = \{t \in \Gamma_K | (x^t)^y = x \text{ for some } y \in G\}$;

$C_{\Gamma_K \times G}(x) = \{(t, y) \in \Gamma_K \times G | x^{(t,y)} = x\}$;

$C_{\Gamma_K}(x) = \{t \in \Gamma_K | x^t = x\}$.

Then $C_{\Gamma_K \times G}(x)$ and $N_G(x)$ are the subgroups of $\Gamma_K \times G$ and G , respectively. And we obviously have the following two lemmas:

Lemma 1.2 $C_{\Gamma_K \times G}(x) \longrightarrow N_G(x)((t, y) \longmapsto y)$ is a surjective homomorphism of groups. In particular, $C_{\Gamma_K \times G}(x)/C_{\Gamma_K}(x) \cong N_G(x)$.

Lemma 1.3 $C_{\Gamma_K \times G}(x) \longrightarrow N_{\Gamma_K}(x)((t, y) \longmapsto t)$ is a surjective homomorphism of groups. In particular, $C_{\Gamma_K \times G}(x)/C_G(x) \cong N_{\Gamma_K}(x)$.

By Lemmas 1.2 and 1.3, we obtain the following result:

Proposition 1.4 The Γ_K -conjugate class $\text{cl}_{\Gamma_K \times G}(x)$ containing x has length:

$$|\text{cl}_{\Gamma_K \times G}(x)| = \frac{|\Gamma_K|}{|C_{\Gamma_K}(x)|} \frac{|G|}{|N_G(x)|} = \frac{|\Gamma_K|}{|N_{\Gamma_K}(x)|} \frac{|G|}{|C_G(x)|}.$$

Let $\chi_1, \chi_2, \dots, \chi_k$ be the full set of irreducible K -characters of G . Let $R_K(G)$ be the ring of generalized K -characters of G , that is, $R_K(G) = \{\sum_{i=1}^k a_i \chi_i | a_i \in \mathbb{Z} (i = 1, \dots, k)\}$, which is a subring of the ring $R(G) = R_L(G)$. As usual, we define an inner product (φ, ψ) by $(\varphi, \psi) = \frac{1}{|G|} \sum_{x \in G} \varphi(x^{-1})\psi(x)$. We say φ and ψ are orthogonal if $(\varphi, \psi) = 0$.

It is well-known that G is an abelian group if and only if any irreducible L -character of G is linear. However, if K is not a splitting field for G , the above conclusion may be false (see Example 2.5 below). In Section 2, we generalize this result to arbitrary field K of characteristic zero. In Section 3, we investigate the orthogonality relations of irreducible K -characters with K being an arbitrary field of characteristic zero.

Throughout this paper, we assume that all $K[G]$ -modules considered are representation modules. For a G -module V , we denote by $\text{Inv}_G V$ the set of G -invariant elements of V . Other notations are standard [3, 4].

2. The degrees of irreducible representations of an abelian group G

Since $\text{char } K = 0$, $K[G]$ is a direct product of simple algebras A_i , corresponding to distinct irreducible $K[G]$ -modules V_i . Set $D_i = \text{End}_{K[G]}(V_i)$, then D_i is a division ring. A_i can be identified with the algebra $\text{End}_{D_i}(V_i)$, i.e., the endomorphisms of the D_i -vector space V_i . If $[V_i : D_i] = n_i$, then $A_i \cong M_{n_i}(D_i)$. The dimension of D_i over its center $K_i = Z(D_i)$ is m_i^2 with m_i the schur index of $K[G]$ -module V_i .

Let V_i be an irreducible A_i -module, $\mathcal{G} = \text{Gal}(L/K)$, $A_i^L = L \otimes_K A_i$, \tilde{V}_i be an irreducible A_i^L -module, and $\chi_i, \tilde{\chi}_i$ be characters defined by V_i, \tilde{V}_i , respectively. Here we understand that $\tilde{\chi}_i$

is a function defined on A_i . Let $K(\tilde{\chi}_i)$ be the field generated by $\{\tilde{\chi}_i(a), a \in A_i\}$ over K . We also denote by $\mathcal{G}_{\tilde{\chi}_i}$ the set of all $\sigma \in \mathcal{G}$ such that $\tilde{\chi}_i^\sigma = \tilde{\chi}_i$, i.e., $\tilde{\chi}_i^\sigma(a) = \tilde{\chi}_i(a)$ for all $a \in A_i$. Thus $\mathcal{G}_{\tilde{\chi}_i} = \{\sigma \in \mathcal{G} \mid \mu^\sigma = \mu \text{ for all } \mu \in K(\tilde{\chi}_i)\}$.

Let G be an abelian group of finite order. Then any irreducible L -character of G is linear. Let $\widehat{G} = \text{Irr}_L(G) = \{\tilde{\chi}_1, \dots, \tilde{\chi}_l\}$. Then \widehat{G} is a group under the multiplication of characters and $G \cong \widehat{G}$ by [5, Problem 2.7]. Since Γ_K and \mathcal{G} act on G and \widehat{G} , respectively, and $\Gamma_K \cong \mathcal{G}$, it is easy to see that these two actions are equivalent under the isomorphism $G \cong \widehat{G}$. Hence the lengths of the corresponding classes are equal to each other. If we denote by C_i the classes of the action of Γ_K on G with x_i as representatives, and denote by \mathcal{K}_i the corresponding classes of the action of \mathcal{G} on \widehat{G} with $\tilde{\chi}_i$ as representatives, then $|\Gamma_K : C_{\Gamma_K}(x_i)| = |\mathcal{G} : \mathcal{G}_{\tilde{\chi}_i}|$. Furthermore we have $|\mathcal{G} : \mathcal{G}_{\tilde{\chi}_i}| = \dim_K Z(D_i)$ by [4, 2.6.2]. Since G is an abelian group, it follows that D_i is commutative and the Schur indices $m_i = 1$ by [3, Proposition 35]. Thus $|\mathcal{G} : \mathcal{G}_{\tilde{\chi}_i}| = \dim_K D_i$.

Lemma 2.1 *Let $K[G] = \bigoplus_{i=1}^k n_i e_i K[G]$, where $e_i \in \pi(K[G])$. Then $\gamma_G = \sum_{i=1}^k \frac{1}{\dim_K D_i} \chi_i(1) \chi_i$, where γ_G is the regular K -character of G .*

Proof Let F_i be an irreducible representation defined by $e_i K[G]$ and χ_i be the irreducible character defined by F_i . Then by the assumption we have $\Gamma_G \sim \sum_{i=1}^k n_i F_i$ and thus $\gamma_G = \sum_{i=1}^k n_i \chi_i$. Since the degree of F_i is equal to $\chi_i(1)$, we have $\chi_i(1) = n_i \dim_K D_i$. Hence $\gamma_G = \sum_{i=1}^k \frac{1}{\dim_K D_i} \chi_i(1) \chi_i$. \square

Theorem 2.2 *With notations as above, if G is an abelian group, then $n_i = 1$ and $m_i = 1$. Conversely, if $n_i = 1$ and $\frac{|\Gamma_K|}{|N_{\Gamma_K}(x_i)|} = \dim_K D_i$, then G is an abelian group.*

Proof By the above analysis, we have $m_i = 1$ and $|\Gamma_K : N_{\Gamma_K}(x_i)| = |\Gamma_K : C_{\Gamma_K}(x_i)| = |\mathcal{G} : \mathcal{G}_{\tilde{\chi}_i}| = \dim_K D_i$. Then by Proposition 1.4 we have

$$|G| = \sum_{i=1}^k |\Gamma_K : N_{\Gamma_K}(x_i)| = \sum_{i=1}^k \dim_K D_i.$$

On the other hand, by Lemma 2.1 we have

$$|G| = \gamma_G(1) = \sum_{i=1}^k \frac{1}{\dim_K D_i} \chi_i(1)^2 = \sum_{i=1}^k n_i^2 \dim_K D_i.$$

Thus the above two equalities yield $n_i = 1$.

Conversely, since $n_i = 1$, by Lemma 2.1 we have

$$|G| = \gamma_G(1) = \sum_{i=1}^k \dim_K D_i.$$

On the other hand, by Proposition 1.4 and the hypothesis we have

$$|G| = \sum_{i=1}^k \frac{|\Gamma_K|}{|N_{\Gamma_K}(x_i)|} \frac{|G|}{|C_G(x_i)|} = \sum_{i=1}^k \dim_K D_i \frac{|G|}{|C_G(x_i)|}.$$

Then the above two equalities imply $|G| = |C_G(x_i)|$. Hence $G = C_G(x_i)$, and it follows that G is abelian.

As an immediate consequence of Theorem 2.2, we get:

Corollary 2.3 *Let G be an abelian group. Then the degree of the irreducible representation defined by the irreducible $K[G]$ -module V_i is $\dim_K D_i$, where $D_i = \text{End}_{K[G]}(V_i)$.*

Remark 2.4 Corollary 2.3 generalizes the well-known Schur' lemma since if V_i is an absolutely irreducible $K[G]$ -module, then $D_i = \text{End}_{K[G]}(V_i) = K$ and thus $\dim_K D_i = 1$.

Example 2.5 Let $G = \langle x | x^3 = 1 \rangle$ be a cyclic group of order 3. Then G has a 2-dimensional representation over \mathbb{R} in which x acts as the rotation through $\frac{2}{3}\pi$. This representation is irreducible since there is no 1-dimensional subspace stable under the group action.

3. The orthogonality relations of K -characters

In this section, we investigate the orthogonality relations of irreducible K -characters of G , where K is an arbitrary field of characteristic zero and thus may not be a splitting field for G .

Lemma 3.1 *Let U, V be $K[G]$ -modules. Then $U^\Lambda \otimes_K V \cong \text{Hom}_K(U, V)$, where U^Λ denotes the contragredient module of U .*

Proof This is Theorem 1.14(ii) in [4].

Lemma 3.2 *Let V be a $K[G]$ -module. Then $\dim_K \text{Inv}_G V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$, where χ_V is the K -character afforded by V .*

Proof Let $a = \frac{1}{|G|} \sum_{g \in G} g \in KG$. Then $ga = a$ for any $g \in G$. It follows that $a^2 = a$ and thus $\rho(a)^2 = \rho(a)$. Hence $\rho(a)$ is similar to a diagonal matrix and the eigenvalues of $\rho(a)$ are 1 or 0. Let $V_1 \subset V$ be the eigenspace corresponding to 1. If $v \in V_1$, then $vg = v$ for any $g \in G$ and thus $v \in \text{Inv}_G V$. Conversely, if $u \in \text{Inv}_G V$, then $|G|u = \sum_{g \in G} gu = \sum_{g \in G} u = |G|u$ and thus $ua = a$, i.e., $u \in V_1$. Hence we have $\text{Inv}_G V = V_1$ and it follows that $\dim_K \text{Inv}_G V = \text{tr}(\rho(a)) = \chi_V(a) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$. \square

Theorem 3.3 *Let U, V be $K[G]$ -modules and χ_U, χ_V the K -characters afforded by U, V , respectively. Then $(\chi_U, \chi_V) = \dim_K \text{Inv}_G \text{Hom}_K(U, V) = \dim_K \text{Hom}_{K[G]}(U, V)$.*

Proof By Lemmas 3.1 and 3.2, we have

$$\begin{aligned} \dim_K \text{Inv}_G \text{Hom}_K(U, V) &= \frac{1}{|G|} \sum_{g \in G} \chi_{\text{Hom}_K(U, V)}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{U^\Lambda \otimes_K V}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{U^\Lambda}(g) \chi_V(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_U(g^{-1}) \chi_V(g) = (\chi_U, \chi_V). \end{aligned}$$

Another equality is clear. \square

Corollary 3.4 (The first orthogonality relation of K -characters) *Let U, V be irreducible $K[G]$ -modules and χ_U, χ_V the K -characters defined by U, V , respectively.*

- (1) *If U is not isomorphic to V , then $(\chi_U, \chi_V) = 0$;*
- (2) *If U is isomorphic to V , then $(\chi_U, \chi_U) = \dim_K \text{End}_{K[G]}(U)$.*

Proof By Schur’s lemma and Theorem 3.3, the conclusions are obvious. \square

Remark 3.5 If U is isomorphic to V , then $\text{End}_{K[G]}(U)$ is a division ring. Let $D = \text{End}_{K[G]}(U)$. If K is not a splitting field for G , then $\dim_K D \geq 1$ (In this case, $\dim_K D = 1$ if and only if U is an absolutely irreducible $K[G]$ -module). If K is a splitting field for G , then $\dim_K D = 1$ by Schur’s lemma. Thus Corollary 3.4 generalizes the first orthogonality relation of K -characters.

Lemma 3.6 *Let $\chi_1, \chi_2, \dots, \chi_k$ be all the distinct irreducible K -characters of G . Then*

- (1) *The χ_i are mutually orthogonal and form a basis of $R_K(G)$;*
- (2) *The χ_i form a basis of the space of functions on G which are constant on Γ_K -classes, and the number of χ_i is equal to the number of Γ_K -classes.*

Proof (1) is Proposition 32 in [3]; (2) is Corollary 2 of Theorem 25 in [3]. \square

Let C_1, C_2, \dots, C_k be all the Γ_K -classes of G and c_1, c_2, \dots, c_k be the representatives of C_1, C_2, \dots, C_k , respectively. Then we have the following result:

Theorem 3.7 (The second orthogonality relation of K -characters) *With notations as above, then we have*

$$\sum_{t=1}^k \frac{1}{\dim_K D_t} \chi_t(c_i^{-1}) \chi_t(c_j) = \delta_{ij} \frac{|N_G(c_i)| |C_{\Gamma_K}(c_i)|}{|\Gamma_K|} = \delta_{ij} \frac{|N_{\Gamma_K}(c_i)| |C_G(c_i)|}{|\Gamma_K|}.$$

Proof We define functions f_i as follows: $f_i(x) = 1$, if $x \in C_i$; $f_i(x) = 0$ otherwise. By Lemma 3.6, we may set $f_i = \sum_{t=1}^k \lambda_t \chi_t$. Then we have $(f_i, \chi_j) = \lambda_j (\chi_j, \chi_j) = \lambda_j \dim_K D_j$. On the other hand, by Proposition 1.4 we get

$$(f_i, \chi_j) = \frac{1}{|G|} \sum_{x \in G} f_i(x) \chi_j(x^{-1}) = \frac{|\Gamma_K|}{|C_{\Gamma_K}(c_i)| |N_G(c_i)|} \chi_j(c_i^{-1}) = \frac{|\Gamma_K|}{|N_{\Gamma_K}(c_i)| |C_G(c_i)|} \chi_j(c_i^{-1}).$$

Thus

$$\lambda_j = \frac{|\Gamma_K|}{\dim_K D_j |C_{\Gamma_K}(c_i)| |N_G(c_i)|} \chi_j(c_i^{-1}) = \frac{|\Gamma_K|}{\dim_K D_j |N_{\Gamma_K}(c_i)| |C_G(c_i)|} \chi_j(c_i^{-1}),$$

where $j = 1, 2, \dots, k$. Hence the conclusion holds since $\delta_{ij} = f_i(c_j) = \sum_{t=1}^k \lambda_t \chi_t(c_j)$. \square

Remark 3.8 If K is a splitting field for G , then $\dim_K D_t = 1, |\Gamma_K| = 1, |C_{\Gamma_K}(c_i)| = 1$ and $N_G(c_i) = C_G(c_i)$. Hence Theorem 3.7 generalizes the ordinary second orthogonality relation of K -characters.

Acknowledgements We thank Prof. Yun FAN for his help and valuable suggestions.

References

- [1] HUGERFORD T W. *Algebra* [M]. Springer-Verlag, New York-Heidelberg, 1980.
- [2] FAN Yun. *On Scott coefficients and block invariants: an approach for nonsplitting case* [J]. *Comm. Algebra*, 1990, **18**(7): 2199–2242.
- [3] SERRE J P. *Linear Representations of Finite Groups* [M]. Springer-Verlag, New York-Heidelberg, 1977.
- [4] NAGAO H, TSUSHIMA Y. *Representations of Finite Groups* [M]. Academic Press, New York, 1992.
- [5] ISAACS I M. *Character Theory of Finite Groups* [M]. Academic Press, New York, 1976.
- [6] KURZWEIL H, STELLMACHER B. *The Theory of Finite Groups: An Introduction* [M]. Springer-Verlag, New York-Heidelberg, 2004.
- [7] HAI Jinke. *On the isomorphisms of the character rings* [J]. *Sci. China Ser. A*, 2005, **48**(1): 88–96.
- [8] HAI Jinke. *A note on the character ring $\overline{R}_K(G)$* [J]. *Southeast Asian Bull. Math.*, 2005, **29**(6): 1057–1062.