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Remarks on Representations of Finite Groups over an Arbitrary Field of Characteristic Zero

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Abstract Let G be a finite group and K a field of characteristic zero. It is well-known that if K is a splitting field for G, then G is abelian if and only if any irreducible representation of G has degree 1. In this paper, we generalize this result to the case that K is an arbitrary field of characteristic zero (that is, K need not be a splitting field for G), and we also obtain the orthogonality relations of irreducible K-characters of G in this case. Our results generalize some well-known theorems.

Keywords Γ_K-action; Γ_K-classes; orthogonality relations.

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1. Introduction

Let G be a finite group and K an arbitrary field of characteristic zero. Let e be the exponent of G, i.e., the least common multiple of the orders of the elements of G, and L be the field generated over K by the eth roots of unity. It is clear that the extension L/K is Galois and that the Galois group $\mathrm{Gal}(L/K)$, which is the group of all K-automorphisms of L, is isomorphic to a subgroup Γ_K of the multiplicative group $(\mathbb{Z}/e\mathbb{Z})^*$ of invertible elements of $\mathbb{Z}/e\mathbb{Z}$ (see [1]). Let ω be an eth root of unity. For $\sigma \in \mathrm{Gal}(L/K)$, there exists a unique element $t \in \Gamma_K$ such that $\sigma(\omega) = \omega^t$, so we write $\sigma = \sigma_t$.

For any $t \in \Gamma_K$, an action on G, denoted by t also, is defined as follows $t : G \longrightarrow G$, $x \longmapsto x^t$. This is well defined as $(t, |G|) = 1$. By sending $t \in \Gamma_K$ to the permutation $x \longmapsto x^t$, we map Γ_K to a permutation group on the underlying set of G. It is easy to see that Γ_K is independent of the choice of ω, but dependent on e (see [2]).

Definition 1.1 ([3]) *Two elements s, $s' \in G$ are said to be Γ_K-conjugate if there exists a $t \in \Gamma_K$ such that s' and s^t are conjugate in G. Γ_K-conjugate is an equivalence relationship, and its classes are called the Γ_K-classes of G.*

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It is an obvious but important fact that the Γ_K-action on G commutes with the G-conjugate action on G, i.e., $\Gamma_K \times G$ acts on G with $x^{(t,y)} = (x^t)^y = (x^y)^t$ for any $(t, y) \in \Gamma_K \times G$.

We shall use the following notations:

For $x \in G$, set $N_G(x) = \{y \in G | (x^t)^y = x \text{ for some } t \in \Gamma_K\}$;

$N_{\Gamma_K}(x) = \{t \in \Gamma_K | (x^t)^y = x \text{ for some } y \in G\}$;

$C_{\Gamma_K \times G}(x) = \{(t, y) \in \Gamma_K \times G | x^{(t,y)} = x\}$;

$C_{\Gamma_K}(x) = \{t \in \Gamma_K | x^t = x\}$.

Then $C_{\Gamma_K \times G}(x)$ and $N_G(x)$ are the subgroups of $\Gamma_K \times G$ and G, respectively. And we obviously have the following two lemmas:

Lemma 1.2 $C_{\Gamma_k \times G}(x) \longrightarrow N_G(x)((t, y) \longmapsto y)$ *is a surjective homomorphism of groups. In particular,* $C_{\Gamma_K \times G}(x)/C_{\Gamma_K}(x) \cong N_G(x)$.

Lemma 1.3 $C_{\Gamma_K \times G}(x) \longrightarrow N_{\Gamma_K}(x)((t, y) \longmapsto t)$ *is a surjective homomorphism of groups. In particular,* $C_{\Gamma_K \times G}(x)/C_G(x) \cong N_{\Gamma_K}(x)$.

By Lemmas 1.2 and 1.3, we obtain the following result:

Proposition 1.4 *The* Γ_K*-conjugate class* $\mathrm{cl}_{\Gamma_K \times G}(x)$ *containing* x *has length:*

$$|\mathrm{cl}_{\Gamma_K \times G}(x)| = \frac{|\Gamma_K|}{|C_{\Gamma_K}(x)|} \frac{|G|}{|N_G(x)|} = \frac{|\Gamma_K|}{|N_{\Gamma_K}(x)|} \frac{|G|}{|C_G(x)|}.$$

Let $\chi_1, \chi_2, \ldots, \chi_k$ be the full set of irreducible K-characters of G. Let $R_K(G)$ be the ring of generalized K-characters of G, that is, $R_K(G) = \{\sum_{i=1}^k a_i \chi_i | a_i \in \mathbb{Z}\ (i = 1, \ldots, k)\}$, which is a subring of the ring $R(G) = R_L(G)$. As usual, we define an inner product (φ, ψ) by $(\varphi, \psi) = \frac{1}{|G|} \sum_{x \in g} \varphi(x^{-1}) \psi(x)$. We say φ and ψ are orthogonal if $(\varphi, \psi) = 0$.

It is well-known that G is an abelian group if and only if any irreducible L-character of G is linear. However, if K is not a splitting field for G, the above conclusion may be false (see Example 2.5 below). In Section 2, we generalize this result to arbitrary field K of characteristic zero. In Section 3, we investigate the orthogonality relations of irreducible K-characters with K being an arbitrary field of charactaristic zero.

Throughout this paper, we assume that all $K[G]$-modules considered are representation modules. For a G-module V, we denote by $\mathrm{Inv}_G V$ the set of G-invariant elements of V. Other notations are standard [3, 4].

2. The degrees of irreducible representations of an abelian group G

Since $\mathrm{char}\, K = 0$, $K[G]$ is a direct product of simple algebras A_i, corresponding to distinct irreducible $K[G]$-modules V_i. Set $D_i = \mathrm{End}_{K[G]}(V_i)$, then D_i is a division ring. A_i can be identified with the algebra $End_{D_i}(V_i)$, i.e., the endomorphisms of the D_i-vector space V_i. If $[V_i : D_i] = n_i$, then $A_i \cong M_{n_i}(D_i)$. The dimension of D_i over its center $K_i = Z(D_i)$ is m_i^2 with m_i the schur index of $K[G]$-module V_i.

Let V_i be an irreducible A_i-module, $\mathcal{G}$= Gal(L/K), $A_i^L = L \otimes_K A_i$, $\tilde{V}_i$ be an irreducible A_i^L-module, and $\chi_i, \tilde{\chi}_i$ be characters defined by V_i,$\tilde{V}_i$, respectively. Here we understand that $\tilde{\chi}_i$

is a function defined on A_i. Let $K(\tilde{\chi_i})$ be the field generated by $\{\tilde{\chi_i}(a), a \in A_i\}$ over K. We also denote by $\mathcal{G}_{\tilde{\chi_i}}$ the set of all $\sigma \in \mathcal{G}$ such that $\tilde{\chi_i}^{\sigma} = \tilde{\chi_i}$, i.e., $\tilde{\chi_i}^{\sigma}(a) = \tilde{\chi_i}(a)$ for all $a \in A_i$. Thus $\mathcal{G}_{\tilde{\chi_i}} = \{\sigma \in \mathcal{G} | \mu^{\sigma} = \mu \text{ for all } \mu \in K(\tilde{\chi_i})\}$.

Let G be an abelian group of finite order. Then any irreducible L-character of G is linear. Let $\widehat{G} = \mathrm{Irr}_L(G) = \{\widetilde{\chi_1}, \dots, \widetilde{\chi_l}\}$. Then $\widehat{G}$ is a group under the multiplication of characters and $G \cong \widehat{G}$ by [5, Problem 2.7]. Since Γ_K and $\mathcal{G}$ act on G and $\widehat{G}$, respectively, and $\Gamma_K \cong \mathcal{G}$, it is easy to see that these two actions are equivalent under the isomorphism $G \cong \widehat{G}$. Hence the lengthes of the corresponding classes are equal to each other. If we denote by C_i the classes of the action of Γ_K on G with x_i as representatives, and denote by $\mathcal{K}_i$ the corresponding classes of the action of $\mathcal{G}$ on $\widehat{G}$ with $\widetilde{\chi_i}$ as representatives, then $|\Gamma_K : C_{\Gamma_K}(x_i)| = |\mathcal{G} : \mathcal{G}_{\widetilde{\chi_i}}|$. Furthermore we have $|\mathcal{G} : \mathcal{G}_{\widetilde{\chi_i}}| = \dim_K Z(D_i)$ by [4, 2.6.2]. Since G is an abelian group, it follows that D_i is commutative and the Schur indices $m_i = 1$ by [3, Proposition 35]. Thus $|\mathcal{G} : \mathcal{G}_{\widetilde{\chi_i}}| = \dim_K D_i$.

Lemma 2.1 *Let $K[G] = \bigoplus_{i=1}^{k} n_i e_i K[G]$, where $e_i \in \pi(K[G])$. Then $\gamma_G = \sum_{i=1}^{k} \frac{1}{\dim_K D_i} \chi_i(1) \chi_i$, where γ_G is the regular K-character of G.*

Proof Let F_i be an irreducible representation defined by $e_i K[G]$ and χ_i be the irreducible character defined by F_i. Then by the assumption we have $\Gamma_G \sim \sum_{i=1}^{k} n_i F_i$ and thus $\gamma_G = \sum_{i=1}^{k} n_i \chi_i$. Since the degree of F_i is equal to $\chi_i(1)$, we have $\chi_i(1) = n_i \dim_K D_i$. Hence $\gamma_G = \sum_{i=1}^{k} \frac{1}{\dim_K D_i} \chi_i(1) \chi_i$. □

Theorem 2.2 *With notations as above, if G is an abelian group, then $n_i = 1$ and $m_i = 1$. Conversely, if $n_i = 1$ and $\frac{|\Gamma_K|}{|N_{\Gamma_K}(x_i)|} = \dim_K D_i$, then G is an abelian group.*

Proof By the above analysis, we have $m_i = 1$ and $|\Gamma_K : N_{\Gamma_K}(x_i)| = |\Gamma_K : C_{\Gamma_K}(x_i)| = |\mathcal{G} : \mathcal{G}_{\tilde{\chi_i}}| = \dim_K D_i$. Then by Proposition 1.4 we have

$$|G| = \sum_{i=1}^{k} |\Gamma_K : N_{\Gamma_K}(x_i)| = \sum_{i=1}^{k} \dim_K D_i.$$

On the other hand, by Lemma 2.1 we have

$$|G| = \gamma_G(1) = \sum_{i=1}^{k} \frac{1}{\dim_K D_i} \chi_i(1)^2 = \sum_{i=1}^{k} n_i^2 \dim_K D_i.$$

Thus the above two equalities yield $n_i = 1$.

Conversely, since $n_i = 1$, by Lemma 2.1 we have

$$|G| = \gamma_G(1) = \sum_{i=1}^{k} \dim_K D_i.$$

On the other hand, by Proposition 1.4 and the hypothesis we have

$$|G| = \sum_{i=1}^{k} \frac{|\Gamma_K|}{|N_{\Gamma_K}(x_i)|} \frac{|G|}{|C_G(x_i)|} = \sum_{i=1}^{k} \dim_K D_i \frac{|G|}{|C_G(x_i)|}.$$

Then the above two equalities imply $|G| = |C_G(x_i)|$. Hence $G = C_G(x_i)$, and it follows that G is abelian.

As an immediate consequence of Theorem 2.2, we get:

Corollary 2.3 *Let G be an abelian group. Then the degree of the irreducible representation defined by the irreducible $K[G]$-module V_i is $\dim_K D_i$, where $D_i = \mathrm{End}_{K[G]}(V_i)$.*

Remark 2.4 Corollary 2.3 generalizes the well-known Schur' lemma since if V_i is an absolutely irreducible $K[G]$-module, then $D_i = \mathrm{End}_{K[G]}(V_i) = K$ and thus $\dim_K D_i = 1$.

Example 2.5 Let $G = \langle x | x^3 = 1\rangle$ be a cyclic group of order 3. Then G has a 2-dimentional representation over $\mathbb{R}$ in which x acts as the rotation through $\frac{2}{3}\pi$. This representation is irreducible since there is no 1-dimentional subspace stable under the group action.

3. The orthogonality relations of K-characters

In this section, we investigate the orthogonality relations of irreducible K-characters of G, where K is an arbitrary field of characteristic zero and thus may not be a splitting field for G.

Lemma 3.1 *Let U, V be $K[G]$-modules. Then $U^\Lambda \otimes_K V \cong \mathrm{Hom}_K(U, V)$, where U^Λ denotes the contragredient module of U.*

Proof This is Theorem 1.14(ii) in [4].

Lemma 3.2 *Let V be a $K[G]$-module. Then $\dim_K \mathrm{Inv}_G V = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$, where χ_V is the K-character afforded by V.*

Proof Let $a = \frac{1}{|G|}\sum_{g\in G} g \in KG$. Then $ga = a$ for any $g \in G$. It follows that $a^2 = a$ and thus $\rho(a)^2 = \rho(a)$. Hence $\rho(a)$ is similar to a diagonal matrix and the eigenvalues of $\rho(a)$ are 1 or 0. Let $V_1 \subset V$ be the eigenspace corresponding to 1. If $v \in V_1$, then $vg = vag = va = v$ for any $g \in G$ and thus $v \in \mathrm{Inv}_G V$. Conversely, if $u \in \mathrm{Inv}_G V$, then $|G|ua = u(\sum_{g\in G} g) = \sum_{g\in G} u = |G|u$ and thus $ua = a$, i.e., $u \in V_1$. Hence we have $\mathrm{Inv}_G V = V_1$ and it follows that $\dim_K \mathrm{Inv}_G V = \mathrm{tr}(\rho(a)) = \chi_V(a) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$. □

Theorem 3.3 *Let U, V be $K[G]$-modules and χ_U, χ_V the K-characters afforded by U, V, respectively. Then $(\chi_U, \chi_V) = \dim_K \mathrm{Inv}_G \mathrm{Hom}_K(U, V) = \dim_K \mathrm{Hom}_{K[G]}(U, V)$.*

Proof By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\dim_K \mathrm{Inv}_G \mathrm{Hom}_K(U, V) &= \frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Hom}_K(U,V)}(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{U^\Lambda\otimes_K V}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_{U^\Lambda}(g)\chi_V(g)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_U(g^{-1})\chi_V(g) = (\chi_U, \chi_V).\end{aligned}$$

Another equality is clear. □

Corollary 3.4 (The first orthogonality relation of K-characters) *Let U, V be irreducible $K[G]$-modules and χ_U, χ_V the K-characters defined by U, V, respectively.*

(1) If U is not isomorphic to V, then $(\chi_U, \chi_V) = 0$;

(2) If U is isomorphic to V, then $(\chi_V, \chi_V) = \dim_K \mathrm{End}_{K[G]}(V)$.

Proof By Schur'lemma and Theorem 3.3, the conclusions are obvious. □

Remark 3.5 If U is isomorphic to V, then $\mathrm{End}_{K[G]}(V)$ is a division ring. Let $D = \mathrm{End}_{K[G]}(V)$. If K is not a splitting field for G, then $\dim_K D \geqslant 1$ (In this case, $\dim_K D = 1$ if and only if V is an absolutely irreducible $K[G]$-module). If K is a splitting field for G, then $\dim_K D = 1$ by Schur's lemma. Thus Corollary 3.4 generalizes the first orthogonality relation of K-characters.

Lemma 3.6 *Let $\chi_1, \chi_2, \dots, \chi_k$ be all the distinct irreducible K-characters of G. Then*

(1) The χ_i are mutually orthogonal and form a basis of $R_K(G)$;

(2) The χ_i form a basis of the space of functions on G which are constant on Γ_K-classes, and the number of χ_i is equal to the number of Γ_K-classes.

Proof (1) is Proposition 32 in [3]; (2) is Corollary 2 of Theorem 25 in [3]. □

Let $C_1, C_2, \dots, C_k$ be all the Γ_K-classes of G and $c_1, c_2, \dots, c_k$ be the representatives of $C_1, C_2, \dots, C_k$, respectively. Then we have the following result:

Theorem 3.7 (The second orthogonality relation of K-characters) *With notations as above, then we have*

$$\sum_{t=1}^{k} \frac{1}{\dim_K D_t} \chi_t(c_i^{-1}) \chi_t(c_j) = \delta_{ij} \frac{|N_G(c_i)||C_{\Gamma_K}(c_i)|}{|\Gamma_K|} = \delta_{ij} \frac{|N_{\Gamma_K}(c_i)||C_G(c_i)|}{|\Gamma_K|}.$$

Proof We define functions f_i as follows: $f_i(x) = 1$, if $x \in C_i$; $f_i(x) = 0$ otherwise. By Lemma 3.6, we may set $f_i = \sum_{t=1}^{k} \lambda_t \chi_t$. Then we have $(f_i, \chi_j) = \lambda_j (\chi_j, \chi_j) = \lambda_j \dim_K D_j$. On the other hand, by Proposition 1.4 we get

$$(f_i, \chi_j) = \frac{1}{|G|} \sum_{x \in G} f_i(x) \chi_j(x^{-1}) = \frac{|\Gamma_K|}{|C_{\Gamma_K}(c_i)||N_G(c_i)|} \chi_j(c_i^{-1}) = \frac{|\Gamma_K|}{|N_{\Gamma_K}(c_i)||C_G(c_i)|} \chi_j(c_i^{-1}).$$

Thus

$$\lambda_j = \frac{|\Gamma_K|}{\dim_K D_j |C_{\Gamma_K}(c_i)||N_G(c_i)|} \chi_j(c_i^{-1}) = \frac{|\Gamma_K|}{\dim_K D_j |N_{\Gamma_K}(c_i)||C_G(c_i)|} \chi_j(c_i^{-1}),$$

where $j = 1, 2, \dots, k$. Hence the conclusion holds since $\delta_{ij} = f_i(c_j) = \sum_{t=1}^{k} \lambda_t \chi_t(c_j)$. □

Remark 3.8 If K is a splitting field for G, then $\dim_K D_t = 1, |\Gamma_K| = 1, |C_{\Gamma_K}(c_i)| = 1$ and $N_G(c_i) = C_G(c_i)$. Hence Theorem 3.7 generalizes the ordinary second orthogonality relation of K-characters.

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