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A Note on FP-Injective Dimension

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Abstract Let R and S be a left coherent ring and a right coherent ring respectively, ${}_R\omega_S$ be a faithfully balanced self-orthogonal bimodule. We give a sufficient condition to show that $l.FP\text{-id}_R(\omega) < \infty$ implies $G\text{-}\dim_\omega(M) < \infty$, where $M \in \operatorname{mod} R$. This result generalizes the result by Huang and Tang about the relationship between the FP-injective dimension and the generalized Gorenstein dimension in 2001. In addition, we get that the left orthogonal dimension is equal to the generalized Gorenstein dimension when $G\text{-}\dim_\omega(M)$ is finite.

Keywords generalized Gorenstein dimension; FP-injective dimension; left orthogonal dimension.

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1. Introduction

Huang and Tang [1] proved that $l.FP\text{-id}_R(\omega) \leq n$ and $r.FP\text{-id}_S(\omega) \leq n$ if and only if every module in $\operatorname{mod} R$ and every module in $\operatorname{mod} S^{op}$ have finite generalized Gorenstein dimension at most n ($G\text{-}\dim_\omega(M) \leq n$ and $G\text{-}\dim_\omega(N) \leq n$, $M \in \operatorname{mod} R$, $N \in \operatorname{mod} S^{op}$). Then it is natural to ask whether $l.FP\text{-id}_R(\omega) \leq n$ (resp. $r.FP\text{-id}_S(\omega) \leq n$) is the sufficient condition or not to get $G\text{-}\dim_\omega(M) \leq n$ ($G\text{-}\dim_\omega(N) \leq n$). Clearly, the answer to this question is negative. In the process of the proof, we need another condition $r.FP\text{-id}_S(\omega) \leq n$. However, we find that this condition is too strong in the proof. So in this paper, we will give another sufficient condition such that $l.FP\text{-id}_R(\omega) \leq n$ implys $G\text{-}\dim_\omega(M) \leq n$.

In Section 1, the main results are given, some definitions and notations are given in Section 2, and some lemmas and the proofs of results are given in Section 3.

Theorem 1.1 *Let R and S be a left coherent ring and a right coherent ring respectively, ${}_R\omega_S$ be a faithfully balanced selforthogonal bimodule. Then $l.FP\text{-id}_R(\omega) \leq n$ and ${}^{\perp}_{R}\omega$ has the ω-torsionless property if and only if $G\text{-}\dim_\omega(M) \leq n$ for any $M \in \operatorname{mod} R$.*

Huang [2] introduced the left orthogonal dimension and gave some results about it. In this paper, we obtain another result of the left orthogonal dimension as follows:

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Theorem 1.2 *For a nonnegative integer n, $l.FP\text{-id}_R(\omega) \leq n$ and ${}^{\perp}_{R}\omega$ has the ω-torsionless property if and only if $G\text{-}\dim_{\omega}(M) = {}^{\perp}_{R}\omega\text{-}\dim_R(M) \leq n$ for any $M \in \operatorname{mod} R$.*

2. Definitions and notations

In this section, we recall some definitions and notations.

Throughout the paper, we assume that all rings are associative with identity elements and all modules being unital.

Definition 2.1 *Let R be a ring and M be a left (resp. right) R-module. M is called finitely presented if there is a finitely generated projective left (resp. right) R-module P and a finitely generated submodule N of P, such that $P/N \cong M$. We use mod R (resp. $\operatorname{mod} R^{op}$) to denote the category of finitely presented left (resp. right) R-modules.*

Definition 2.2 *A ring R is called a left (resp. right) coherent ring if every finitely generated submodule of a finitely presented left (resp. right) R-module also is finitely presented.*

Definition 2.3 *A left (resp.right) R-module A is called FP-injective if $\operatorname{Ext}^1_R(F, A) = 0$ for every finitely presented left (resp.right) R-module F. Let $l.FP\text{-id}_R(\omega)$ (resp. $r.FP\text{-id}_R(\omega)$) denote the smallest integer $n \geq 0$ such that $\operatorname{Ext}^{n+1}_R(F, A) = 0$ for every finitely presented left (resp. right) R-module F.*

For FP-selfinjective, one can refer to Huang [5], and for the more background knowledge, one can read Anderson and Fuller [6].

We call $\operatorname{Hom}_R({}_RA, {}_R\omega_S)$ (resp. $\operatorname{Hom}_S(A_S, {}_R\omega_S)$) the dual module of A with respect to ${}_R\omega_S$, and denote either of these modules by A^*. For a homomorphism f between R-module (resp. S^{op}-modules), we put $f^* = \operatorname{Hom}(f, {}_R\omega_S)$.

Let $\sigma_A : A \longrightarrow A^{**}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$, and $f \in A^*$ be the canonical evaluation homomorphism. If σ_A is a monomorphism, then A is called an ω-torsionless module. If σ_A is an isomorphism, then A is called an ω-reflexive module.

Definition 2.4 *Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod} R$. $\mathcal{X}$ is said to have the ω-torsionless property (resp. the ω-reflexive property), if every module in $\mathcal{X}$ is ω-torsionless (resp. ω-reflexive).*

Definition 2.5 *Let $\omega \in \operatorname{mod} R$ be a selforthogonal module and $X \in \operatorname{mod} R$. X is said to be left orthogonal with ω if $\operatorname{Ext}^i_R(X, \omega) = 0$ for any $i \geq 1$. We use ${}^{\perp}_{R}\omega$ to denote the subcategory of $\operatorname{mod} R$ consisting of the modules which are left orthogonal with ω. And an exact sequence $\cdots \to X_n \to \cdots \to X_0 \to X \to 0$ is called a left orthogonal resolution of X if all $X_i \in {}^{\perp}_{R}\omega$.*

Let $M \in \operatorname{mod} R$ and n be a nonnegative integer. If M has a left orthogonal resolution: $0 \to X_n \to \cdots \to X_0 \to M \to 0$, then set ${}^{\perp}_{R}\omega\text{-}\dim_R(M) = \inf\{n \mid 0 \to X_n \to \cdots \to X_0 \to M \to 0$ is a left orthogonal resolution of $M\}$. If no such a resolution exists, set ${}^{\perp}_{R}\omega\text{-}\dim_R(M) = \infty$. We call ${}^{\perp}_{R}\omega\text{-}\dim_R(M)$ left orthogonal dimension of M.

Definition 2.6 ([4]) *A module M in $\operatorname{mod} R$ is said to have generalized Gorenstein dimension* *zero (with respect to ω), denoted by $G\text{-}\dim_\omega(M) = 0$, if the following conditions hold:*

(1) M is ω-reflexive;

(2) $\mathrm{Ext}^i_R(M,\omega) = \mathrm{Ext}^i_S(M^*,\omega) = 0$ for any $i \geq 1$.

We use $\mathcal{G}_\omega$ to denote the full subcategory of $\mathrm{mod}\, R$ consisting of the modules with generalized Gorenstein dimension zero.

Definition 2.7 ([4]) *For any $n \geq 0$, $M \in \mathrm{mod}\, R$, M is said to have generalized Gorenstein dimension at most n (with respect to ω), denoted by $G\text{-}\dim_\omega(M) \leq n$, if there exists an exact sequence $0 \to M_n \to \cdots \to M_1 \to M_0 \to M \to 0$ in $\mathrm{mod}\, R$ with $G\text{-}\dim_\omega(M_i) = 0$ for any $0 \leq i \leq n$.*

3. The proofs of main results

We firstly recall some notions in [1].

Let $N \in \mathrm{mod}\, S^{op}$. Suppose $0 \to N \xrightarrow{\delta_0} I_0 \xrightarrow{\delta_1} I_1 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_i} I_i \to \cdots$ is an exact sequence with all I_i FP-injective S^{op}-modules. Such an exact sequence is called an FP-injective resolution of N. If there is a positive integer n such that $\mathrm{Im}\delta_n$ has a decomposition $\bigoplus_{j=1}^m W_j$ with each W_j isomorphic to a direct summand of some $\mathrm{Im}\delta_{i_j}$ with $i_j < n$, then the above exact sequence is called an FP-injective resolution of N ultimately closed at n. An ultimately closed FP-injective resolution of N means an FP-injective resolution of N ultimately closed at n for some n.

It is clear that if $r.FP\text{-id}_S(\omega) < \infty$, then the minimal FP-injective resolution of ω_S is ultimately closed.

Lemma 3.1 ([1, Theorem 2.4]) *Suppose ω_S has an FP-injective resolution ultimately closed at n. If $M \in$modR satisfies $\mathrm{Ext}^i_R(M,\omega) = 0$ for any $1 \leq i \leq n$, then M is ω-reflexive.*

By this Lemma, ${}^{\perp}_R\omega$ has the ω-reflexive property, which implies that ${}^{\perp}_R\omega$ has the ω-torsionless property. Therefore we obtain that ${}^{\perp}_R\omega$ has the ω-torsionless property, which is a necessary condition of $r.FP\text{-id}_S(\omega) < \infty$.

Now, we need the following important property:

Proposition 3.1 *The following statements are equivalent:*

(1) ${}^{\perp}_R\omega$ has the ω-torsionless property;

(2) ${}^{\perp}_R\omega$ has the ω-reflexive property;

(3) ${}^{\perp}_R\omega = \mathcal{G}_\omega$.

Proof It follows from Proposition 2.3 of Huang [2]. □

Proof of the Theorem 1.1 Assume that $l.FP\text{-id}_R(\omega) \leq n$. For any $M \in \mathrm{mod}\, R$, we get an exact sequence in $\mathrm{mod}\, R$: $0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where P_i is finite generated projective for any $0 \leq i \leq n-1$. Then $\mathrm{Ext}^j_R(K,\omega) \cong \mathrm{Ext}^{n+j}_R(M,\omega) = 0$ for any $j \geq 1$. Therefore $K \in {}^{\perp}_R\omega$. Since ${}^{\perp}_R\omega$ has the ω-torsionless property, by Proposition 3.1, we have ${}^{\perp}_R\omega = \mathcal{G}_\omega$. And $K \in \mathcal{G}_\omega$, hence $G\text{-}\dim_\omega(M) \leq n$.

Conversely, let M be any module in $\mathrm{mod}\, R$ and $G\text{-}\dim_\omega(M) \leq n$. Then there exists an exact sequence in $\operatorname{mod} R$ with $G\text{-}\dim_\omega(X_i) = 0$ for any $0 \leq i \leq n$: $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$. So we have $\operatorname{Ext}_R^{n+j}(M, \omega) \cong \operatorname{Ext}_R^j(X_n, \omega) = 0$ for any $j \geq 1$, which implies $l.FP - \operatorname{id}_R(\omega) \leq n$. Because $\operatorname{Ext}_R^j(X_i, \omega) = 0$ for any $j \geq 1$ and $0 \leq i \leq n$, we have $X_i \in {}^{\perp}_{R}\, \omega$. Since $G\text{-}\dim_\omega(X_i) = 0$, ${}^{\perp}_{R}\omega = \mathcal{G}_\omega$. Then by Proposition 3.1, ${}^{\perp}_{R}\omega$ has the ω-torsionless property. □

Corollary 3.1 *$l.FP\text{-}\operatorname{id}_R(\omega) \leq n$ and $r.FP\text{-}\operatorname{id}_S(\omega) \leq n$ if and only if $G\text{-}\dim_\omega(M) \leq n$ and $G\text{-}\dim_\omega(N) \leq n$ ($M \in \operatorname{mod} R$, $N \in \operatorname{mod} S^{op}$).*

Remark From this corollary, we generalize the result of Huang and Tang [1] about the relationship between the FP-injective dimension and the generalized Gorenstein dimension.

In the following, we give the relationship between the generalized Gorenstein dimension and the left orthogonal dimension.

Lemma 3.1 *Let $M \in \operatorname{mod} R$. Then ${}^{\perp}_{R}\omega\text{-}\dim_R(M) \leq n$ if and only if ${}^{\perp}_{R}\omega\text{-}\dim_R(\Omega^n(M)) = 0$, where $\Omega^n(M)$ denotes the nth syzygy module of M (note $\Omega^0(M) = M$).*

Proof See Lemma 6 of Huang [3]. □

Lemma 3.2 *Let $M \in \operatorname{mod} R$ and ${}^{\perp}_{R}\omega\text{-}\dim_R(M) < \infty$. Then*

$$ {}^{\perp}_{R}\omega\text{-}\dim_R(M) = \sup\{i | \operatorname{Ext}_R^i(M, \omega) \neq 0\}. $$

Proof See Lemma 7 of Huang [3]. □

Lemma 3.3 *Let $M \in \operatorname{mod} R$. Then $G\text{-}\dim_\omega(M) \leq n < \infty$ if and only if $\Omega^n(M) \in \mathcal{G}_\omega$.*

Lemma 3.4 *For any $M \in \operatorname{mod} R$, $G\text{-}\dim_\omega(M) \leq n < \infty$ if and only if*

$$ G\text{-}\dim_\omega(M) = \sup\{i | \operatorname{Ext}_R^i(M, \omega) \neq 0\}. $$

Proof The Proofs of Lemmas 3.3 and 3.4 are similar to those of Lemmas 6 and 7 of Huang [3]. And we omit the proofs here. □

Proof of Theorem 1.2 The sufficiency follows from Theorem 1.1. Now we prove the necessity: because ${}^{\perp}_{R}\omega$ has the ω-torsionless property, $l.FP\text{-}\operatorname{id}_R(\omega) \leq n$, we have ${}^{\perp}_{R}\omega = \mathcal{G}_\omega$, and $G\text{-}\dim_\omega(M) \leq n$. Therefore, by Lemma 3.4, we obtain that $G\text{-}\dim_\omega(M) = \sup\{i | \operatorname{Ext}_R^i(M, \omega) \neq 0\}$. If $G\text{-}\dim_\omega(M) = m$, then $\operatorname{Ext}_R^m(M, \omega) \neq 0$ and $\operatorname{Ext}_R^{m+1}(M, \omega) = 0$. So by ${}^{\perp}_{R}\omega\text{-}\dim_R(M) = \sup\{i | \operatorname{Ext}_R^i(M, \omega) \neq 0\}$, we have ${}^{\perp}_{R}\omega\text{-}\dim_R(M) = G\text{-}\dim_\omega(M) = m$. □

Corollary 3.2 *For any $M \in \operatorname{mod} R$, if $G\text{-}\dim_\omega(M) < \infty$, then ${}^{\perp}_{R}\omega\text{-}\dim_R(M) = G\text{-}\dim_\omega(M)$.*

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