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A New Hilbert-Type Integral Inequality with Parameters

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Abstract In this paper it is shown that a new Hilbert-type integral inequality can be established by introducing two parameters m ($m \in N$) and λ ($\lambda > 0$). And the constant factor expressed by the Bernoulli number and π is proved to be the best possible. And then some important and especial results are enumerated. As applications, some equivalent forms are given.

Keywords Hilbert-type integral inequality; hyperbolic cosecant function; Bernoulli number; weight function; best constant.

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1. Introduction and lemmas

Let $f(x),\ g(x) \in L^2(0,\ +\infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y}) f(x)\, g(y)}{x - y} \mathrm{d}x\mathrm{d}y \le \pi^2 \Big\{\int_0^\infty f^2(x)\, \mathrm{d}x\Big\}^{\frac{1}{2}} \Big\{\int_0^\infty g^2(x)\, \mathrm{d}x\Big\}^{\frac{1}{2}} \tag{1}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x + y} \mathrm{d}x\mathrm{d}y \le \pi \Big\{\int_0^\infty f^2(x)\, \mathrm{d}x\Big\}^{\frac{1}{2}} \Big\{\int_0^\infty g^2(x)\, \mathrm{d}x\Big\}^{\frac{1}{2}}. \tag{2}$$

They are the famous Hilbert integral inequalities, where the constant factor π^2 and π are the best possible [1, 2]. This inequality (1) was extended in the paper [3]. Owing to the importance of the Hilbert inequality and the Hilbert type inequality in analysis and applications, some mathematicians have been studying them. Recently, various improvements and extensions of (1) and (2) appear in a lot of papers [4–9]. Specially, Gao and Hsu enumerated more than 40 research articles in the paper [4]. The aim of the present paper is to extend (1) and to build some new Hilbert-type integral inequality by introducing two parameters and by using the technique of analysis, and to discuss the constant factor which is related to Bernoulli number, and then to give some important and especial results, and study some equivalent forms of them.

In order to prove our main results, we need the following lemmas.

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Lemma 1 *Let $a > 0$ and n be a nonnegative integer. Then*

$$\int_0^\infty x^n e^{-ax}\mathrm{d}x = \frac{n!}{a^{n+1}}.$$

This result has been given in the paper [10, p. 226, formula 1053], and it can be obtained by applying the integration by parts.

Lemma 2 *Let m be a positive integer. Then*

$$S = \sum_{k=1}^\infty \frac{1}{(2k-1)^{2m}} = \frac{(2^{2m}-1)\pi^{2m}}{2(2m)!}B_m, \tag{3}$$

where the $B_{m's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc..

Proof It is known from the paper [11] that

$$S_1 = \sum_{k=1}^\infty \frac{1}{k^{2m}} = \frac{2^{2m-1}\pi^{2m}}{(2m)!}B_m,$$

where the $B_{m's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc.. It is easy to deduce that

$$S_2 = \sum_{k=1}^\infty \frac{1}{(2k)^{2m}} = \frac{1}{2^{2m}}S_1.$$

Notice that $S = S_1 - S_2$. The relation (3) follows.

Lemma 3 *Let $a > 0$ and m be a positive integer. Then*

$$\int_0^\infty \frac{x^{2m-1}}{\sinh ax}\mathrm{d}x = \frac{(2^{2m}-1)\pi^{2m}}{2ma^{2m}}B_m, \tag{4}$$

where the $B_{m's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc..

Proof Expanding the hyperbolic cosecant function $\frac{1}{\sinh ax}$, and then using Lemma 1, we have

$$\begin{aligned}\int_0^\infty \frac{x^{2m-1}}{\sinh ax}\mathrm{d}x &= 2\int_0^\infty \frac{x^{2m-1}e^{-ax}}{1-e^{-2ax}}\mathrm{d}x = 2\int_0^\infty x^{2m-1}e^{-ax}\sum_{k=0}^\infty e^{-2kax}\mathrm{d}x \\ &= 2\sum_{k=1}^\infty \int_0^\infty x^{2m-1}e^{-(2k-1)ax}\mathrm{d}x = \frac{2(2m-1)!}{a^{2m}}\sum_{k=1}^\infty \frac{1}{(2k-1)^{2m}}.\end{aligned}$$

By Lemma 2, we obtain (4) at once.

By the way, there is an error in the paper [10, p. 260, formula 1566]: Namely the integral

$$\int_0^\infty \frac{x^m}{\sinh ax}\mathrm{d}x = \frac{(2^{m+1}-1)m!}{2^m a^{m+1}}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{m+1}}$$

is wrong. It should be

$$\int_0^\infty \frac{x^m}{\sinh ax}\mathrm{d}x = \frac{(2^{m+1}-1)m!}{2^m a^{m+1}}\sum_{k=1}^\infty \frac{1}{k^{2m}}.$$

Applying this correct result, it is easy to verify the formulas 1562-1565 in the paper [10, p. 259]. These are omitted here. □

2. Main result

In this section, we will prove our assertions by using the above Lemmas.

Theorem 1 *Let f and g be two real functions, and m be a positive integer and $\lambda > 0$. If $0 \le \int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x < +\infty$ and $0 \le \int_0^\infty x^{1-\lambda} g^2(x)\mathrm{d}x < +\infty$, then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^{2m-1} f(x)g(y)}{x^\lambda - y^\lambda}\mathrm{d}x\mathrm{d}y \le C_\lambda(m)\Big\{\int_0^\infty x^{1-\lambda} f^2\left(x\right)\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty x^{1-\lambda} g^2\left(x\right)\mathrm{d}x\Big\}^{\frac{1}{2}}, \quad (5)$$

where the constant factor C_λ is defined by

$$C_\lambda(m) = \frac{2^{2m-1}(2^{2m}-1)\pi^{2m}}{m\lambda^{2m}} B_m \quad (6)$$

and the $B_{m's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc.. And the constant factor $C_\lambda(m)$ is the best possible. And the equality holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof We can apply the Cauchy inequality to estimate the left-hand side of (5) as follows

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1} f(x)g(y)}{x^\lambda - y^\lambda}\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty \Big(\frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda - y^\lambda}\Big)^{\frac{1}{2}} \left(\tfrac{x}{y}\right)^{\frac{2-\lambda}{4}} f(x) \Big(\frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda - y^\lambda}\Big)^{\frac{1}{2}} \left(\tfrac{y}{x}\right)^{\frac{2-\lambda}{4}} g(y)\mathrm{d}x\mathrm{d}y \\
&\le \Big(\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda - y^\lambda} \left(\tfrac{x}{y}\right)^{\frac{2-\lambda}{2}} f^2(x)\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}} \Big(\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda - y^\lambda} \left(\tfrac{y}{x}\right)^{\frac{2-\lambda}{2}} g^2(y)\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}} \\
&= \Big(\int_0^\infty \omega(x) f^2(x)\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_0^\infty \omega(x) g^2(x)\mathrm{d}x\Big)^{\frac{1}{2}}, \qquad (7)
\end{aligned}$$

where $\omega(x) = \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda - y^\lambda} \left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}}\mathrm{d}y$.

By using Lemma 3, it is easy to deduce that

$$\begin{aligned}
\omega(x) &= \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1}}{x^\lambda\left(1 - (\frac{y}{x})^\lambda\right)} \left(\tfrac{x}{y}\right)^{1-\frac{\lambda}{2}}\mathrm{d}y = -x^{1-\lambda} \int_0^\infty u^{\frac{\lambda}{2}-1} (\ln u)^{2m-1} \tfrac{1}{1-u^\lambda}\mathrm{d}u \\
&= -x^{1-\lambda} \int_{-\infty}^\infty \frac{t^{2m-1} e^{\frac{\lambda}{2}t}}{1 - e^{\lambda t}}\mathrm{d}t = x^{1-\lambda} \int_{-\infty}^\infty \frac{t^{2m-1}}{e^{\frac{\lambda}{2}t} - e^{-\frac{\lambda}{2}t}}\mathrm{d}t = x^{1-\lambda} \int_0^\infty \frac{t^{2m-1}}{\sinh(\frac{\lambda}{2}t)}\mathrm{d}t \\
&= x^{1-\lambda} \frac{(2^{2m}-1)\pi^{2m}}{2m(\frac{\lambda}{2})^{2m}} B_m, \qquad (8)
\end{aligned}$$

where the $B_{m's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc..

It follows from (7) and (8) that

$$\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^{2m-1} f(x)g(y)}{x^\lambda - y^\lambda}\mathrm{d}x\mathrm{d}y \le C_\lambda(m)\Big\{\int_0^\infty x^{1-\lambda} f^2\left(x\right)\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty x^{1-\lambda} g^2\left(x\right)\mathrm{d}x\Big\}^{\frac{1}{2}}, \quad (9)$$

where the constant factor $C_\lambda(m)$ is given by (6).

If (9) takes the form of the equality, then there exist a pair of non-zero constants c_1 and c_2

such that

$$c_1\frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda}f^2\left(x\right)(\frac{x}{y})^{1-\frac{\lambda}{2}}=c_2\frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda}g^2\left(y\right)(\frac{y}{x})^{1-\frac{\lambda}{2}}\text{ a.e. on }(0,+\infty)\times(0,+\infty).$$

Then we have

$$c_1x^{2-\lambda}f^2\left(x\right)=c_2y^{2-\lambda}g^2\left(y\right)=C_0.\ \text{(constant)}\quad\text{a.e. on }\ (0,+\infty)\times(0,+\infty).$$

Without loss of generality, we suppose that $c_1\neq0$. Then

$$\int_0^\infty x^{1-\lambda}f^2\left(x\right)\mathrm{d}x=\frac{C_0}{c_1}\int_0^\infty x^{-1}\mathrm{d}x.$$

This contradicts that $0<\int_0^\infty x^{1-\lambda}f^2\left(x\right)\mathrm{d}x<+\infty$. Hence it is impossible to take the equality in (9). This shows the equality contained in (5) holds if and only if $f(x)=0$, or $g(x)=0$. So the inequality (5) is valid.

It remains to need only to show that $C_\lambda\left(m\right)$ in (5) is the best possible.

$\forall\,0<\varepsilon<1$. Define two functions by

$$\tilde f\left(x\right)=\begin{cases}0, & x\in(0,\ 1)\\ x^{-\frac{2-\lambda+\varepsilon}{2}}, & x\in[1,\ \infty)\end{cases}\quad\text{and}\quad\tilde g\left(y\right)=\begin{cases}0, & y\in(0,\ 1)\\ y^{-\frac{2-\lambda+\varepsilon}{2}}, & y\in[1,\ \infty)\end{cases}.$$

It is easy to deduce that

$$\int_0^{+\infty}x^{1-\lambda}\tilde f^2\left(x\right)\mathrm{d}x=\int_0^{+\infty}y^{1-\lambda}\tilde g^2\left(y\right)\mathrm{d}y=\tfrac{1}{\varepsilon}.$$

If $C_\lambda\left(m\right)$ is not the best possible, then there exists $K>0$, such that

$$\begin{aligned}H(\lambda,m)&=\int_0^\infty\int_0^\infty\tfrac{(\ln\frac{x}{y})^{2m-1}\tilde f(x)\tilde g(y)}{x^\lambda-y^\lambda}\mathrm{d}x\mathrm{d}y=\int_1^\infty\int_1^\infty\tfrac{(\ln\frac{x}{y})^{2m-1}\tilde f(x)\tilde g(y)}{x^\lambda-y^\lambda}\mathrm{d}x\mathrm{d}y\\&\le K\Big(\int_1^\infty x^{1-\lambda}\tilde f^2\left(x\right)\mathrm{d}x\Big)^{\frac12}\Big(\int_1^\infty y^{1-\lambda}\tilde g^2\left(y\right)\mathrm{d}y\Big)^{\frac12}=\frac{K}{\varepsilon}<\frac{C_\lambda\left(m\right)}{\varepsilon}.\end{aligned}\tag{10}$$

On the other hand, we have

$$\begin{aligned}H(\lambda,m)&=\int_0^\infty\int_0^\infty\tfrac{(\ln\frac{x}{y})^{2m-1}\tilde f(x)\tilde g(y)}{x^\lambda-y^\lambda}\mathrm{d}x\mathrm{d}y\\&=\int_1^\infty\int_1^\infty\frac{\{x^{-\frac{2-\lambda+\varepsilon}{2}}\}\{(\ln\frac{x}{y})^{2m-1}y^{-\frac{2-\lambda+\varepsilon}{2}}\}}{x^\lambda-y^\lambda}\mathrm{d}x\mathrm{d}y\\&=\int_1^\infty\Big\{\int_1^\infty\frac{(\ln\frac{x}{y})^{2m-1}y^{-\frac{2-\lambda+\varepsilon}{2}}}{x^\lambda\left(1-(\frac{y}{x})^\lambda\right)}\mathrm{d}y\Big\}\{x^{-\frac{2-\lambda+\varepsilon}{2}}\}\mathrm{d}x\\&=\int_1^\infty\Big\{\int_{1/x}^\infty\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^\lambda}\mathrm{d}u\Big\}\{x^{-1-\varepsilon}\}\mathrm{d}x\\&=\int_1^\infty\Big\{\int_{1/x}^1\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^\lambda}\mathrm{d}u\Big\}\{x^{-1-\varepsilon}\}\mathrm{d}x+\\&\quad\int_1^\infty\Big\{\int_1^\infty\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^\lambda}\mathrm{d}u\Big\}\{x^{-1-\varepsilon}\}\mathrm{d}x\end{aligned}$$

$$=\int_0^1\Big\{\int_{1/u}^{\infty}x^{-1-\varepsilon}\mathrm{d}x\Big\}\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^{\lambda}}\mathrm{d}u+$$

$$\int_1^{\infty}\Big\{\int_1^{\infty}\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^{\lambda}}\mathrm{d}u\Big\}\{x^{-1-\varepsilon}\}\mathrm{d}x$$

$$=\frac{1}{\varepsilon}\int_0^1\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda-\varepsilon}{2}}}{1-u^{\lambda}}\mathrm{d}u+\frac{1}{\varepsilon}\int_1^{\infty}\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda+\varepsilon}{2}}}{1-u^{\lambda}}\mathrm{d}u. \tag{11}$$

When ε is sufficiently small, we obtain from (11) that

$$\begin{aligned}H(\lambda,m)=&\frac{1}{\varepsilon}\Big(\int_0^1\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda}{2}}}{1-u^{\lambda}}\mathrm{d}u+o_1(1)\Big)+\frac{1}{\varepsilon}\Big(\int_1^{\infty}\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda}{2}}}{1-u^{\lambda}}\mathrm{d}u+o_2(1)\Big)\\
=&\frac{1}{\varepsilon}\Big(\int_0^{\infty}\frac{(\ln\frac{1}{u})^{2m-1}u^{-\frac{2-\lambda}{2}}}{1-u^{\lambda}}\mathrm{d}u+o(1)\Big)\\
=&\frac{1}{\varepsilon}\Big(-\int_0^{\infty}\frac{(\ln u)^{2m-1}u^{\frac{\lambda}{2}-1}}{1-u^{\lambda}}\mathrm{d}u+o(1)\Big),\quad\varepsilon\to0.\end{aligned}$$

By Lemma 3, we have

$$H(\lambda,m)=\frac{1}{\varepsilon}\Big(\int_0^{\infty}\frac{t^{2m-1}}{\sinh(\frac{\lambda}{2}t)}\mathrm{d}t+o(1)\Big)=\frac{C_{\lambda}(m)}{\varepsilon}+o(1),\quad\varepsilon\to0. \tag{12}$$

Evidently, the inequality (12) is in contradiction with (10). Therefore, the constant factor $C_\lambda(m)$ in (5) is the best possible. Thus the proof of Theorem is completed. □

By Theorem 2.1, we can obtain some especial and interesting results.

In particular, when $\lambda=m=1$, we have $C_1(1)=\pi^2$, and the inequality (5) can be reduced to (1), which shows that Theorem 2.1 is an extension of (1).

Corollary 1 *If* $0\le\int_0^{\infty}f^2(x)\mathrm{d}x<+\infty$ *and* $0\le\int_0^{\infty}g^2(x)\mathrm{d}x<+\infty$, *then*

$$\int_0^{\infty}\int_0^{\infty}\frac{(\ln\frac{x}{y})^3f(x)g(y)}{x-y}\mathrm{d}x\mathrm{d}y\le2\pi^4\Big\{\int_0^{\infty}f^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^{\infty}g^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}, \tag{13}$$

where the constant factor $2\pi^4$ *is the best possible. And the equality contained in (13) holds if and only if* $f(x)=0$, *or* $g(x)=0$.

Corollary 2 *If* $0\le\int_0^{\infty}x^{-1}f^2(x)\mathrm{d}x<+\infty$ *and* $0\le\int_0^{\infty}x^{-1}g^2(x)\mathrm{d}x<+\infty$, *then*

$$\int_0^{\infty}\int_0^{\infty}\frac{(\ln\frac{x}{y})f(x)g(y)}{x^2-y^2}\mathrm{d}x\mathrm{d}y\le\frac{\pi^2}{4}\Big\{\int_0^{\infty}x^{-1}f^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^{\infty}x^{-1}g^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}, \tag{14}$$

where the constant factor $\frac{\pi^2}{4}$ *is the best possible. And the equality contained in (14) holds if and only if* $f(x)=0$, *or* $g(x)=0$.

Corollary 3 *If* $0\le\int_0^{\infty}\sqrt{x}f^2(x)\mathrm{d}x<+\infty$ *and* $0\le\int_0^{\infty}\sqrt{x}g^2(x)\mathrm{d}x<+\infty$, *then*

$$\int_0^{\infty}\int_0^{\infty}\frac{(\ln\frac{x}{y})f(x)g(y)}{\sqrt{x}-\sqrt{y}}\mathrm{d}x\mathrm{d}y\le4\pi^2\Big\{\int_0^{\infty}\sqrt{x}f^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^{\infty}\sqrt{x}g^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}, \tag{15}$$

where the constant factor $4\pi^2$ *is the best possible. And the equality contained in (15) holds if and only if* $f(x)=0$, *or* $g(x)=0$.

Corollary 4 *If $0 \le \int_0^\infty x^{-1} f^2(x)\mathrm{d}x < +\infty$ and $0 \le \int_0^\infty x^{-1} g^2(x)\mathrm{d}x < +\infty$, then*

$$\int_0^\infty \int_0^\infty \frac{(\ln \frac{x}{y})^3 f(x)g(y)}{x^2-y^2}\mathrm{d}x\mathrm{d}y \le \frac{\pi^4}{8}\Big\{\int_0^\infty x^{-1} f^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty x^{-1} g^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}, \tag{16}$$

where the constant factor $\frac{\pi^4}{8}$ is the best possible. And the equality contained in (16) holds if and only if $f(x) = 0$,or $g(x) = 0$.

Similarly, we can also establish a lot of new inequalities. They are omitted here.

3. Some equivalent forms

As applications, we will build some new inequalities.

Theorem 2 *Let f be a real function, and m be a positive integer and $\lambda > 0$. If*

$$0 \le \int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x < +\infty,$$

then

$$\int_0^\infty y^{\lambda-1}\Big\{\int_0^\infty \frac{(\ln(\frac{x}{y}))^{2m-1}}{x^\lambda-y^\lambda} f(x)\,\mathrm{d}x\Big\}^2\mathrm{d}y \le (C_\lambda(m))^2\int_0^\infty x^{1-\lambda} f^2(x)\,\mathrm{d}x, \tag{17}$$

where $C_\lambda(m)$ is defined by (6) and the constant factor $(C_\lambda(m))^2$ in (17) is the best possible. And the equality contained in (17) holds if and only if $f(x) = 0$. And the inequality (17) is equivalent to (5).

Proof First, we assume that the inequality (5) is valid. Setting a real function $g(y)$ as

$$g(y) = y^{\lambda-1}\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,\mathrm{d}x, \quad y \in (0, +\infty).$$

By using (5), we have

$$\begin{aligned}
&\int_0^\infty y^{\lambda-1}\Big\{\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,\mathrm{d}x\Big\}^2\mathrm{d}y = \int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,g(y)\mathrm{d}x\mathrm{d}y\\
&\le (C_\lambda(m))\Big\{\int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty y^{1-\lambda} g^2(y)\mathrm{d}y\Big\}^{\frac{1}{2}}\\
&= (C_\lambda(m))\Big\{\int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty y^{\lambda-1}\Big(\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2m-1}}{x^\lambda-y^\lambda} f(x)\mathrm{d}x\Big)^2\mathrm{d}y\Big\}^{\frac{1}{2}}.
\end{aligned} \tag{18}$$

It follows from (18) that the inequality (17) is valid after some simplifications.

On the other hand, assume that the inequality (17) keeps valid. Applying Cauchy's inequality and (17), we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,g(y)\,\mathrm{d}x\mathrm{d}y = \int_0^\infty y^{\frac{\lambda-1}{2}}\Big\{\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,\mathrm{d}x\Big\} y^{\frac{1-\lambda}{2}} g(y)\,\mathrm{d}y\\
&\le \Big\{\int_0^\infty y^{\lambda-1}\Big(\int_0^\infty \frac{(\ln\frac{x}{y})^{2m-1}}{x^\lambda-y^\lambda} f(x)\,\mathrm{d}x\Big)^2\mathrm{d}y\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty y^{1-\lambda} g^2(y)\,\mathrm{d}y\Big\}^{\frac{1}{2}}\\
&\le \Big\{(C_\lambda(m))^2\int_0^\infty x^{1-\lambda} f^2(x)\,\mathrm{d}x\Big\}^{\frac{1}{2}}\Big\{\int_0^\infty y^{1-\lambda} g^2(y)\,\mathrm{d}y\Big\}^{\frac{1}{2}}
\end{aligned}$$

$$= (C_\lambda(m)) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)\,\mathrm{d}x \right\}^{\frac{1}{2}} \left\{ \int_0^\infty y^{1-\lambda} g^2(y)\,\mathrm{d}y \right\}^{\frac{1}{2}}. \tag{19}$$

Therefore the inequality (17) is equivalent to (5).

If the constant factor $(C_\lambda(m))^2$ in (17) is not the best possible, then it is known from (19) that the constant factor $C_\lambda(m)$ in (5) is not the best possible either. This is a contradiction. It is obvious that the equality contained in (17) holds if and only if $f(x) = 0$. Theorem is proved. □

Corollary 5 *Let f be a real function. If $0 \le \int_0^\infty f^2(x)\mathrm{d}x < +\infty$, then*

$$\int_0^\infty \left\{ \int_0^\infty \frac{(\ln(\frac{x}{y}))^3}{x-y} f(x)\,\mathrm{d}x \right\}^2 \mathrm{d}y \le 4\pi^8 \int_0^\infty f^2(x)\,\mathrm{d}x, \tag{20}$$

where the constant factor $4\pi^8$ is the best possible. And the equality contained in (20) holds if and only if $f(x) = 0$.

And the inequality (20) is equivalent to (13). Its proof is similar to that of Theorem 2. Hence it is omitted.

Similarly, we can also establish some new inequalities which are respectively equivalent to the inequalities (14)–(16). They are omitted here.

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