Journal of Mathematical Research & Exposition
May, 2011, Vol. 31, No. 3, pp. 509–520
DOI:10.3770/j.issn:1000-341X.2011.03.017
Http://jmre.dlut.edu.cn

A Class of Oscillatory Singular Integrals with Hardy Kernels on Triebel-Lizorkin Spaces and Besov Spaces

Yao Ming NIU[1,*], **Shuang Ping TAO**[2]

1. *Faculty of Mathematics, Baotou teachers college, Inner Mongolia* 014030*, P. R. China;*
2. *College of Mathematics and Information Science, Northwest Normal University, Gansu* 730070*, P. R. China*

Abstract In this paper, the boundedness is obtained on the Triebel-Lizorkin spaces and the Besov spaces for a class of oscillatory singular integrals with Hardy kernels.

Keywords oscillatory singular integrals; Triebel-Lizorkin spaces; Besov spaces; Hardy kernel.

Document code A
MR(2010) Subject Classification 42B25; 42B35
Chinese Library Classification O174.2

1. Introduction

Let $\Omega(x)$ be a measurable function on $\mathbf{R}^n$ satisfying the following conditions:

$$\Omega(\lambda x)=\Omega(x);\quad \text{for any}\quad \lambda>0\quad \text{and}\ x\in R^n\backslash\{0\}; \tag{1.1}$$

$$\int_{S^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0, \tag{1.2}$$

where S^{n-1} is the unit sphere in $\mathbf{R}^n$ with normalized Lebesgue measure $\mathrm{d}\sigma$. Let $P(x)$ be a real valued polynomial on $\mathbf{R}^n$. The oscillatory singular integral operator T is defined on the test function space $S(\mathbf{R}^n)$ by

$$Tf(x)=\text{p.v.}\int_{\mathbf{R}^n}e^{iP(x-y)}\frac{\Omega(x-y)}{|x-y|^n}f(y)\mathrm{d}y. \tag{1.3}$$

Ricci and Stein in [1] proved that if $\Omega\in C^1(S^{n-1})$ with the conditions (1.1) and (1.2), then T is bounded on $L^p(\mathbf{R}^n)$ $(1<p<\infty)$, and the norm of $L^p(\mathbf{R}^n)$ of T depends only on the degree of $P(x)$, not its coefficients. Later, Lu and Zhang in [2] improved the result under a weaker condition $\Omega\in L^r(S^{n-1})$ $(1<r\le\infty)$. Moreover, Fan and Pan in [3] proved if $\Omega\in H^1(S^{n-1})$, then T is still bounded on $L^p(\mathbf{R}^n)$ $(1<p<\infty)$. On the other hand, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is a unified setting of many well-known function spaces, i.e., $L^p(\mathbf{R}^n)=\dot{F}_p^{0,2}(\mathbf{R}^n)$, Sobolev spaces $L^p_\alpha(\mathbf{R}^n)=\dot{F}_p^{\alpha,2}(\mathbf{R}^n)$, when $1<p<\infty$, and Hardy spaces $H^p(\mathbf{R}^n)=\dot{F}_p^{0,2}(\mathbf{R}^n)$ when $0<p\le1$. Recently, Chen, Jia and Jiang in [4] showed that T is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$

Received October 12, 2009; Accepted October 3, 2010
Supported by the National Natural Science Foundation of China (Grant No. 11071250).
* Corresponding author
E-mail address: nymmath@126.com (Y. M. NIU); taosp@nwnu.edu.cn (S. P. TAO)

under a further weaker condition $\Omega \in L\log^+ L(S^{n-1})$. The following theorem is the main result in [4]:

Theorem A *Let $\alpha \in \mathbf{R}$, $1 < p, q < \infty$, $P(x)$ be a polynomial with $\nabla P(0) = 0$, and T be defined as in (1.3). If $\Omega \in L\log^+ L(S^{n-1})$ and satisfies conditions (1.1) and (1.2), then T is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$, that is*

$$\|Tf\|_{\dot{F}_p^{\alpha,q}} \le C(1 + \|\Omega\|_{L\log^+ L(S^{n-1})})\|f\|_{\dot{F}_p^{\alpha,q}},$$

where C is a constant which depends only on the degree of $P(x)$ but not its coefficients.

When $P(x)$ is of degree $N = 0$, the phase function in T is identically zero and T is the classical singular integral operator of convolution type. In this case, it was proved that T is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ and $\dot{B}_p^{\alpha,q}(\mathbf{R}^n)$ in [5] as $\Omega \in H^1(S^{n-1})$. With the following fact

$$\bigcup_{r>1} L^r(S^{n-1}) \subseteq L\log^+ L(S^{n-1}) \subseteq H^1(S^{n-1}),$$

it is natural to ask whether T defined as in (1.3) is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ or not as $\Omega \in H^1(S^{n-1})$. In this paper, we will study this problem. It is commonly known that Triebel-Lizorkin spaces are much harder to work with than Besov spaces due to their particular structure.

Suppose that $P(s) = P_N(s)$ is a real polynomial on $\mathbf{R}$ of degree N, the oscillatory singular integral operator T_Ω is defined on the test function space $S(\mathbf{R}^n)$ by

$$T_{\Omega,P}f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(|x-y|)} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y. \tag{1.4}$$

Specially, for $\beta \in \mathbf{R}$, $\beta \ne 0, 1$, the oscillatory singular integral operator T_Ω is defined on the test function space $S(\mathbf{R}^n)$ by

$$T_{\Omega,\beta}f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i|x-y|^\beta} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y. \tag{1.5}$$

In [6] and [7], Chanillo, Kurts and Sampson studied the $L^p(\omega)$ $(1 < p < \infty)$ and weighted weak type (1, 1) boundedness of operator $Tf(x) = \text{p.v.}(1 + |\cdot|)^{-1}e^{i|\cdot|^\beta} * f(x)$, where $\omega \in A_p$. As shown in [7], the same results are also true for the operator defined in (1.5) with standard C-Z kernel. In [8], Chen and Jiang showed that $T_{\Omega,\beta}$ defined in (1.5) is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ as $\Omega \in L\log^+ L(S^{n-1})$.

The aim of this note is to investigate the boundedness of the oscillatory singular integral operators T_Ω and T with the Hardy kernels on the Triebel-Lizorkin spaces and the Besov spaces. Before stating our main results, we recall the definitions of the Triebel-Lizorkin spaces and the Besov spaces.

Definition 1.1 *Let $\phi \in C_0^\infty(\mathbf{R}^n)$ and $\text{supp}(\phi) \subset \{x : 1/2 \le |x| \le 2\}$ such that $0 \le \phi \le 1$, and $\phi(x) > c > 0$, when $3/5 \le |x| \le 5/3$. Write $\phi_l(x) = \phi(2^l x)$ and $\sum_{l=-\infty}^{+\infty} \phi_l^2(x) = 1$ when $x \ne 0$. Denote $S_l f = \Phi_l * f$, where $\hat{\Phi}_l(\xi) = \phi_l(\xi)$. Let $\Psi \in S(\mathbf{R}^n)$ with $\text{supp}(\Psi) \subset \{\xi : |\xi| \le 2\}$ and $|\hat{\Psi}| \ge c > 0$ as $|\xi| \le \frac{5}{3}$. Let $P(\mathbf{R}^n)$ denote the class of polynomials on $\mathbf{R}^n$. Then the* *homogeneous Triebel-Lizorkin spaces are defined by*

$$\dot{F}_p^{\alpha,q} = \{f \in S'(R^n)/P(R^n) : \|f\|_{\dot{F}_p^{\alpha,q}(R^n)} = \Big\|\big(\sum_{l\in\mathbf{Z}} 2^{-l\alpha q}|S_l f|^q\big)^{\frac{1}{q}}\Big\|_p < \infty\},$$

and the inhomogeneous Triebel-Lizorkin spaces are defined by

$$F_p^{\alpha,q}(R^n) = \{f \in S'(R^n) : \|f\|_{F_p^{\alpha,q}(R^n)} = \Big\|\big(\sum_{l\geq 1} 2^{-l\alpha q}|S_l f|^q\big)^{\frac{1}{q}}\Big\|_p + \|\Psi * f\|_p < \infty\}.$$

And the homogeneous Besov spaces $\dot{B}_p^{\alpha,q}(R^n)$ *are defined by*

$$\dot{B}_p^{\alpha,q}((R^n) = \{f \in S'((R^n)/P(R^n) : \|f\|_{\dot{B}_p^{\alpha,q}(R^n)} = \big(\sum_{l\in\mathbf{Z}} 2^{-l\alpha q}\|S_l f\|_{L^p}^q\big)^{\frac{1}{q}} < \infty\},$$

and the inhomogeneous Besov spaces are defined by

$$B_p^{\alpha,q}(R^n) = \{f \in S'(R^n) : \|f\|_{B_p^{\alpha,q}(R^n)} = \big(\sum_{l\geq 1} 2^{-l\alpha q}\|S_l f\|_{L^p}^q\big)^{\frac{1}{q}} + \|\Psi * f\|_p < \infty\}.$$

Let S_l^* be the dual operator of S_l. It is easy to see that

$$\|f\|_{\dot{F}_p^{\alpha,q}(R^n)} \sim \Big\|\big(\sum_{l\in\mathbf{Z}} 2^{-l\alpha q}|S_l^* f|^q\big)^{\frac{1}{q}}\Big\|_p.$$

The following properties of Triebel-Lizorkin spaces and the Besov spaces are well-known. Let $1 < p,\ q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then we have

(1) $\dot{F}_p^{0,2} = H^p$ for $0 < p \leq 1$, $\dot{F}_p^{0,2} = \dot{B}_p^{0,2} = L^p$ for $1 < p < \infty$ and $\dot{F}_\infty^{0,2} = \mathrm{BMO}$;
(2) $F_p^{\alpha,q} \sim \dot{F}_p^{\alpha,q} \bigcap L^p$ and $\|f\|_{F_p^{\alpha,q}} \sim \|f\|_{L^p} + \|f\|_{\dot{F}_p^{\alpha,q}}$, for $\alpha > 0$;
(3) $B_p^{\alpha,q} \sim \dot{B}_p^{\alpha,q} \bigcap L^p$ and $\|f\|_{B_p^{\alpha,q}} \sim \|f\|_{L^p} + \|f\|_{\dot{B}_p^{\alpha,q}}$, for $\alpha > 0$;
(4) $(F_p^{\alpha,q})^* = F_{p'}^{-\alpha,q'}$ and $(\dot{F}_p^{\alpha,q})^* = \dot{F}_{p'}^{-\alpha,q'}$;
(5) $(B_p^{\alpha,q})^* = B_{p'}^{-\alpha,q'}$ and $(\dot{B}_p^{\alpha,q})^* = \dot{B}_{p'}^{-\alpha,q'}$;
(6) $\dot{F}_p^{\alpha,q_1} \subset \dot{F}_p^{\alpha,q_2}$ and $F_p^{\alpha,q_1} \subset F_p^{\alpha,q_2}$, if $q_1 \leq q_2$.

The main results of this note are in the following.

Theorem 1.1 *Let* $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$, *and* $P(s) = P_N(s)$ *be a real polynomial on* $\mathbf{R}$ *of degree* N $(N \geq 2)$. *If* $\Omega \in H^1(S^{n-1})$ *and satisfies the conditions (1.1) and (1.2), then* $T_{\Omega,P}$ *defined as in (1.4) is bounded on* $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$, *that is,*

$$\|T_{\Omega,P} f\|_{\dot{F}_p^{\alpha,q}} \leq C\|f\|_{\dot{F}_p^{\alpha,q}}, \tag{1.6}$$

where C *is a constant which depends only on the degree of* P *but not its coefficients.*

Since the operator $T_{\Omega,P}$ is bounded on $L^p(\mathbf{R}^n)$ (see [3]), by applying Theorem 1.1 and the properties (2), (4), we have the following corollary about the inhomogeneous Triebel-Lizorkin spaces.

Corollary 1.1 *Let* $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$. *Let* $T_{\Omega,P}$, Ω *be defined as in Theorem 1.1. Then* $T_{\Omega,P}$ *is bounded on* $F_p^{\alpha,q}(\mathbf{R}^n)$, *that is,*

$$\|T_{\Omega,P} f\|_{F_p^{\alpha,q}} \leq C\|f\|_{F_p^{\alpha,q}},$$

where C *is a constant which depends only on the degree of* $P(x)$ *but not its coefficients.*

Theorem 1.2 *Let $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$, $\beta > 0$ ($\beta \neq 0, 1$). If $\Omega \in H^1(S^{n-1})$ and satisfies the conditions (1.1) and (1.2), then $T_{\Omega,\beta}$ defined as in (1.5) is bounded on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$, that is,*

$$\|T_{\Omega,\beta}f\|_{\dot{F}_p^{\alpha,q}} \leq C\|f\|_{\dot{F}_p^{\alpha,q}}. \tag{1.7}$$

Noting the following fact

$$\bigcup_{r>1} L^r(S^{n-1}) \subseteq L\log^+ L(S^{n-1}) \subseteq H^1(S^{n-1}),$$

we see that Theorem 1.2 improves Theorem 1.1 in [7].

Theorem 1.3 *Let $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$. Let $P(x)$ be a real valued polynomial on $\mathbf{R}^n$. If $\Omega \in H^1(S^{n-1})$ and satisfies the conditions (1.1) and (1.2), then T defined as in (1.3) is bounded on $\dot{B}_p^{\alpha,q}(\mathbf{R}^n)$, that is,*

$$\|Tf\|_{\dot{B}_p^{\alpha,q}} \leq C\|f\|_{\dot{B}_p^{\alpha,q}},$$

where C is a constant which depends only on the degree of $P(x)$ but not its coefficients.

Since the operator T is bounded on $L^p(\mathbf{R}^n)$ (see [3]), by applying Theorem 1.3 and the properties (3), (5), we have the following corollary about the inhomogeneous Besov spaces.

Corollary 1.2 *Let $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$. Let T, Ω and $P(x)$ be defined as in Theorem 1.3. Then T is bounded on $B_p^{\alpha,q}(\mathbf{R}^n)$, that is,*

$$\|Tf\|_{B_p^{\alpha,q}} \leq C\|f\|_{B_p^{\alpha,q}},$$

where C is a constant which depends only on the degree of $P(x)$ but not its coefficients.

Noting that $(\dot{F}_p^{\alpha_0,q_0}, \dot{F}_p^{\alpha_1,q_1})_{\theta,q} = \dot{B}_p^{\alpha_1,q_1}$ for $0 < \theta < 1$, we can obtain the following results from Theorem 1.2 and the properties (3), (5).

Theorem 1.4 *Let $\alpha \in \mathbf{R}$, $1 < p,\ q < \infty$. Let $T_{\Omega,\beta}$, Ω, and β be defined as in Theorem 1.2. Then*

(i) $\|T_{\Omega,\beta}f\|_{\dot{B}_p^{\alpha,q}} \leq C\|f\|_{\dot{B}_p^{\alpha,q}}$, *for* $\alpha \in \mathbf{R}$;

(ii) $\|T_{\Omega,\beta}f\|_{B_p^{\alpha,q}} \leq C\|f\|_{B_p^{\alpha,q}}$, *for* $\alpha > 0$.

In the next section we shall introduce some notations and lemmas which will be used in our proofs. In the last section we shall give the proofs of Theorems 1.1 and 1.3.

2. Preliminary lemmas

Let us begin with recalling the definition of the Hardy space $H^1(S^{n-1})$. The Poisson kernel on S^{n-1} is defined by $P_{ty'}(x') = (1-t^2)/|ty'-x'|^n$ with $0 \leq t < 1$ and $x', y' \in S^{n-1}$. Then the Hardy space $H^1(S^{n-1})$ is defined by

$$H^1(S^{n-1}) = \Big\{\bar{\omega} \in L^1(S^{n-1}) : P^+\bar{\omega}(x') =: \sup_{0\leq t<1} \Big|\int_{S^{n-1}} \bar{\omega}(y')P_{tx'}(y')\mathrm{d}\sigma(y')\Big| \in L^1(S^{n-1})\Big\}$$

with the norm $\|\bar{\omega}\|_{H^1(S^{n-1})} =: \|P^+\bar{\omega}\|_{L^1(S^{n-1})}$ for $\bar{\omega} \in H^1(S^{n-1})$.

An exceptional atom is an L^∞ function $a(\cdot)$ satisfying $\|a\|_{L^\infty(S^{n-1})} \le 1$. A regular q-atom is an L^q $(1 < q \le \infty)$ function $a(\cdot)$ that satisfies:

$$\operatorname{supp}(a) \subset S^{n-1} \bigcap \{y \in \mathbf{R}^n : |y - \xi'| < \rho \ \text{ for some } \xi' \in S^{n-1} \text{ and } \rho \in (0,1]\}; \tag{2.1}$$

$$\int_{S^{n-1}} a(x')\mathrm{d}\sigma(x') = 0; \tag{2.2}$$

and

$$\|a\|_q \le \rho^{(n-1)(1/q-1)}. \tag{2.3}$$

If $\Omega \in H^1(S^{n-1})$, then it has the following atomic decomposition [12]

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.4}$$

where $\sum_{j=1}^{\infty} |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$ and the a_j's are either exceptional atoms or regular q-atoms. In particular, if $\Omega \in H^1(S^{n-1})$ has the mean zero property (1.2), then all the atoms a_j in (2.4) can be chosen to be regular q-atoms for a fixed q, $1 < q \le \infty$.

In the rest of this paper, for any non-zero $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$, we write $\xi/|\xi| = \xi' = (\xi_1', \xi_2', \ldots, \xi_n') \in S^{n-1}$. Let $a(\cdot)$ be a regular ∞-atom in $H^1(S^{n-1})$ whose support satisfies $\operatorname{supp}(a) \subset S^{n-1} \cap B(\xi', \rho)$. Set

$$E_a(s,\xi') = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s) \int_{S^{n-2}} a\big(s, (1-s^2)^{1/2}\tilde{y}\big)\mathrm{d}\sigma(\tilde{y}), \quad n \ge 3, \tag{2.5}$$

and

$$e_a(s,\xi') = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\big[a\big(s,(1-s^2)^{1/2}\big) + a\big(s, -(1-s^2)^{1/2}\big)\big], \quad n = 2.$$

The following lemmas are needed in the next section.

Lemma 2.1 ([13]) *There exists a constant $c > 0$, independent of a, such that $E_a(s,\xi')$ is an ∞-atom on $\mathbf{R}$. That is, cE_a satisfies*

$$\|cE_a\|_{L^\infty} \le 1/r(\xi'), \operatorname{supp}(E_a) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi'))$$

and

$$\int_{\mathbf{R}} E_a(s,\xi')\mathrm{d}s = 0,$$

where $r(\xi') = |\xi|^{-1}|A_\rho \xi|$ and $A_\rho(\xi) = (\rho^2\xi_1, \rho\xi_2, \ldots, \rho\xi_n)$.

Lemma 2.2 ([13]) *For $1 < q < 2$, there exists a constant $c > 0$, independent of a, such that $ce_a(s,\xi')$ is a q-atom on $\mathbf{R}$, the center of whose support is ξ_1' and the radius $r(\xi') = |\xi|^{-1}(\rho^4\xi_1^2 + \rho^2\xi_2^2)^{1/2}$.*

Lemma 2.3 ([1]) *Let $\phi(t)$ be real-valued and smooth function in (a,b) and $|\phi^{(k)}(t)| \ge 1$ for all $t \in (a,b)$. Then*

$$\Big|\int_a^b e^{i\lambda\phi(t)}\mathrm{d}t\Big| \le C_k \lambda^{-\frac{1}{k}}$$

holds when (i) $k \ge 2$ or (ii) $k = 1$, $\phi'(t)$ is monotonic. The bound C_k is independent of ϕ and λ.

Lemma 2.4 *Denote* $\sigma_k(x) = e^{iP(|x|)}|x|^{-n}a(x')\chi(2^{k-1} < |x| \le 2^k)$, *where* $P(s) = P_N(s) = \sum_{m=0}^{N-1}\beta_m s^m + s^N$ *is a real polynomial on* $\mathbf{R}$ *of degree* N, a *is an* ∞*-atom. Then there exists a positive constant* θ *such that*

$$|\widehat{\sigma_k}(\xi)| \le C\min\{|A_\rho 2^k\xi|^{\theta}, |A_\rho 2^k\xi|^{-1/4}\}, \tag{2.6}$$

where C *is independent of* $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$ *and* $\rho > 0$, *only depends on the degree of* P *but not its coefficients.*

Proof We will only prove the case $n \ge 3$, since the proof for $n = 2$ is essentially the same (using Lemma 2.2 instead of Lemma 2.1).

Firstly, we verify that $\widehat{\sigma_k}(\xi)$ satisfies

$$|\widehat{\sigma_k}(\xi)| \le C|A_\rho 2^k\xi|^{-1/4}.$$

For any $\xi \neq 0$, we choose a rotation O such that $O(\xi) = |\xi|\mathbf{1} = |\xi|(1, 0, \ldots, 0)$. Let $y' = (s, y'_2, y'_3, \ldots, y'_n)$. Then it is easy to see that

$$\widehat{\sigma_k}(\xi) = \int_{I_k} t^{-1}\int_{S^{n-1}} a(O^{-1}y')e^{-it|\xi|\langle 1,y'\rangle + iP_N(t)}\mathrm{d}\sigma(y')\mathrm{d}t,$$

where $I_k = [2^{k-1}, 2^k]$, O^{-1} is the inverse of O. Now $a(O^{-1}y')$ is again an ∞-atom with support in $B(\xi', \rho) \cap S^{n-1}$. Therefore, we get

$$\widehat{\sigma_k}(\xi) = \int_{I_k} t^{-1}\int_R E_a(s, \xi')e^{-it|\xi|s + iP_N(t)}\mathrm{d}s\mathrm{d}t,$$

where $E_a(s, \xi')$ is a function defined as (2.5). By Lemma 2.1, without loss of generality we may assume that $E_a(s, \xi')$ is a q-atom with support in $(-2r(\xi'), 2r(\xi'))$ for $1 < q < 2$. Thus $A(s) = r(\xi')E_a(r(\xi')s, \xi')$ is a q-atom with support in the interval $(-1, 1)$. After changing variables, we have

$$\widehat{\sigma_k}(\xi) = \int_{I_k} t^{-1}\int_R A(s)e^{-itr(\xi')|\xi|s + iP_N(t)}\mathrm{d}s\mathrm{d}t. \tag{2.7}$$

From (2.7) and Hölder's inequality, we have

$$|\widehat{\sigma_k}(\xi)| \le C2^{-k/2}\Big\{\int_{I_k}\Big|\int_R A(s)e^{-itr(\xi')|\xi||s}\mathrm{d}s\Big|^2\mathrm{d}t\Big\}^{1/2}.$$

Let

$$N_k = \Big\{\int_{I_k}\Big|\int_R A(s)e^{-itr(\xi')|\xi|s}\mathrm{d}s\Big|^2\mathrm{d}t\Big\}^{1/2}.$$

To estimate N_k, we choose a function $\psi \in C^\infty(R)$ satisfying $\psi(t) \equiv 1$ for $|t| \le 1$, $\psi(t) \equiv 0$ for $|t| > 2$. Define T_k by

$$(T_k f)(t) = \chi_{I_k}(t)\int_R e^{-itr(\xi')|\xi|s}\psi(s)f(s)\mathrm{d}s.$$

Then

$$T_kT_k^* f(t) = \int_R L(t, s)f(s)\mathrm{d}s,$$

where

$$L(t, s) = \int_R e^{-iv[(s-t)r(\xi')|\xi|]}\psi(v)^2\mathrm{d}v\chi_{I_k}(t)\chi_{I_k}(s).$$

We can easily get

$$|L(t,s)| \le C\chi_{I_k}(t)\chi_{I_k}(s). \tag{2.8}$$

On the other hand, by integration by parts, we have

$$|L(t,s)| \le C(|s-t|r(\xi')|\xi|)^{-1}\chi_{I_k}(t)\chi_{I_k}(s). \tag{2.9}$$

Thus, by (2.8) and (2.9), we have

$$|L(t,s)| \le C(|s-t|r(\xi')|\xi|)^{-1/2}\chi_{I_k}(t)\chi_{I_k}(s).$$

Therefore,

$$\sup_{s>0}\int_R |L(t,s)|\mathrm{d}t \cong \sup_{t>0}\int_R |L(t,s)|\mathrm{d}s \cong 2^k(2^k r(\xi')|\xi|)^{-1/2}. \tag{2.10}$$

It follows that

$$\|T_k\|_2 \le C2^{k/2}(2^k|A_\rho\xi|)^{-1/4}.$$

Thus

$$|\widehat{\sigma_k}(\xi)| \le C(2^k|A_\rho\xi|)^{-1/4}.$$

By using Hölder's inequality, we have

$$|\widehat{\sigma_k}(\xi)| \le C\Big\{\int_{I_k}\Big|\int_R A(s)e^{-itr(\xi'))|\xi||s}\mathrm{d}s\Big|^2\mathrm{d}t\Big\}^{1/2}.$$

We have from (2.8) and (2.10),

$$|\widehat{\sigma_k}(\xi)| \le C2^{k/2}. \tag{2.11}$$

Therefore, after changing variables and by the cancelation property of $A(.)$, we obtain that

$$\begin{aligned}
\widehat{\sigma_k}(\xi) &= \int_{I_k}\int_R A(s)e^{iP_N(t)}\{e^{-it\mathrm{r}(\xi')|\xi|s}-1\}\mathrm{d}st^{-1}\mathrm{d}t\\
&= \int_1^2 e^{i(\sum_{m=0}^{N-1}\beta_m(2^kt)^m+(2^kt)^N)}\int_R A(s)\{e^{-i2^kt\mathrm{r}(\xi')|\xi|s}-1\}\mathrm{d}st^{-1}\mathrm{d}t\\
&= \int_R\int_1^2 e^{i(\sum_{m=0}^{N-1}\beta_m(2^kt)^m+(2^kt)^N+2^kt\mathrm{r}(\xi')|\xi|\tau s)}\mathrm{d}tA(s)2^kr(\xi')|\xi|si\mathrm{d}s.
\end{aligned}$$

Noting that

$$(\frac{\partial}{\partial t})^N\Big(\sum_{m=0}^{N-1}\beta_m(2^kt)^m+(2^kt)^N+2^kt\mathrm{r}(\xi')|\xi|\tau s\Big) \ge C2^k,$$

where $\tau\in(0,1)$ and C is a constant depending on N, by Lemma 2.3, we obtain

$$\begin{aligned}
|\widehat{\sigma_k}(\xi)| &\le \int_R\Big|\int_1^2 e^{i(\sum_{m=0}^{N-1}\beta_m(2^kt)^m+(2^kt)^N+2^kt\mathrm{r}(\xi'|\xi|\tau s)}\mathrm{d}t\Big|A(s)2^kr(\xi'|\xi||s|\mathrm{d}s\\
&\le C2^{k(-1/N)}2^k|A_\rho\xi| \le C2^{k(1-1/N)}|A_\rho\xi|.
\end{aligned}$$

It follows that

$$|\widehat{\sigma_k}(\xi)| \le C2^{k(1-1/N)}|A_\rho\xi|. \tag{2.12}$$

Therefore, by (2.11) and (2.12), for $\theta=N/(N+2)\in(0,1)$, we have

$$|\widehat{\sigma_k}(\xi)| \le C2^{k(1-1/N)\theta}|A_\rho\xi|^\theta 2^{[k(1-\theta)/2]}.$$

Thus,

$$|\widehat{\sigma_k}(\xi)| \le C(2^k|A_\rho\xi|)^\theta,$$

where $\theta = N/(N+2)$. So the proof of Lemma 2.4 is completed. □

Lemma 2.5 *Denote* $\sigma'_k(x) = e^{i|x|^\beta}|x|^{-n}a(x')\chi\ (2^{k-1} < |x| \le 2^k)$, *where* a *is an* ∞*-atom and* $\beta > 0$ $(\beta \ne 1)$. *Then*

$$|\widehat{\sigma'_k}(\xi)| \le C\min\{|A_\rho 2^k\xi|, |A_\rho 2^k\xi|^{-1/4}\},$$

where C *is independent of* $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$ *and* $\rho > 0$, *only depends on* β.

Proof Similarly to the proof of Lemma 2.4, we have $\widehat{\sigma'_k}(\xi)$ satisfies

$$|\widehat{\sigma'_k}(\xi)| \le C|A_\rho 2^k\xi|^{-1/4}.$$

By Hölder's inequality, we have

$$|\widehat{\sigma'_k}(\xi)| \le C\Big\{\int_{I_k}\Big|\int_R A(s)e^{-itr(\xi')|\xi||s}\mathrm{d}s\Big|^2\mathrm{d}t\Big\}^{1/2}.$$

Similarly to estimate (2.11), we have

$$|\widehat{\sigma'_k}(\xi)| \le C2^{k/2}. \tag{2.13}$$

Since

$$(\frac{\partial}{\partial t})^2(t^\beta - t|\xi|\tau s) \ge C2^{k(\beta-2)},$$

where $\tau \in (0,1)$ and C is a constant depending on β. So by the cancelation property of $A(\cdot)$, we have

$$\begin{aligned}\widehat{\sigma'_k}(\xi) &= \int_{I_k}\int_R A(s)e^{it^\beta}\{e^{-it|\xi|s}-1\}\mathrm{d}st^{-1}\mathrm{d}t\\ &= \int_R\int_{I_k} e^{it^\beta - t|\xi|\tau s}\mathrm{d}tA(s)i|\xi|s\mathrm{d}s.\end{aligned}$$

Since

$$(\frac{\partial}{\partial t})^2(t^\beta - t|\xi|\tau s) \ge C2^{k(\beta-2)},$$

where $\tau \in (0,1)$ and C is a constant depending on β. Therefore, by Lemma 2.3, we obtain

$$|\widehat{\sigma'_k}(\xi)| \le \int_R\Big|\int_{2^k}^{2^{k+1}} e^{it^\beta - -t|\xi|\tau s}\mathrm{d}t\Big|A(s)|\xi||s|\mathrm{d}s \le C2^{k(1-\beta/2)}|A_\rho\xi|.$$

It follows that

$$|\widehat{\sigma_k}(\xi)| \le C2^{k(1-\beta/2)}|A_\rho\xi|. \tag{2.14}$$

Therefore, by (2.13) and (2.14) for $\theta = \beta/(\beta+4) \in (0,1)$, we have

$$|\widehat{\sigma'_k}(\xi)| \le C2^{k(1-\beta/2)\theta}|A_\rho\xi|^\theta 2^{[k(1-\theta)/2]}.$$

That is,

$$|\widehat{\sigma'_k}(\xi)| \le C(2^k|A_\rho\xi|)^\theta,$$

where $\theta = \beta/(\beta+4)$. Thus, the proof of Lemma 2.5 is completed. □

Lemma 2.6 ([5]) *If $\psi \in S(\mathbf{R}^n)$ with $\mathrm{supp}(\psi) \subset \{x : 1/2 \le |x| \le 2\}$ and for $k \in \mathbf{Z}$, define the multiplier S_k by $\widehat{S_k f}(\xi) = \psi(2^k\xi)\hat{f}(\xi)$. Then for $1 < p,\ q < \infty$, we have*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|S_k f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of j and k.

Lemma 2.7 ([5]) *Suppose $\psi \in S(\mathbf{R}^n)$ with $\mathrm{supp}(\psi) \subset \{x : 1/2 \le |x| \le 2\}$. Denote $A_\rho x = (\rho^2 x_1, \rho x_1, \dots, \rho x_n)$ for $\rho > 0$ and $x \in \mathbf{R}^n$. Define the multiplier $S_{k,\rho}$ by $\widehat{S_{k,\rho} f}(x) = \psi_{k,\rho}(x)\hat{f}(x)$, where $k \in \mathbf{Z}$ and $\psi_{k,\rho}(\xi) = \psi(2^k A_\rho x)$. Then for $1 < p,\ q < \infty$,*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|S_{k,\rho} f_j|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of j, ρ and k.

Let M_Ω denote the rough maximal operator with $\Omega \in L^1(S^{n-1})$ defined by

$$M_\Omega f(x) = \sup_{r>0}\frac{1}{r^n}\int_{|y|<r}|\Omega(y)||f(x-y)|\mathrm{d}y.$$

Lemma 2.8 ([5]) *Let $1 < p,\ q < \infty$, $\{f_j\} \in L^p(l^q)$ and $\Omega \in L^1(S^{n-1})$. Then*

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}|M_\Omega f_j|^q\Big)^{1/q}\Big\|_{L^p} \le C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of Ω and $\{f_j\}$.

Lemma 2.9 *Let $1 < p,\ q < \infty$, $h(x)$ be a function on $\mathbf{R}^+$. Denote*

$$\sigma_k(x) = e^{ih(|x|)}|x|^{-n}|\Omega(x')|\chi(2^{k-1} < |x| \le 2^k).$$

Suppose $\{(\sum_{k\in\mathbf{Z}}|g_{k,j}|^2)^{1/2}\}_j \in L^p(l^q)$ and $\Omega \in L^1(S^{n-1})$. Then

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p} \le C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p},$$

where C is independent of Ω and $\{g_{k,j}\}$.

Proof Note that

$$\sup_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}| \le \sup_{k\in\mathbf{Z}}|\sigma_k| * |g_{k,j}| \le M_\Omega(\sup_{k\in\mathbf{Z}}|g_{k,j}|).$$

Therefore, we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\big(\sup_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}|\big)^q\Big)^{1/q}\Big\|_{L^p} \le \Big\|\Big(\sum_{j\in\mathbf{Z}}\big(M_\Omega(\sup_{k\in\mathbf{Z}}|g_{k,j}|)\big)^q\Big)^{1/q}\Big\|_{L^p}.$$

By Lemma 2.8, we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\big(\sup_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}|\big)^q\Big)^{1/q}\Big\|_{L^p} \le C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}\big(\sup_{k\in\mathbf{Z}}|g_{k,j}|\big)^q\Big)^{1/q}\Big\|_{L^p}. \tag{2.15}$$

On the other hand, there exists a sequence $\{h_j\} \in L^{p'}(l^{q'})$ such that

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\big(\sum_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}|\big)^q\Big)^{1/q}\Big\|_{L^p} = \Big|\sum_{j\in\mathbf{Z}}\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|\sigma_k * g_{k,j}|(x)h_j(x)\mathrm{d}x\Big|$$

$$\leq \sum_{j\in\mathbf{Z}}\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|g_{k,j}|(x)\sup_{k\in\mathbf{Z}}|\widetilde{\sigma_k}*|h_j|(x)\mathrm{d}x$$
$$\leq C\sum_{j\in\mathbf{Z}}\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}|g_{k,j}(x)||M_{\tilde{\Omega}}h_j(x)|\mathrm{d}x,$$

where $\widetilde{\sigma_k}(x)=\sigma_k(-x)$ and $\widetilde{\Omega}(x)=\Omega(-x)$. By Hölder's inequality and (2.4), we have

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\sigma_k*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p}\leq C\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p}\Big\|\Big(\sum_{j\in\mathbf{Z}}|M_{\tilde{\Omega}}h_j|\Big)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}}$$
$$\leq C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p}\Big\|\Big(\sum_{j\in\mathbf{Z}}|h_j|\Big)^{q'}\Big)^{1/q'}\Big\|_{L^{p'}}.$$

It follows that

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\sigma_k*g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p}\leq C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|g_{k,j}|\Big)^q\Big)^{1/q}\Big\|_{L^p}. \tag{2.16}$$

By an interpolating between (2.15) and (2.16) [14], we obtain

$$\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|\sigma_k*g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}\leq C\|\Omega\|_1\Big\|\Big(\sum_{j\in\mathbf{Z}}\Big(\sum_{k\in\mathbf{Z}}|g_{k,j}|^2\Big)^{q/2}\Big)^{1/q}\Big\|_{L^p}.$$

Thus, the proof of Lemma 2.9 is completed. □

3. Proofs of theorems

Proof of Theorem 1.1 For the given polynomial $P(s)=P_N(s)=\sum_{m=0}^N\beta_m s^m$, without loss of generality we may assume that $\beta_N=1$. Otherwise, let $A=|\beta_N|^{\frac{1}{N}}$. Write

$$P_N(|x|)=\sum_{m=0}^N\frac{\beta_m}{A^m}(A|x|)^m:=Q(A|x|).$$

Then by making a change of variable, we have

$$T_{\Omega,P}f(\frac{x}{A})=\text{p.v.}\int_{\mathbf{R}^n}e^{iQ(|x-y|)}\frac{\Omega(x-y)}{|x-y|^n}f(\frac{y}{A})\mathrm{d}y.$$

Noticing that $\|f(\frac{\cdot}{A})\|_{\dot{F}_p^{\alpha,q}}\sim A^{-\alpha+\frac{p}{n}}\|f(\cdot)\|_{\dot{F}_p^{\alpha,q}}$, thus we only need considering the case $A=1$. By (2.4), we have

$$T_{\Omega,P}f(x)=\sum_{j=1}^{\infty}\lambda_jT_jf(x), \tag{3.1}$$

where

$$T_jf(x)=e^{iP(|\cdot|)}\frac{a(\cdot)}{|\cdot|^n}*f(x).$$

Therefore,

$$T_jf(x)=\sum_{k\in\mathbf{Z}}\sigma_k*f(x),$$

where $\sigma_k(x)=e^{iP(|x|)}\frac{a_j(x')}{|x|^n}\chi(2^{k-1}<|x|\leq 2^k)$. We take $\psi\in S(\mathbf{R}^n)$ with $\mathrm{supp}(\psi)\subset\{x\in\mathbf{R}^n 1/2\leq|x|\leq 2\}$. In addition, we may claim ψ to satisfy $\sum_{k\in\mathbf{Z}}(\psi(2^k\xi))^2=1$ for $\xi\neq 0$.

Therefore by the definition of $S_{k,\rho}$, $f=\sum_{k\in\mathbf{Z}} S_{k,\rho}(S_{k,\rho}f)$ for any $f\in S(\mathbf{R}^n)$. We have

$$\begin{aligned} T_j f(x) &= \sum_{k\in\mathbf{Z}} \sigma_k * \big(\sum_{i\in\mathbf{Z}} S_{i+k,\rho}(S_{i+k,\rho}f)\big) \\ &= \sum_{i\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} S_{i+k,\rho}(\sigma_k * S_{i+k,\rho}f) =: \sum_{i\in\mathbf{Z}} B_i f(x). \end{aligned}$$

It follows that

$$\|T_j f\|_{\dot F_p^{\alpha,q}} \le \sum_{i\in\mathbf{Z}} \|B_i f\|_{\dot F_p^{\alpha,q}}. \tag{3.2}$$

With a similar argument to the proof of Theorem 1 in [5], by Lemmas 2.6, 2.7 and 2.9, we easily get

$$\Big\|\big(\sum_l |B_i f_l|^q\big)^{1/q}\Big\|_{L^p} \le C\Big\|\big(\sum_l |f_l|^q\big)^{1/q}\Big\|_{L^p}, \tag{3.3}$$

where $1<p,\ q<\infty$, $\alpha\in\mathbf{R}$ and C is independent of i, l and ρ. Therefore by (3.3), we have

$$\begin{aligned} \|B_i f\|_{\dot F_p^{\alpha,q}} &\le \Big\|\big(\sum_{l\in\mathbf{Z}} 2^{-l\alpha q}|\Phi_l * B_i f)|^q\big)^{\frac{1}{q}}\Big\|_{L^p} \\ &\le \Big\|\big(\sum_{l\in\mathbf{Z}} |B_i(2^{-l\alpha}\Phi_l * f|^q\big)^{\frac{1}{q}}\Big\|_{L^p} \\ &\le C\Big\|\big(\sum_{l\in\mathbf{Z}} 2^{-l\alpha q}|\Phi_l * f)|^q\big)^{\frac{1}{q}}\Big\|_{L^p}. \end{aligned}$$

It follows that

$$\|B_i f\|_{\dot F_p^{\alpha,q}} \le C\|f\|_{\dot F_p^{\alpha,q}}. \tag{3.4}$$

By properties (1), Lemma 2.4 and Plancherel's theorem, we obtain

$$\|B_i f\|_{F_2^{0,2}} \le C2^{-\delta|i|}\|f\|_{F_2^{0,2}}, \tag{3.5}$$

where C is independent of i and ρ.

Therefore, by an interpolating between (3.4) and (3.5), there exists $\eta>0$ such that

$$\|B_i f\|_{\dot F_p^{\alpha,q}} \le C2^{-\eta|i|}\|f\|_{\dot F_p^{\alpha,q}}. \tag{3.6}$$

By (3.1), (3.2) and (3.6), we have

$$\begin{aligned} \|T_{\Omega,P} f\|_{\dot F_p^{\alpha,q}} &\le \sum_{j=1}^{\infty} \lambda_j \|T_j f\|_{\dot F_p^{\alpha,q}} \\ &\le \sum_{j=1}^{\infty} \lambda_j \sum_{i\in\mathbf{Z}} \|B_i f\|_{\dot F_p^{\alpha,q}} \\ &\le C\sum_{j=1}^{\infty} \lambda_j \sum_{i\in\mathbf{Z}} \|f\|_{\dot F_p^{\alpha,q}} \\ &\le C\|\Omega\|_{H^1(S^{n-1})}\|f\|_{\dot F_p^{\alpha,q}}. \end{aligned}$$

This completes the proof of Theorem 1.1. □

From the proof of Theorem 1.1, by Lemmas 2.5, 2.6, 2.7 and 2.9, Theorem 1.2 can be easily proved. Here, we omit the details.

Proof of Theorem 1.3 By Theorem 9.1 in [3], we know T is L^p boundedness for $1 < p < \infty$. Therefore, for $1 < p$, $q < \infty$, we have

$$\Big(\sum_{l\in\mathbf{Z}}\|Tf_l\|_p^q\Big)^{1/q}\le C\Big(\sum_{l\in\mathbf{Z}}\|f_l\|_p^q\Big)^{1/q}.$$

We take Φ_l as Definition 1.1. Therefore, for any $1 < p, q < \infty$ and $\alpha \in \mathbf{R}$, we obtain

$$\begin{aligned}\|Tf\|_{\dot{B}_p^{\alpha,q}} &= \Big(\sum_{l\in\mathbf{Z}}2^{-l\alpha q}\|\Phi_l * Tf\|_p^q\Big)^{1/q}\le\Big(\sum_{l\in\mathbf{Z}}\|T(2^{-l\alpha}\Phi_l * f)\|_p^q\Big)^{1/q}\\ &\le C\Big(\sum_{l\in\mathbf{Z}}2^{-l\alpha q}\|\Phi_l * f\|_p^q\Big)^{1/q}=C\|f\|_{\dot{B}_p^{\alpha,q}}.\end{aligned}$$

This completes the proof of Theorem 1.3. □

References

[1] RICCI F, STEIN E M. *Harmonic analysis on nilpotent groups and singular integrals. I. Oscillatory integrals* [J]. J. Funct. Anal., 1987, **73**(1): 179–194.

[2] LU Shanzhen, ZHANG Yan. *Criterion on L^p-boundedness for a class of oscillatory singular integrals with rough kernels* [J]. Rev. Mat. Iberoamericana, 1992, **8**(2): 201–219.

[3] FAN Dashan, PAN Yibiao. *Singular integral operators with rough kernels supported by subvarieties* [J]. Amer. J. Math., 1997, **119**(4): 799–839.

[4] CHEN Jiecheng, JIA Houyu, JIANG Liya. *Boundedness of rough oscillatory singular integral on Triebel-Lizorkin spaces* [J]. J. Math. Anal. Appl., 2005, **306**(2): 385–397.

[5] CHEN Yanping, DING Yong. *Rough singular integrals on Triebel-Lizorkin space and Besov space* [J]. J. Math. Anal. Appl., 2008, **347**(2): 493–501.

[6] CHANILLO S, KURTZ D S, SAMPSON G. *Weighted L^p estimates for oscillating kernels* [J]. Ark. Mat., 1983, **21**(2): 233–257.

[7] CHANILLO S, KURTZ D S, SAMPSON G. *Weighted weak $(1,1)$ and weighted L^p estimates for oscillating kernels* [J]. Trans. Amer. Math. Soc., 1986, **295**(1): 127–145.

[8] JIANG Liya, CHEN Jiecheng. *A class of oscillatory singular integrals on Triebel-Lizorkin spaces* [J]. Appl. Math. J. Chinese Univ. Ser. B, 2006, **21**(1): 69–78.

[9] TRIEBEL H. *Theory of Function Spaces* [M]. Birkhäuser Verlag, Basel, 1983.

[10] TRIEBEL H. *Theory of Function Spaces (II)* [M]. Birkhäuser Verlag, Basel, 1992.

[11] FRAZIER M, JAWERTH B. *Decomposition of Besov spaces* [J]. Indiana Univ. Math. J., 1985, **34**(4): 777–799.

[12] COLZANI. Hardy space on sphere [D]. Ph. D. Thesis, Washington University, St. Louis, 1982.

[13] FAN Dashan, PAN Yibiao. *A singular integral operator with rough kernel* [J]. Proc. Amer. Math. Soc., 1997, **125**(12): 3695–3703.

[14] BERGH J, LÖFSTRÖM J. *Interpolation Spaces* [M]. Springer-Verlag, Berlin-New York, 1976.