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New Rapidly Convergent Series Concerning $\zeta(2k+1)$

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Abstract Values of new series

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k)!}\alpha^{2n},\quad \sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k+1)!}\beta^{2n}$$

are given concerning $\zeta(2k+1)$, where k is a positive integer, α can be taken as 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/6, 5/6 and β can be taken as 1, 1/2. Some previous results are included as special cases in the present paper and new series converges more rapidly than those exsiting results for $\alpha=1/3$, or $\alpha=1/4$, or $\alpha=1/6$.

Keywords Riemann zeta function; rapidly convergent series.

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1. Introduction

Riemann zeta function is defined as follows

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s},\quad \mathrm{Re}(s)>1. \tag{1.1}$$

For $\mathrm{Re}(s)\le 1$, $s\ne 1$, $\zeta(s)$ is defined as the analytic continuation of (1.1). Therefore, $\zeta(s)$ is analytic for all complex plane except for a simple pole at $s=1$ with residue 1. Based on this, we have

$$\zeta(0)=-\frac{1}{2},\quad \zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\quad n=1,2,3,\ldots,$$

and

$$\zeta(-2n)=0,\quad \zeta(-2n+1)=-\frac{B_{2n}}{2n},$$

where the rational numbers B_{2n} are the Bernoulli numbers, that is, $\zeta(2n)$ can be expressed as a rational multiple of π^{2n}. There is no analogous closed evaluation for $\zeta(2n+1)$, and various series and integral representations have been found (for example see [1–3]).

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In 1978 Apéry proved another remarkable result that $\zeta(3)$ is an irrational number based on rapidly converging series [4–6]

$$\zeta(3)=\frac{5}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^3\binom{2k}{k}}.$$

There exist related simple formulae for $\zeta(2)$ and $\zeta(4)$, but expressions for $\zeta(2n+1)$, $n\geq 2$, are much more complicated [5, 7, 8].

Ewell [9] found a new simple series

$$\zeta(3)=-\frac{4\pi^2}{7}\sum_{k=0}^{\infty}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}, \tag{1.2}$$

and Ewell [10] showed that there exists a multiple series representation of $\zeta(n)$ in the form

$$\zeta(n)=-\frac{2^{n-2}\pi^2}{2^n-1}\sum_{m=0}^{\infty}\frac{(-1)^m A_{2m}(n-2)\pi^{2m}}{(2m+2)!},\quad n=3,4,5,\ldots, \tag{1.3}$$

where the coefficients $A_{2m}(n)$ are the finite sums which involve multinomial coefficients and the Bernoulli numbers. Ewell's method was modified by Zhang and Williams in [11]. They obtained several series analogous to that in [3] and found a new formula for $\zeta(2n+1), n\geq 2$. Although still complicated, their representation is simpler than that given by [10]. Again Ewell [12] deduced a new series representation for $\zeta(2n+1)$ in a determinantal form. Further, Cvijović and Klinowski [13] found two series representations for $\zeta(2n+1)$, $n\geq 1$, that is

$$\zeta(2n+1)=(-1)^n\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\Big[\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{\pi^{2k}(2n-2k)!}+\sum_{k=0}^{\infty}\frac{\zeta(2k)(2k)!}{2^{2k}(2k+2n)!}\Big] \tag{1.4}$$

and

$$\zeta(2n+1)=(-1)^n\frac{4(2\pi)^{2n}}{(2n+1)!}\sum_{k=0}^{\infty}R_{2n+1,k}\zeta(2k), \tag{1.5}$$

where

$$R_{2n+1,k}=\sum_{m=0}^{2n}\binom{2n}{m}\frac{(2n+1)B_{2n-m}}{2^{2k+m+1}(2k+m+1)(m+1)},\quad k=0,1,2,\ldots,$$

which are of rapid convergence.

Now in the present paper, by taking the advantage of Fourier's expansion

$$\log\big(2\mathrm{sin}\frac{\theta}{2}\big)=-\sum_{\mathrm{n}=1}^{\infty}\frac{\mathrm{cosn}\theta}{\mathrm{n}},\quad 0<\theta<2\pi, \tag{1.6}$$

and constructing integrals

$$\int_0^{\alpha}(\alpha-x)^k\log(2\mathrm{sin}\pi x)\mathrm{d}x,\quad k=1,2,3,\ldots,$$

we give a family of new series

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k)!}\alpha^{2n},\quad \sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k+1)!}\beta^{2n}, \tag{1.7}$$

which concern $\zeta(2l+1)$, $l = 1, 2, 3, \ldots, k$, where k is a positive integer, α can be taken as 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/6, 5/6, and β can be taken as 1, 1/2. All main results in [9–13] are included as our special cases and new series converges more rapidly than those in [9–13] for α =1/3, or α =1/4, or α =1/6.

2. Results and proofs

The results and proofs are as follows.

Theorem 2.1 *For any positive integral k and $0 < \alpha \le 1$, we have the formula*

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k)!}\alpha^{2n} = \frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}\zeta(2l+1)}{(2\pi\alpha)^{2l}(2k-2l)!} + \frac{(-1)^k}{2(2\pi\alpha)^{2k}}\sum_{n=1}^{\infty}\frac{\cos(2n\pi\alpha)}{n^{2k+1}} + \frac{1}{2(2k)!}\left(\log(2\pi\alpha) + \sum_{l=1}^{2k}(-1)^l l^{-1}\binom{2k}{l}\right). \tag{2.1}$$

Proof Recalling Euler's formula

$$\sin\pi x = \pi x\prod_{k=1}^{\infty}(1-\frac{x^2}{k^2}), \quad 0 < x < 1, \tag{2.2}$$

and taking logarithms on both sides of (2.2), we have

$$\begin{aligned}\log(2\sin\pi x) - \log(2\pi x) &= \sum_{k=1}^{\infty}\log(1-\frac{x^2}{k^2}) = -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{2n}}{nk^{2n}}\\ &= -\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)\frac{x^{2n}}{n} = -\sum_{n=1}^{\infty}\frac{x^{2n}\zeta(2n)}{n}.\end{aligned} \tag{2.3}$$

Multiplying $(\alpha - x)^{2k-1}$ in both sides of (2.3), and integrating from $x = 0$ to $x = \alpha$, we have

$$\begin{aligned}&\int_0^{\alpha}(\alpha-x)^{2k-1}\log(2\sin\pi x)\mathrm{d}x - \int_0^{\alpha}(\alpha-x)^{2k-1}\log(2\pi x)\mathrm{d}x\\ &= -\int_0^{\alpha}(\alpha-x)^{2k-1}\sum_{n=1}^{\infty}\frac{x^{2n}\zeta(2n)}{n}\mathrm{d}x,\end{aligned} \tag{2.4}$$

where $0 < \alpha \le 1$ and k is a positive integer. By Fourier expansion (1.6), replacing x by $\frac{\theta}{2\pi}$ and integrating by parts, we deduce that

$$\begin{aligned}&\int_0^{\alpha}(\alpha-x)^{2k-1}\log(2\sin\pi x)\mathrm{d}x\\ &= \frac{1}{(2\pi)^{2k}}\int_0^{2\pi\alpha}(2\pi\alpha-\theta)^{2k-1}\log(2\sin\frac{\theta}{2})\mathrm{d}\theta\\ &= -\frac{1}{(2\pi)^{2k}}\int_0^{2\pi\alpha}(2\pi\alpha-\theta)^{2k-1}\sum_{n=1}^{\infty}\frac{\cos n\theta}{n}\mathrm{d}\theta\\ &= -\frac{1}{(2\pi)^{2k}}\int_0^{2\pi\alpha}(2\pi\alpha-\theta)^{2k-1}d(\sum_{n=1}^{\infty}\frac{\sin n\theta}{n^2})\end{aligned}$$

$$= \frac{2k-1}{(2\pi)^{2k}} \int_0^{2\pi\alpha} (2\pi\alpha - \theta)^{2k-2} d(\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^3})$$
$$= -\frac{(2k-1)\alpha^{2k-2}}{(2\pi)^2}\zeta(3) + \frac{(2k-1)!}{(2\pi)^{2k}(2k-3)!} \int_0^{2\pi\alpha} (2\pi\alpha - \theta)^{2k-3} d(\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^4})$$
$$= \cdots$$
$$= \sum_{l=1}^{k} \frac{(-1)^l (2k-1)!\zeta(2l+1)}{(2\pi)^{2l}(2k-2l)!}\alpha^{2k-2l} + \frac{(-1)^{k-1}(2k-1)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{\cos(2n\pi\alpha)}{n^{2k+1}}, \tag{2.5}$$

and

$$-\int_0^{\alpha} (\alpha - x)^{2k-1} \log(2\pi x) \mathrm{d}x = \frac{1}{2k} \int_0^{\alpha} \log(2\pi x) d[(\alpha - x)^{2k} - \alpha^{2k}]$$
$$= -\frac{\alpha^{2k}}{2k} \left(\log(2\pi\alpha) + \sum_{l=1}^{2k} (-1)^l l^{-1} \binom{2k}{l} \right). \tag{2.6}$$

On the other hand, integrating by parts, we deduce that

$$-\int_0^{\alpha} (\alpha - x)^{2k-1} \sum_{n=1}^{\infty} \frac{x^{2n}\zeta(2n)}{n} \mathrm{d}x = -2(2k-1)! \sum_{n=1}^{\infty} \frac{(2n-1)!\zeta(2n)}{(2n+2k)!}\alpha^{2n+2k}. \tag{2.7}$$

Combining (2.4), (2.5), (2.6) and (2.7) gives the formula (2.1).

Theorem 2.2 *For any positive integral k and $0 < \beta \leq 1$, we have the formula*

$$\sum_{n=1}^{\infty} \frac{(2n-1)!\zeta(2n)}{(2n+2k+1)!}\beta^{2n} = \frac{1}{2} \sum_{l=1}^{k} \frac{(-1)^{l-1}\zeta(2l+1)}{(2\pi\beta)^{2l}(2k-2l+1)!} + \frac{(-1)^k}{2(2\pi\beta)^{2k+1}} \sum_{n=1}^{\infty} \frac{\sin(2n\pi\beta)}{n^{2k+2}} +$$
$$\frac{1}{2(2k+1)!} \left(\log(2\pi\beta) + \sum_{l=1}^{2k+1} (-1)^l l^{-1} \binom{2k+1}{l} \right). \tag{2.8}$$

Proof Similarly to Theorem 2.1, multiplying $(\beta - x)^{2k}$ in both sides of (2.3), and integrating from $x = 0$ to $x = \beta$, we deduce that Theorem 2.2 holds.

Noting that

$$\sum_{n=1}^{\infty} \frac{\cos(2n\pi)}{n^{2k+1}} = \zeta(2k+1),$$
$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{2k+1}} = -\frac{2^{2k}-1}{2^{2k}}\zeta(2k+1),$$
$$\sum_{n=1}^{\infty} \frac{\cos(2n\pi/3)}{n^{2k+1}} = \sum_{n=1}^{\infty} \frac{\cos(4n\pi/3)}{n^{2k+1}} = -\frac{3^{2k}-1}{2 \times 3^{2k}}\zeta(2k+1),$$
$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/2)}{n^{2k+1}} = \sum_{n=1}^{\infty} \frac{\cos(3n\pi/2)}{n^{2k+1}} = -\frac{2^{2k}-1}{2^{4k+1}}\zeta(2k+1),$$
$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n^{2k+1}} = \sum_{n=1}^{\infty} \frac{\cos(5n\pi/3)}{n^{2k+1}} = \frac{(2^{2k}-1)(3^{2k}-1)}{2^{2k+1}3^{2k}}\zeta(2k+1),$$

and making $\alpha = 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$ in (2.1), and making $\beta = 1, \frac{1}{2}$ in (2.8), we obtain the following formulas.

Corollary 2.1 *For any positive integral* k*, we have*

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k-1}\frac{(-1)^{l-1}\zeta(2l+1)}{(2\pi)^{2l}(2k-2l)!}+\frac{1}{2(2k)!}\left(\log 2\pi+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.9}$$

where the finite sum on the right-hand side is 0 when $k=1$*,*

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{2^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}\zeta(2l+1)}{\pi^{2l}(2k-2l)!}+\frac{(-1)^{k-1}(2^{2k}-1)\zeta(2k+1)}{2(2\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\pi+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.10}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{3^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}3^{2l}\zeta(2l+1)}{(2\pi)^{2l}(2k-2l)!}+\frac{(-1)^{k-1}(3^{2k}-1)\zeta(2k+1)}{4(2\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{2\pi}{3}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.11}$$

$$\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)!\zeta(2n)}{3^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}3^{2l}\zeta(2l+1)}{(4\pi)^{2l}(2k-2l)!}+\frac{(-1)^{k-1}(3^{2k}-1)\zeta(2k+1)}{4(4\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{4\pi}{3}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.12}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{4^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}2^{2l}\zeta(2l+1)}{\pi^{2l}(2k-2l)!}+\frac{(-1)^{k-1}(2^{2k}-1)\zeta(2k+1)}{4(2\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{\pi}{2}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.13}$$

$$\sum_{n=1}^{\infty}\frac{3^{2n}(2n-1)!\zeta(2n)}{4^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}2^{2l}\zeta(2l+1)}{(3\pi)^{2l}(2k-2l)!}+\frac{(-1)^{k-1}(2^{2k}-1)\zeta(2k+1)}{4(6\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{3\pi}{2}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.14}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{6^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}3^{2l}\zeta(2l+1)}{\pi^{2l}(2k-2l)!}+\frac{(-1)^{k}(2^{2k}-1)(3^{2k}-1)\zeta(2k+1)}{4(2\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{\pi}{3}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.15}$$

$$\sum_{n=1}^{\infty}\frac{5^{2k}(2n-1)!\zeta(2n)}{6^{2n}(2n+2k)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}3^{2l}\zeta(2l+1)}{(5\pi)^{2l}(2k-2l)!}+\frac{(-1)^{k}(2^{2k}-1)(3^{2k}-1)\zeta(2k+1)}{4(10\pi)^{2k}}+\frac{1}{2(2k)!}\left(\log\frac{5\pi}{3}+\sum_{l=1}^{2k}(-1)^{l}l^{-1}\binom{2k}{l}\right), \tag{2.16}$$

and

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{(2n+2k+1)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}\zeta(2l+1)}{(2\pi)^{2l}(2k-2l+1)!}+\frac{1}{2(2k+1)!}\left(\log 2\pi+\sum_{l=1}^{2k+1}(-1)^{l}l^{-1}\binom{2k+1}{l}\right), \tag{2.17}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{2^{2n}(2n+2k+1)!}=\frac{1}{2}\sum_{l=1}^{k}\frac{(-1)^{l-1}\zeta(2l+1)}{\pi^{2l}(2k-2l+1)!}+\frac{1}{2(2k+1)!}\left(\log\pi+\sum_{l=1}^{2k+1}(-1)^{l}l^{-1}\binom{2k+1}{l}\right). \tag{2.18}$$

Corollary 2.2 *For any positive integral k, we have*

$$\sum_{n=1}^{\infty}\frac{(2n)!\zeta(2n)}{2^{2n}(2n+2k)!}=\sum_{l=1}^{k-1}\frac{(-1)^{l}l\zeta(2l+1)}{\pi^{2l}(2k-2l)!}+\frac{(-1)^{k}k(2^{2k+1}-1)\zeta(2k+1)}{(2\pi)^{2k}}+\frac{1}{2(2k)!}, \tag{2.19}$$

where the finite sum on the right-hand side is 0 when $k=1$.

Proof Replacing k by $k-1$ in (2.18), we have

$$\sum_{n=1}^{\infty}\frac{(2n-1)!\zeta(2n)}{2^{2n}(2n+2k-1)!}=\frac{1}{2}\sum_{l=1}^{k-1}\frac{(-1)^{l-1}\zeta(2l+1)}{\pi^{2l}(2k-2l-1)!}+\frac{1}{2(2k-1)!}\left(\log\pi+\sum_{l=1}^{2k-1}(-1)^{l}l^{-1}\binom{2k-1}{l}\right). \tag{2.20}$$

Noting that

$$\frac{(2n-1)!}{(2n+2k-1)!}-\frac{2k(2n-1)!}{(2n+2k)!}=\frac{(2n)!}{(2n+2k)!}$$

with (2.10) and (2.20), we obtain (2.19).

As special case for (2.9)–(2.19), we get a family of $\zeta(3)$:

$$\zeta(3)=2\pi^2\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(n+1)(n+2)(2n+1)(2n+3)}-\frac{1}{6}\log 2\pi+\frac{25}{72}\right), \tag{2.21}$$

$$\zeta(3)=\frac{2\pi^2}{7}\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}n(n+1)(2n+1)}-\log\pi+\frac{3}{2}\right), \tag{2.22}$$

$$\zeta(3)=\frac{2\pi^2}{13}\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{3^{2n}n(n+1)(2n+1)}-\log\frac{2\pi}{3}+\frac{3}{2}\right), \tag{2.23}$$

$$\zeta(3)=\frac{8\pi^2}{13}\left(\sum_{n=1}^{\infty}\frac{2^{2n}\zeta(2n)}{3^{2n}n(n+1)(2n+1)}-\log\frac{4\pi}{3}+\frac{3}{2}\right), \tag{2.24}$$

$$\zeta(3)=\frac{4\pi^2}{35}\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{4^{2n}n(n+1)(2n+1)}-\log\frac{\pi}{2}+\frac{3}{2}\right), \tag{2.25}$$

$$\zeta(3)=\frac{36\pi^2}{35}\left(\sum_{n=1}^{\infty}\frac{3^{2n}\zeta(2n)}{4^{2n}n(n+1)(2n+1)}-\log\frac{3\pi}{2}+\frac{3}{2}\right), \tag{2.26}$$

$$\zeta(3) =\frac{\pi^2}{12}\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{6^{2n}n(n+1)(2n+1)}-\log\frac{\pi}{3}+\frac{3}{2}\right), \tag{2.27}$$

$$\zeta(3) =\frac{25\pi^2}{12}\left(\sum_{n=1}^{\infty}\frac{5^{2n}\zeta(2n)}{6^{2n}n(n+1)(2n+1)}-\log\frac{5\pi}{3}+\frac{3}{2}\right), \tag{2.28}$$

$$\zeta(3) =2\pi^2\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(n+1)(2n+1)(2n+3)}-\frac{1}{3}\log 2\pi+\frac{11}{18}\right), \tag{2.29}$$

$$\zeta(3) =\frac{\pi^2}{2}\left(\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}n(n+1)(2n+1)(2n+3)}-\frac{1}{3}\log\pi+\frac{11}{18}\right), \tag{2.30}$$

$$\zeta(3) =-\frac{4\pi^2}{7}\left(\sum_{n=0}^{\infty}\frac{\zeta(2n)}{2^{2n}(2n+1)(2n+2)}\right), \tag{2.31}$$

(2.31) is (1.4). So does $\zeta(5)$, and so on. □

Remark We note that (2.19) is exactly (1.4). Also, making another form combining (2.10) and (2.18), we get (1.5). For example, combining (2.22) and (2.30), we deduce that

$$\zeta(3) = -\frac{8\pi^2}{5}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{2^{2n}(2n+1)(2n+2)(2n+3)}. \tag{2.32}$$

Combining (2.31) and (2.32) gives

$$\zeta(3) = -\frac{\pi^2}{3}\sum_{n=0}^{\infty}\frac{(2n+5)\zeta(2n)}{2^{2n}(2n+1)(2n+2)(2n+3)}, \tag{2.33}$$

that is (17a) in [13].

References

[1] RAMANUJAN S. *Notebooks of Srinivasa Ramanujan (2 vols.)* [M]. Tata Institute of Fundamental Research, Bombay, 1957.

[2] BERNDT B C. *Modular transformations and generalizations of several formulae of Ramanujan* [J]. Rocky Mountain J. Math., 1977, **7**(1): 147–189.

[3] BERNDT B. *Ramanujan's Notebooks, Part II* [M]. Springer, New York, 1989.

[4] APÉRY R. *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$ [J]. Astérisque, 1979, **61**: 11–13.

[5] COHEN H. *G'enéralisation d'une construction de R. Apéry* [J]. Bull. Soc. Math. France, 1981, **109**(3): 269–281. (in French)

[6] A J Van der Poorten. *A proof of Euler missed* $\cdots$ *Apéry's proof of the irrationality of* $\zeta(3)$ [J]. Math. Intelligencer, 1979, **1**(4): 195–203.

[7] LESHCHINER D. *Some new identities for* $\zeta(k)$ [J]. J. Number Theory, 1981, **13**(3): 355–362.

[8] BUTZER P L, MARKETT C, SCHMIDT M. *Stirling numbers, central factorial numbers, representation of the Riemann zeta function* [J]. Resultate Math., 1991, **19**: 257–274.

[9] EWELL J A. *A new series representation for* $\zeta(3)$ [J]. Amer. Math. Monthly, 1990, **97**(3): 219–220.

[10] EWELL J A. *On values of the Riemann zeta function at integral arguments* [J]. Canad. Math. Bull., 1991, **34**(1): 60–66.

[11] ZHANG Nanyue, WILLIAMS K S. *Some series representation of* $\zeta(2n+1)$ [J]. Rocky Mountain J. Math., 1993, **23**(4): 1581–1592.

[12] EWELL J A. *On the zeta function values* $\zeta(2n+1), k=1,2,\ldots$ [J]. Rocky Mountain J. Math., 1995, **25**(3): 1003–1012.

[13] CVIJOVIĆ D, KLINOWSKI J. *New rapidly convergent series representation for* $\zeta(2n+1)$ [J]. Pro. Amer. Math. Soc., 1997, **125**(5): 1263–1271.