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Stability of (p, Y)-Operator Frames

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Abstract In this paper we study the stability of (p, Y)-operator frames. We firstly discuss the relations between p-Bessel sequences (or p-frames) and (p, Y)-operator Bessel sequences (or (p, Y)-operator frames). Through defining a new union, we prove that adding some elements to a given (p, Y)-operator frame, the resulted sequence will be still a (p, Y)-operator frame. We obtain a necessary and sufficient condition for a sequence of compound operators to be a (p, Y)-operator frame. Lastly, we show that (p, Y)-operator frames for X are stable under some small perturbations.

Keywords p-frame; (p, Y)-operator Bessel sequence; (p, Y)-operator frame; perturbation; Banach space.

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1. Introduction

In 1946, Gabor [26] discussed a decomposition of signals in terms of elementary signals. In 1952, Duffin and Schaeffer [21] generalized Gabor's fundamental idea and firstly introduced the notion of frames in a Hilbert space when they studied nonharmonic Fourier analysis. But their work was not continued until 1986 when Daubechies, Grossman and Meyer [19] found that the functions in $L^2(\mathbb{R})$ could be expressed as the series which was similar to the orthonormal basis by using the theory of frames and applied the theory of frames to wavelet and Gabor transform. From then on, the theory of frames began to be studied widely and deeply [4, 6, 8, 10, 18]. Recently, the theory of frames for Hilbert spaces has been generalized in several directions [3, 9, 11, 12, 17, 25, 29–32] and applied to signal processing, image processing, data compressing and sampling theory and so on [26–28].

In 1990s, Grochenig, Aldroubi, Sung and Tang began to study the theory of frames in Banach spaces [27]. They introduced two kinds of notions of frames in a Banach space: Banach frames

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and p-frames ($1 < p < \infty$). Sun [35, 36] introduced and discussed the concept of g-frames for a Hilbert space, which generalizes the concepts of frames [6], pseudo frames [34], oblique frames [13, 22], outer frames [1], bounded quasi-projectors [23, 24] and frames of subspaces [3, 15]. Cao [5] generalized the concept of g-frames to the concept of (p, Y)-operator frames for a Banach space X. Hence, the concept of (p, Y)-operator frame for a Banach space generalizes all of the concepts of frames.

In this paper, we study the stability of (p, Y)-operator frames for a Banach space. At first, we review the concepts of p-frames, (p, Y)-operator Bessel sequences and (p, Y)-operator frames for a Banach space X. We discuss the relations between p-Bessel sequences (or p-frames) and (p, Y)-operator Bessel sequences (or (p, Y)-operator frames). Moreover, by defining a new union, we prove that adding some elements to a given (p, Y)-operator frame, the resulted sequence will be still a (p, Y)-operator frame. We obtain a necessary and sufficient condition for a sequence of compound operators to be a (p, Y)-operator frame. Lastly, we show that (p, Y)-operator frames for X are stable under some small perturbations.

Throughout this paper, let Λ and Γ be infinite countable sets, $\mathbb{F}$ be the field $\mathbb{C}$ of all complex numbers, or the field $\mathbb{R}$ of all real numbers, and $\mathcal{F}(\Lambda)$ be the set of all nonempty subsets of Λ. In what follows, X, Y are Banach spaces over $\mathbb{F}$, X^* is the dual space of X. Let $B(X, Y)$ denote the Banach space of all bounded linear operators from X into Y, and $B(X) = B(X, X)$.

Recall that

$$\ell^p(Y) = \{\{y_n\}_{n\in\Lambda} : y_n \in Y \ (\forall n \in \Lambda) \ \text{ with } \ \|\{y_n\}_{n\in\Lambda}\|_p < \infty\}, \tag{1}$$

where

$$\|\{y_n\}_{n\in\Lambda}\|_p = \Big(\sum_{n\in\Lambda} \|y_n\|^p\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{2}$$

$$\|\{y_n\}_{n\in\Lambda}\|_\infty = \sup_{n\in\Lambda} \|y_n\|. \tag{3}$$

It is easy to check that $(\ell^p(Y), \|\cdot\|_p)$ becomes a Banach space over $\mathbb{F}$.

Definition 1.1 ([27]) *Let $\{f_i\}_{i\in\Lambda} \subset X^*$, $1 < p < \infty$. Then $\{f_i\}_{i\in\Lambda} \subset X^*$ is a p-Bessel sequence for X, if there exists a positive constant B such that*

$$\Big(\sum_{i\in\Lambda} |\langle f, f_i\rangle|^p\Big)^{1/p} \le B\|f\|, \quad \forall f \in X. \tag{4}$$

We say that $\{f_i\}_{i\in\Lambda} \subset X^*$ is a p-frame for X, if there exist two positive constants A, B such that

$$A\|f\| \le \Big(\sum_{i\in\Lambda} |\langle f, f_i\rangle|^p\Big)^{1/p} \le B\|f\|, \quad \forall f \in X. \tag{5}$$

Definition 1.2 ([5]) *Let $T = \{T_i\}_{i\in\Lambda} \subset B(X, Y)$, $1 \le p \le \infty$. Then $\{T_i\}_{i\in\Lambda} \subset B(X, Y)$ is a (p, Y)-operator Bessel sequence in X, if there exists a positive constant B such that*

$$\|\{T_i f\}_{i\in\Lambda}\|_p \le B\|f\|, \quad \forall f \in X. \tag{6}$$

Definition 1.3 ([5]) *Let $T = \{T_i\}_{i\in\Lambda} \subset B(X, Y)$, $1 \le p \le \infty$. Then $\{T_i\}_{i\in\Lambda} \subset B(X, Y)$ is a (p, Y)-operator frame in X, if there exist two positive constants A, B such that*

$$A\|f\| \le \|\{T_i f\}_{i\in\Lambda}\|_p \le B\|f\|, \quad \forall f \in X. \tag{7}$$

We denote by $B^p_X(Y)$ and $F^p_X(Y)$ the sets of all (p, Y)-operator Bessel sequences and (p, Y)-operator frames for X, respectively. For every $T \in F^p_X(Y)$, define

$$B_T = \inf\{B > 0|\ B \text{ satisfies } (1.7)\}, \tag{8}$$

$$A_T = \sup\{A > 0|\ A \text{ satisfies } (1.7)\}, \tag{9}$$

which are called the upper and lower frame bounds of T, respectively.

This paper is organized as follows. At first, we review the concepts of p-frames, (p, Y)-operator Bessel sequences and (p, Y)-operator frames for a Banach space X. In Section 2, we discuss the relations between p-Bessel sequences (or p-frames) and (p, Y)-operator Bessel sequences (or (p, Y)-operator frames). By defining a new union, we prove that adding some elements to a given (p, Y)-operator frame, the resulted sequence will be still a (p, Y)-operator frame. Moreover, we obtain a necessary and sufficient condition for a sequence of compound operators to be a (p, Y)-operator frame. In Section 3, we show that (p, Y)-operator frames for X are stable under some small perturbations.

2. Some properties of (p, Y)-operator frames

It was proved in [5] that $B^p_X(Y)$ is a Banach space. Let $T = \{T_n\}_{n\in\Lambda} \in B^p_X(Y)$, $S = \{S_n\}_{n\in\Lambda} \in B^p_X(Y)$ and $\lambda \in \mathbb{F}$. We define

$$T + S = \{T_n + S_n\}_{n\in\Lambda}, \lambda T = \{\lambda T_n\}_{n\in\Lambda}, \tag{10}$$

$$\|T\| = \sup_{\|f\|\le 1} \|\{T_n f\}_{n\in\Lambda}\|_p. \tag{11}$$

Moreover, for every f in X, put $Tf = \{T_n f\}_{n\in\Lambda}$. Then

$$\|Tf\|_p \le \|T\|\|f\|, \quad \forall f \in X.$$

The following result was given in [5], which gives some characterizations of (p, Y)-operator Bessel sequences. But there exists a bug in the proof for (3) $\Rightarrow$ (4) of the theorem. Next, we give a revised proof of it.

Theorem 2.1 *Let $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ and $T = \{T_i\}_{i\in\Lambda} \subset B(X, Y)$. Then the following statements are equivalent.*

1) $T \in B^p_X(Y)$.

2) $\forall x \in X, \sum_{i\in\Lambda} \|T_i x\|^p < \infty$.

3) $\forall \{y^*_i\}_{i\in\Lambda} \in \ell^q(Y^*)$, $\sum_{i\in\Lambda} T^*_i y^*_i$ converges in X^*.

4) The operator $S_T : \ell^q(Y^*) \to X^*$ given by

$$S_T\{y^*_i\}_{i\in\Lambda} = \sum_{i\in\Lambda} T^*_i y^*_i \tag{12}$$

is well-defined and bounded.

Proof 3) $\Rightarrow$ 4). Let 3) be valid. Then the operator $S_T : \ell^q(Y^*) \to X^*$ is well-defined and linear. Take $\Lambda_n \in \mathcal{F}(\Lambda)$ such that $\Lambda_n \subset \Lambda_{n+1}$ for $n = 1, 2, \dots$ and $\Lambda = \cup_{n=1}^{\infty} \Lambda_n$. Define $B_n : \ell^q(Y^*) \to X^*$ by $B_n\{y_i^*\}_{i\in\Lambda} = \sum_{i\in\Lambda} \chi_{\Lambda_n}(i) T_i^* y_i^*$, then $B_n \in B(\ell^q(Y^*), X^*)$ for all $n = 1, 2, \dots$. For every $\{y_i^*\}_{i\in\Lambda} \in \ell^q(Y^*)$, we compute that $\lim_{n\to\infty} B_n\{y_i^*\}_{i\in\Lambda} = \sum_{i\in\Lambda} T_i^* y_i^* = S_T\{y_i^*\}_{i\in\Lambda}$. By the Banach-Steinhaus Theorem, we know that S_T is bounded. The proof is completed. $\Box$

The following theorem is a generalization of Theorem 2.2 in [5], which gives a relation between (p, Y)-operator Bessel sequences and p-Bessel sequences.

Theorem 2.2 *Let $1 \le p < \infty$, $T = \{T_i\}_{i\in\Lambda}$ be a (p, Y)-operator Bessel sequence for X. Then for every family $\{y_i^*\}_{i\in\Lambda} \in \ell^\infty(Y^*)$, the family $\{T_i^* y_i^*\}_{i\in\Lambda}$ is a p-Bessel sequence for X.*

Proof $\forall \{y_i^*\}_{i\in\Lambda} \in \ell^\infty(Y^*)$ and $\forall f \in X$, we compute that

$$\begin{aligned}\sum_{i\in\Lambda} |\langle f, T_i^* y_i^*\rangle|^p &= \sum_{i\in\Lambda} |\langle T_i f, y_i^*\rangle|^p \le \sum_{i\in\Lambda} (\|T_i f\|^p \cdot \|y_i^*\|^p) \\ &\le \Big(\sum_{i\in\Lambda} \|T_i f\|^p\Big) \cdot \Big(\sup_{i\in\Lambda} \|y_i^*\|\Big)^p \le B_T^p \|f\|^p \cdot \|\{y_i^*\}\|_\infty^p.\end{aligned}$$

This shows that $\{T_i^* y_i^*\}_{i\in\Lambda}$ is a p-Bessel sequence for X. The proof is completed. $\Box$

Corollary 2.3 *Let $1 \le p \le \infty$, $T = \{T_i\}_{i\in\Lambda}$ be a (p, Y)-operator Bessel sequence for X. Then for every family $\{y_i^*\}_{i\in\Lambda} \in \ell^q(Y^*)$, the family $\{T_i^* y_i^*\}_{i\in\Lambda}$ is a p-Bessel sequence for X.*

Proof Use $\ell^q(Y^*) \subset \ell^\infty(Y^*)$. $\Box$

Theorem 2.4 *Let $T = \{T_i\}_{i\in\Lambda} \in B_X^p(Y)$ and there exists a family $\{y_i^*\}_{i\in\Lambda} \in \ell^\infty(Y^*)$ such that $g = \{T_i^* y_i^*\}_{i\in\Lambda}$ is a p-frame for X. Then $T = \{T_i\}_{i\in\Lambda} \in F_X^p(Y)$.*

Proof Since $g = \{T_i^* y_i^*\}_{i\in\Lambda}$ is a p-frame for X, $\forall f \in X$,

$$A_g^p \|f\|^p \le \sum_{i\in\Lambda} |\langle f, T_i^* y_i^*\rangle|^p = \sum_{i\in\Lambda} |\langle T_i f, y_i^*\rangle|^p \le \sum_{i\in\Lambda} \|T_i f\|^p \|\{y_i^*\}_{i\in\Lambda}\|_\infty^p,$$

thus, $\forall f \in X$,

$$\Big(\sum_{i\in\Lambda} \|T_i f\|^p\Big)^{1/p} \ge \frac{A_g}{\|\{y_i^*\}_{i\in\Lambda}\|_\infty} \|f\|.$$

On the other hand, since $T = \{T_i\}_{i\in\Lambda} \in B_X^p(Y)$, $T = \{T_i\}_{i\in\Lambda} \in F_X^p(Y)$. The proof is completed. $\Box$

Definition 2.5 *Let $T_1 = \{T_i^1\}_{i\in\Lambda}$, $T_2 = \{T_i^2\}_{i\in\Lambda}, \dots, T_n = \{T_i^n\}_{i\in\Lambda} \subset B(X, Y)$ $(2 \le n < \infty)$ and $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$, where $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ are disjoint infinite subsets of Λ. Then there exist n bijections*

$$\alpha_1 : \Lambda \to \Lambda_1, \alpha_2 : \Lambda \to \Lambda_2, \dots, \alpha_n : \Lambda \to \Lambda_n.$$

Define $\{H_i\}_{i\in\Lambda}$ by

$$H_i = \begin{cases} T^1_{\alpha_1^{-1}(i)}, & i \in \Lambda_1; \\ T^2_{\alpha_2^{-1}(i)}, & i \in \Lambda_2; \\ \cdots \\ T^n_{\alpha_n^{-1}(i)}, & i \in \Lambda_n, \end{cases} \tag{13}$$

called the disjoint union of $\{T_i^1\}_{i\in\Lambda}, \{T_i^2\}_{i\in\Lambda}, \dots, \{T_i^n\}_{i\in\Lambda}$, denoted by

$$\{T_i^1\}_{i\in\Lambda} \sqcup \{T_i^2\}_{i\in\Lambda} \sqcup \cdots \sqcup \{T_i^n\}_{i\in\Lambda}.$$

Theorem 2.6 *Let $T = \{T_i\}_{i\in\Lambda} \in F_X^p(Y)$ and $S = \{S_i\}_{i\in\Lambda} \subset B(X, Y)$. Then $H = \{H_i\}_{i\in\Lambda} = \{T_i\}_{i\in\Lambda} \sqcup \{S_i\}_{i\in\Lambda} \in F_X^p(Y)$ if and only if $S = \{S_i\}_{i\in\Lambda} \in B_X^p(Y)$. In that case, $A_T \le A_H \le B_H \le (B_T^p + B_S^p)^{\frac{1}{p}}$.*

Proof $\Leftarrow$. Assume $S = \{S_i\}_{i\in\Lambda} \in B_X^p(Y)$. $\forall f \in X$, we have

$$\begin{aligned}\sum_{i\in\Lambda} \|H_i f\|^p &= \sum_{i\in\Lambda_1} \|H_i f\|^p + \sum_{i\in\Lambda_2} \|H_i f\|^p = \sum_{i\in\Lambda_1} \|T_{\alpha_1^{-1}(i)} f\|^p + \sum_{i\in\Lambda_2} \|S_{\alpha_2^{-1}(i)} f\|^p \\ &= \sum_{i\in\Lambda} \|T_i f\|^p + \sum_{i\in\Lambda} \|S_i f\|^p \le (B_T^p + B_S^p)\|f\|^p.\end{aligned}$$

On the other hand,

$$\sum_{i\in\Lambda} \|H_i f\|^p \ge \sum_{i\in\Lambda} \|T_i f\|^p \ge A_T^p \|f\|^p, \quad \forall f \in X.$$

Therefore, $\{H_i\}_{i\in\Lambda} \in F_X^p(Y)$ with $A_T \le A_H \le B_H \le (B_T^p + B_S^p)^{\frac{1}{p}}$.

$\Rightarrow$. Assume $H = \{H_i\}_{i\in\Lambda} \in F_X^p(Y)$. Then $\forall f \in X$, we have

$$\sum_{i\in\Lambda} \|S_i f\|^p \le \sum_{i\in\Lambda} \|H_i f\|^p \le B_H^p \|f\|^p.$$

Therefore, we conclude that

$$S = \{S_i\}_{i\in\Lambda} \in B_X^p(Y).$$

The proof is completed. □

Theorem 2.7 *Let $T = \{T_i\}_{i\in\Lambda} \subset B(X, Y)$, $S = \{S_j\}_{j\in\Gamma} \in F_Y^p(Z)$. Then the following statements are equivalent.*

1) $\{T_i\}_{i\in\Lambda} \in F_X^p(Y)$.

2) $ST := \{S_j T_i\}_{(i,j)\in\Lambda\times\Gamma} \in F_X^p(Z)$. In that case, $A_S A_T \le A_{ST} \le B_{ST} \le B_S B_T$.

Proof 1) $\Rightarrow$ 2). Let 1) hold. $\forall f \in X$,

$$\sum_{(i,j)\in\Lambda\times\Gamma} \|S_j T_i f\|^p = \sum_{i\in\Lambda}\sum_{j\in\Gamma} \|S_j T_i f\|^p \le \sum_{i\in\Lambda} B_S^p \|T_i f\|^p \le B_S^p B_T^p \|f\|^p$$

and

$$\sum_{(i,j)\in\Lambda\times\Gamma} \|S_j T_i f\|^p = \sum_{i\in\Lambda}\sum_{j\in\Gamma} \|S_j T_i f\|^p \ge \sum_{i\in\Lambda} A_S^p \|T_i f\|^p \ge A_S^p A_T^p \|f\|^p.$$

This shows that $\{S_j T_i\}_{i\in\Lambda, j\in\Gamma} \in F_X^p(Z)$ with $A_S A_T \le A_{ST} \le B_{ST} \le B_S B_T$.

2) $\Rightarrow$ 1). Let 2) be valid. $\forall f \in X$, since $S = \{S_j\}_{j\in\Gamma} \in F_Y^p(Z)$, we conclude that

$$A_S\|T_i f\| \leq (\sum_{j\in\Gamma} \|S_j T_i f\|^p)^{1/p} \leq B_S\|T_i f\|, \quad \forall i \in \Lambda,$$

$$A_S^p\|T_i f\|^p \leq \sum_{j\in\Gamma} \|S_j T_i f\|^p \leq B_S^p\|T_i f\|^p, \quad \forall i \in \Lambda.$$

Thus

$$A_S^p \sum_{i\in\Lambda} \|T_i f\|^p \leq \sum_{i\in\Lambda}\sum_{j\in\Gamma} \|S_j T_i f\|^p \leq B_S^p \sum_{i\in\Lambda} \|T_i f\|^p.$$

So, we get that, $\forall f \in X$,

$$\sum_{i\in\Lambda} \|T_i f\|^p \leq 1/A_S^p \sum_{(i,j)\in\Lambda\times\Gamma} \|S_j T_i f\|^p \leq \frac{B_{ST}^p}{A_S^p}\|f\|^p,$$

and

$$\sum_{i\in\Lambda} \|T_i f\|^p \geq 1/B_S^p \sum_{(i,j)\in\Lambda\times\Gamma} \|S_j T_i f\|^p \geq \frac{A_{ST}^p}{B_S^p}\|f\|^p.$$

Therefore,

$$T = \{T_i\}_{i\in\Lambda} \in F_X^p(Y).$$

The proof is completed. □

Theorem 2.8 *Suppose that* $\{L_j\}_{j=1}^n \subset B(Y, Z)$, $T = \{T_i\}_{i\in\Lambda} \in F_X^p(Y)$, *and there exists* m $(1 \leq m \leq n)$ *such that* L_m *is bounded below, then* $H = \{H_k\}_{k\in\Lambda} = \bigsqcup_{j=1}^n \{L_j T_i\}_{i\in\Lambda} \in F_X^p(Z)$ *with*

$$cA_T \leq A_H \leq B_H \leq \Big(\sum_{j=1}^n \|L_j\|^p\Big)^{\frac{1}{p}} B_T.$$

Proof Since L_m is bounded below, there exists $c > 0$ such that

$$c\|y\| \leq \|L_m y\| \leq \|L_m\|\|y\|, \quad \forall y \in Y.$$

Thus, $\forall f \in X$, we get that

$$c^p A_T^p\|f\|^p \leq c^p \sum_{i\in\Lambda} \|T_i f\|^p \leq \sum_{j=1}^n \sum_{i\in\Lambda} \|L_j T_i f\|^p = \sum_{k\in\Lambda} \|H_k f\|^p.$$

On the other hand,

$$\sum_{j=1}^n \sum_{i\in\Lambda} \|L_j T_i f\|^p \leq \Big(\sum_{j=1}^n \|L_j\|^p\Big)\Big(\sum_{i\in\Lambda} \|T_i f\|^p\Big) \leq \Big(\sum_{j=1}^n \|L_j\|^p\Big) B_T^p\|f\|^p.$$

Consequently, $H \in F_X^p(Z)$ with $cA_T \leq A_H \leq B_H \leq (\sum_{j=1}^n \|L_j\|^p)^{\frac{1}{p}} B_T$. The proof is completed. □

An interesting question is whether the inverse of Theorem 2.8 is true.

Note that following Definition 2.5, n in Theorem 2.8 is assumed to be ≥ 2. In fact, it is easy to prove that $\{LT_i\}_{i\in\Lambda} \in F_X^p(Z)$ where $n = 1$.

3. The perturbations of (p, Y)-operator frames

Theorem 3.1 *Let $T = \{T_i\}_{i\in\Lambda} \in B^p_X(Y)$.*

1) Suppose that $T = \{T_i\}_{i\in\Lambda} \in F^p_X(Y)$, $S = \{S_i\}_{i\in\Lambda} \in B^p_X(Y)$ with $\|S\| < A_T$, then $T+S \in F^p_X(Y)$ and $T-S \in F^p_X(Y)$.

2) Suppose that there exists $S = \{S_i\}_{i\in\Lambda} \in B^p_X(Y)$ such that $T+S \in F^p_X(Y)$ and $\|S\| < A_{T+S}$, then $T = \{T_i\}_{i\in\Lambda} \in F^p_X(Y)$.

Proof 1) $\forall f \in X$, we get

$$\|(T+S)f\|_p = \|Tf + Sf\|_p \le \|T\|\|f\| + \|S\|\|f\| \le (B_T + \|S\|)\|f\|,$$

and

$$\|(T+S)f\|_p = \|Tf + Sf\|_p \ge \|Tf\|_p - \|Sf\|_p \ge (A_T - \|S\|)\|f\|.$$

Thus, $T+S \in F^p_X(Y)$. Similarly, we can get that $T-S \in F^p_X(Y)$.

2) Use 1).

The proof is completed. □

Theorem 3.2 *Let $T = \{T_i\}_{i\in\Lambda} \in F^p_X(Y)$, $S = \{S_i\}_{i\in\Lambda} \subset B(X,Y)$. Suppose that there exist constants $\lambda_1, \lambda_2, \mu \ge 0$, such that $\max\{\lambda_1 + \mu/A_T, \lambda_2\} < 1$, and the following condition is satisfied:*

$$\|(T-S)f\|_p \le \lambda_1\|Tf\|_p + \lambda_2\|Sf\|_p + \mu\|f\|, \quad \forall f \in X, \tag{14}$$

then $S = \{S_i\}_{i\in\Lambda} \in F^p_X(Y)$ with

$$A_S \ge A_T\left(1 - \frac{\lambda_1 + \lambda_2 + \mu/A_T}{1+\lambda_2}\right), \quad B_S \le B_T\left(1 + \frac{\lambda_1 + \lambda_2 + \mu/A_T}{1-\lambda_2}\right).$$

Proof Since $T = \{T_i\}_{i\in\Lambda} \in F^p_X(Y)$, we get that

$$\|f\| \le \frac{1}{A_T}\|Tf\|_p, \quad \forall f \in X,$$

it follows from (14) that

$$\|(T-S)f\|_p \le (\lambda_1 + \mu/A_T)\|Tf\|_p + \lambda_2\|Sf\|_p. \tag{15}$$

By the triangle inequality, we have

$$\|(T-S)f\|_p \ge \|Tf\|_p - \|Sf\|_p. \tag{16}$$

Hence, by (15) and (16), we have

$$(1+\lambda_2)\|Sf\|_p \ge (1 - \lambda_1 - \mu/A_T)\|Tf\|_p \ge (1 - \lambda_1 - \mu/A_T)A_T\|f\|.$$

Therefore,

$$\|Sf\|_p \ge \frac{1 - \lambda_1 - \mu/A_T}{1+\lambda_2}A_T\|f\|,$$

that is

$$\|Sf\|_p \ge A_T\left(1 - \frac{\lambda_1 + \lambda_2 + \mu/A_T}{1+\lambda_2}\right)\|f\|.$$

On the other hand, by the triangle inequality, we have

$$\|(T-S)f\|_p \geq \|Sf\|_p - \|Tf\|_p. \tag{17}$$

Hence, by (15) and (17), we have

$$(1-\lambda_2)\|Sf\|_p \leq (1+\lambda_1+\mu/A_T)\|Tf\|_p \leq (1+\lambda_1+\mu/A_T)B_T\|f\|,$$

hence

$$\|Sf\|_p \leq B_T(1+\frac{\lambda_1+\lambda_2+\mu/A_T}{1-\lambda_2})\|f\|.$$

Consequently, $S=\{S_i\}_{i\in\Lambda}\in F_X^p(Y)$ with

$$A_S \geq A_T(1-\frac{\lambda_1+\lambda_2+\mu/A_T}{1+\lambda_2}),\quad B_S \leq B_T(1+\frac{\lambda_1+\lambda_2+\mu/A_T}{1-\lambda_2}).$$

The proof is completed. □

In [2], a perturbation of a g-frame was introduced.

Definition 3.3 ([2]) *Let $\{T_i\}_{i\in\Lambda}$ be a g-frame for U with respect to $\{V_i\}_{i\in\Lambda}$, $0\leq\lambda_1<1$, $0\leq\lambda_2<1$ and $\{c_i\}_{i\in\Lambda}$ a sequence of positive numbers with $\sum_{i\in\Lambda}{c_i}^2<\infty$. Then the family $\{S_i\}_{i\in\Lambda}$ is a $(\lambda_1,\lambda_2,\{c_i\}_{i\in\Lambda})$-perturbation of $\{T_i\}_{i\in\Lambda}$ if*

$$\|T_if-S_if\| \leq \lambda_1\|T_if\|+\lambda_2\|S_if\|+c_i\|f\|,\quad \forall i\in\Lambda,\ \forall f\in U.$$

Now we generalize this concept to the case of (p,Y)-operator frames.

Definition 3.4 *Let $\{T_i\}_{i\in\Lambda}\in F_X^p(Y)$, $0\leq\lambda_1<1$, $0\leq\lambda_2<1$, and $\{c_i\}_{i\in\Lambda}$ a sequence of positive numbers with $c=\sum_{i\in\Lambda}{c_i}^p<\infty$. Then the family $S=\{S_i\}_{i\in\Lambda}\subset B(X,Y)$ is a $(\lambda_1,\lambda_2,\{c_i\}_{i\in\Lambda})$-perturbation of $\{T_i\}_{i\in\Lambda}$, if*

$$\|T_if-S_if\| \leq \lambda_1\|T_if\|+\lambda_2\|S_if\|+c_i\|f\|,\quad \forall i\in\Lambda,\ \forall f\in X. \tag{18}$$

Theorem 3.5 *Let $T=\{T_i\}_{i\in\Lambda}\in F_X^p(Y)$ and S be a $(\lambda_1,\lambda_2,\{c_i\}_{i\in\Lambda})$-perturbation of $\{T_i\}_{i\in\Lambda}$. If $(1-\lambda_1)A_T>c^{1/p}$. Then $S=\{S_i\}_{i\in\Lambda}\in F_X^p(Y)$ with*

$$A_S \geq \frac{(1-\lambda_1)A_T-c^{1/p}}{1+\lambda_2},\quad B_S \leq \frac{(1+\lambda_1)B_T+c^{1/p}}{1-\lambda_2}.$$

Proof $\forall f\in X$, we get from (18) that

$$\begin{aligned}\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p} \leq&\Big(\sum_{i\in\Lambda}(\|T_if\|+\|(T_i-S_i)f\|)^p\Big)^{1/p}\\ \leq&\Big(\sum_{i\in\Lambda}(\|T_if\|+\lambda_1\|T_if\|+\lambda_2\|S_if\|+c_i\|f\|)^p\Big)^{1/p}\\ \leq&(1+\lambda_1)\Big(\sum_{i\in\Lambda}\|T_if\|^p\Big)^{1/p}+\lambda_2\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}+c^{1/p}\|f\|.\end{aligned}$$

Hence,

$$(1-\lambda_2)\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p} \leq \Big((1+\lambda_1)B_T+c^{1/p}\Big)\|f\|,\quad \forall f\in X,$$

which yields

$$\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}\le\frac{(1+\lambda_1)B_T+c^{1/p}}{1-\lambda_2}\|f\|. \tag{19}$$

Similarly, by (18) we have

$$\begin{aligned}\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}\ge&\Big(\sum_{i\in\Lambda}(\|T_if\|-\|(T_i-S_i)f\|)^p\Big)^{1/p}\\ \ge&\Big(\sum_{i\in\Lambda}(\|T_if\|-\lambda_1\|T_if\|-\lambda_2\|S_if\|-c_i\|f\|)^p\Big)^{1/p}\\ \ge&(1-\lambda_1)\Big(\sum_{i\in\Lambda}\|T_if\|^p\Big)^{1/p}-\lambda_2\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}-c^{1/p}\|f\|.\end{aligned}$$

Hence,

$$(1+\lambda_2)\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}\ge\Big((1-\lambda_1)A_T-c^{1/p}\Big)\|f\|,\quad\forall f\in X.$$

Finally,

$$\Big(\sum_{i\in\Lambda}\|S_if\|^p\Big)^{1/p}\ge\frac{(1-\lambda_1)A_T-c^{1/p}}{1+\lambda_2}\|f\|. \tag{20}$$

By (19) and (20), we get that $S=\{S_i\}_{i\in\Lambda}\in F_X^p(Y)$ with

$$A_S\ge\frac{(1-\lambda_1)A_T-c^{1/p}}{1+\lambda_2},\quad B_S\le\frac{(1+\lambda_1)B_T+c^{1/p}}{1-\lambda_2}.$$

The proof is completed. □

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