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Symplectic Cyclic Actions on Elliptic Surfaces

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Abstract Let $X = E(n)$ be the relatively minimal elliptic surface with rational base, where $n \geq 2$. In this paper, several pseudofree, homologically trivial, symplectic cyclic actions by groups whose orders are 2, 3, 5 and 7 on X are studied.

Keywords symplectic action; fixed-point set; elliptic surface; locally linear.

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1. Introduction

Let $G = \mathbb{Z}_p$ be the cyclic group of order p, where $p = 2, 3, 5$ or 7. Let $X = E(n)$ be the relatively minimal elliptic surface with rational base, and $n \geq 2$ in this paper unless we announce particularly. The elliptic surface $E(n)$ is defined as the n-fold fiber sum of copies of $E(1)$, where $E(1)$ is $\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$ being equipped with an elliptic fibration. The second Betti number of X is denoted by $b_2(X)$, the signature of X by $\sigma(X)$, and the Euler characteristic of X by $\chi(X)$. We also denote by $\sigma(X/G)$ and $\chi(X/G)$ the signature and Euler characteristic of the quotient manifold respectively. An elliptic surface is Kähler, and can be equipped with a symplectic structure which is provided by the complex structure and the Kähler metric. The elliptic surfaces are minimal [8].

When studying actions on manifold of a finite group, one can consider an induced action on some algebraic invariants associated with the manifold, and it is often important and beneficial. Furthermore, a central problem is to describe the structure of the fixed-point set and the action around it. Since the G-signature theorem of Atiyah-Singer [1] imposes restrictions on the symmetries of 4-manifolds through the fixed-point set and the action around it, it is useful to consider the fixed-point set together with the G-signature theorem. For a spin manifold, the G-index theorem for Dirac operators can be used to do calculation. In [3], Chen and Kwasik determined the structure of the fixed-point set for a symplectic cyclic action of prime order on a minimal symplectic 4-manifold M with $c_1^2 = 0$ and $b_2^+ \geq 2$, which induces a trivial action on $H^2(M;\mathbb{Q})$. Recall that a symplectic action means a smooth finite group action which preserves some symplectic structure on the 4-manifold [3]. In this paper, the symplectic $\mathbb{Z}_p$-actions of prime orders

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no more than 7 on elliptic surfaces under certain conditions are studied. We can see that the action is trivial if the order of the cyclic group is 2. It is showed that not every simply connected 4-manifold admits pseudofree and homologically trivial $\mathbb{Z}_3$-actions by the G-signature theorem in [6], then we have the non-existent result. We also describe the fixed-point set structure when the order of the action is 5. It can be seen that there exist a locally linear, pseudofree $\mathbb{Z}_5$-action to realize the corresponding fixed point data by using the realization theorem of Edmonds and Ewing in [7]. When $p = 7$, we attempt to determine the fixed-point set of the action in example 3.3, and more conditions may be needed. Our main results are stated as follows.

Theorem 1.1 *Let $G = \mathbb{Z}_p$.*

1) For $p = 2$, if an elliptic surface $X = E(n)$ admits a homologically trivial (over $\mathbb{Q}$ coefficients), pseudofree, symplectic G-action. Then the action is trivial.

2) For $p = 3$, there is no homologically trivial (over $\mathbb{Q}$ coefficients), pseudofree, symplectic G-action on X.

Theorem 1.2 *Let $G = \mathbb{Z}_5$, and $X = E(n)$ be an elliptic surface, and let X admit a nontrivial, pseudofree action of G, such that the symplectic structure is preserved under the action and the induced action on $H^2(X;\mathbb{Q})$ is trivial. Then the fixed-point set of the G-action consists of $4n$ fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_5^k z_1, \mu_5^{2k} z_2)$ and $8n$ fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_5^{-k} z_1, \mu_5^{4k} z_2)$ for some $k \neq 0 \bmod 5$.*

2. Preliminaries and tools

In this section we collect some G-index theorems and some results which will be used (See [4] for more details of G-index theorems). The formulas in G-index theorems allow us to relate the fixed-point set structure of the group action with the induced representation on the rational cohomology of the manifold. We need to make use of a result of Chen and Kwasik in [3] which is a complete description of the structure of the fixed-point set for a symplectic cyclic action of prime order on a minimal symplectic 4-manifold under some conditions. We list it below as Theorem 2.5.

Let X be a closed, oriented smooth 4-manifold, and let cyclic group $G \equiv \mathbb{Z}_p$ of prime order act on X effectively via orientation-preserving diffeomorphisms. Then the fixed-point set F, if nonempty, will consist of isolated points and surfaces. If a generator g of G is fixed, each fixed point $m \in F$ is associated with a nonzero integers pair (a_m, b_m), where $-p < a_m, b_m < p$, and they are uniquely determined up to a change of order or a change of sign simultaneously, such that the induced g-action on the tangent space at m is given by the complex linear transformation $(z_1, z_2) \mapsto (\mu_p^{a_m} z_1, \mu_p^{b_m} z_2)$, where $\mu_p = \exp(\frac{2\pi i}{p})$. For each connected surface $Y \subset F$, the action of g on the normal bundle of Y in X is given by $z \mapsto \mu_p^{c_Y} z$ for an integer c_Y with $0 < c_Y < p$, which is uniquely determined up to a sign modulo p.

Theorem 2.1 (Lefschetz Fixed Point Theorem) *Let $T : X \to X$ generate an action of $\mathbb{Z}_l$ on X, a closed, simply-connected 4-manifold. Then $L(T, X) = \chi(F)$, where $\chi(F)$ is the Euler* *characteristic of the fixed-point set F and $L(T,X)$ is the Lefschetz number of the map T, which is defined by*

$$L(T,X)=\sum_{k=0}^{4}(-1)^k\mathrm{tr}(g)|_{H^k(X;\mathbb{R})}.$$

For our cases in this paper, the formula in the theorem can be written as

$$\chi(F)=2+\mathrm{trace}[T_*:H_2(X)\to H_2(X)]=2+b_2(X)=\chi(X).$$

Theorem 2.2 (G-signature Theorem) *Set*

$$\sigma(g,X)=\mathrm{tr}(g)|_{H^{2,+}(X;\mathbb{R})}-\mathrm{tr}(g)|_{H^{2,-}(X;\mathbb{R})},$$

then

$$\sigma(g,X)=\sum_{m\in F}-\cot(\frac{a_m\pi}{p})\cdot\cot(\frac{b_m\pi}{p})+\sum_{Y\subset F}\csc^2(\frac{c_Y\pi}{p})\cdot(Y\cdot Y),$$

where $Y\cdot Y$ denotes the self-intersection number of Y.

The weaker version of the G-Signature Theorem is more often used since it is convenient for calculation [4, 10].

Theorem 2.3 (G-signature Theorem - the weaker version) *Set*

$$|G|\cdot\sigma(X/G)=\sigma(X)+\sum_{m\in F}\mathrm{def}_m+\sum_{Y\subset F}\mathrm{def}_Y,$$

where the terms def_m *and* def_Y *are called signature defects. They are given by the following formulae:*

$$\mathrm{def}_m=\sum_{k=1}^{p-1}\frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}$$

if the local representation of G at m is given by $(z_1,z_2)\mapsto(\mu_p^k z_1,\mu_p^{kq}z_2)$, and

$$\mathrm{def}_Y=\frac{p^2-1}{3}\cdot(Y\cdot Y).$$

Note that the G-signature theorem is also valid for locally linear, topological actions of prime orders in dimension 4 ([6, 9]). In our case of pseudofree actions on $E(n)$, the terms def_Y in the formula vanish.

If the 4-manifold X is spin, and the G-action on X lifts to the spin structures on X. The index of Dirac operator $\mathbb{D}$ is a useful tool for our aim. For each $g\in G$, one can define the Spin-number of g by

$$\mathrm{Spin}(g,X)=\mathrm{tr}(g)|_{\mathrm{Ker}\mathbb{D}}-\mathrm{tr}(g)|_{\mathrm{Coker}\mathbb{D}}.$$

Let ${V_k}^+$, ${V_k}^-$ be the eigenspaces of g on $\mathrm{Ker}\mathbb{D}$ and $\mathrm{Coker}\mathbb{D}$ with eigenvalue μ_p^k, respectively. We can write $\mathrm{Ker}\mathbb{D}=\bigoplus_{k=0}^{p-1}{V_k}^+$, $\mathrm{Coker}\mathbb{D}=\bigoplus_{k=0}^{p-1}{V_k}^-$, then

$$\mathrm{Spin}(g,X)=\sum_{k=0}^{p-1}d_k\mu_p^k,$$

where $d_k\equiv\dim_{\mathbb{C}}V_k^+-\dim_{\mathbb{C}}V_k^-$. Note that d_0 is even, and $d_k=d_{p-k}$ for $1\le k\le p-1$ when the order of group G is odd.

Theorem 2.4 (G-index theorem for Dirac operators [1]) *Assume further that the action of G on X is spin and that there are only isolated fixed points. Then the Spin-number* $\mathrm{Spin}(g, X) = \sum_{k=0}^{p-1} d_k \mu_p^k$ *is given in terms of the fixed point set structure by the following formula*

$$\mathrm{Spin}(g, X) = -\sum_{m \in F} \epsilon(g, X) \cdot \frac{1}{4} \csc(\frac{a_m \pi}{p}) \cdot \csc(\frac{b_m \pi}{p}),$$

where $\epsilon(g, X) = \pm 1$ depends on the fixed point m and the lifting of the action of g to the spin structure. Precisely, if the action of G preserves an almost complex structure on X (e.g. the action of G is via symplectic symmetries), then $\epsilon(g, X)$ is given by

$$\epsilon(g, X) = (-1)^{k(g,m)},$$

where $k(g, m)$ is defined by the equation

$$k(g, m) \cdot p = 2r_m + a_m + b_m,$$

for some r_m satisfying $0 \leq r_m < p$.

The structure of the fixed-point set for a symplectic cyclic action of prime order on a minimal symplectic 4-manifold X with $c_1^2 = 0$ and $b_2^+ \geq 2$, which induces a trival action on $H^2(X; \mathbb{Q})$, was described by Chen and Kwasik [3]. We invoke the result for the case of pseudofree actions below.

Theorem 2.5 ([3]) *Let X be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrival, pseudofree action of $G = \mathbb{Z}_p$, where p is a prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(X; \mathbb{Q})$ is trival. Then the set of fixed points of G can be divided into groups, each of which belongs to the following five possible types.*

1) One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \bmod p$, i.e., with representation contained in $SL_2(\mathbb{C})$.

2) Two fixed points with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ for some $k \neq 0 \bmod p$, respectively. This type of fixed points occurs only when $p > 5$.

3) Three fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ and the other two with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ for some $k \neq 0 \bmod p$. This type of fixed points occurs only when $p > 3$.

4) Four fixed points, one with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ and the other three with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ for some $k \neq 0 \bmod p$. This type of fixed points occurs only when $p > 3$.

5) Three fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ for some $k \neq 0 \bmod p$. This type of fixed points occurs only when $p = 3$.

The rigidity for the corresponding homologically trivial actions is as follows, and it shows that symplectic symmetries are more restrictive than topological ones.

Theorem 2.6 ([3]) *Let X be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a homologically trivial (over $\mathbb{Q}$ coefficients), pseudofree, symplectic $\mathbb{Z}_p$-action for a prime*

p. Then the following conclusions hold.

(a) The action is trivial if $p \neq 1 \bmod 4$, $p \neq 1 \bmod 6$, and the signature of X is nonzero, then for infinitely many primes p the manifold X does not admit any such nontrivial $\mathbb{Z}_p$-actions.

(b) The action is trivial as long as there is a fixed point of type 1) in Theorem 2.5.

3. Symplectic cyclic actions

In this section we prove Theorems 1.1 and 1.2, and we consider a symplectic cyclic action of order 7 on the elliptic surfaces.

Proof of Theorem 1.1 1) Recall that the elliptic surface $E(n)$ is the minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$. If it admits a homologically trivial (over $\mathbb{Q}$ coefficients), pseudofree, symplectic $\mathbb{Z}_2$-action, then the $\mathbb{Z}_2$-action is trivial by Theorem 2.6.

2) If $X = E(n)$ admits such a $\mathbb{Z}_3$-action. The fixed-point set will consist of two types of points under the consideration of local representation. We denote by x_1 the number of points of type (1,1) and x_2 the number of points of type (1,2). Let m and n be one fixed point of type (1,1) and (1,2). Then $\mathrm{def}_m = -\frac{1}{3}$, $\mathrm{def}_n = \frac{1}{3}$. The formula in Theorem 2.3 is rewritten as $2 \cdot \sigma(X) = \frac{2}{3}(x_2 - x_1)$. Together with the Lefschetz fixed point theorem, we have the following inequality, $3|\sigma(X)| \leq \chi(F) = \chi(X) = 2 + b_2(X)$, or $|\sigma(X)| \leq \frac{(b_2(X)+2)}{3}$. Obviously, the elliptic surfaces do not satisfy it. □

Remark 3.1 Edmonds pointed out that many closed, simply-connected 4-manifolds do not support pseudofree, homologically trivial, locally linear $\mathbb{Z}_3$-actions in [6]. Such as the E_8 manifold, the Kummer surface, a connected sum of $n > 1$ copies of $+\mathbb{C}P^2$ or $+Ch$, etc.

Lemma 3.2 ([3]) *Let $\mathrm{def}_{(k)}$ be the total signature defect contributed by one group of fixed points of type (k) in Theorem 2.5, where $k = 1, \ldots, 4$. Then we have*

1) $\mathrm{def}_{(1)} = \frac{1}{3}(p-1)(p-2)$ *for all $p > 1$.*

2) $\mathrm{def}_{(2)} = -8r$ *if $p = 6r+1$,* $\mathrm{def}_{(2)} = 8r+8$ *if $p = 6r+5$.*

3) $\mathrm{def}_{(3)} = -8r$ *if $p = 4r+1$,* $\mathrm{def}_{(3)} = 2$ *if $p = 4r+3$.*

4) $\mathrm{def}_{(4)} = -8r$ *if $p = 3r+1$,* $\mathrm{def}_{(4)} = -4r$ *if $p = 3r+2$.*

Proof of Theorem 1.2 By the assumption that the G-action on $X = E(n)$ is pseudofree and with the G-signature theorem, we have

$$|G| \cdot \sigma(X/G) = \sigma(X) + \sum_{m \in F} \mathrm{def}_m,$$

where $G = \mathbb{Z}_5$, and $F = X^G$ denotes the fixed-point set.

The induced action on $H^2(X;\mathbb{Q})$ is trivial, then $\sigma(X/G) = \sigma(X)$ and the Lefschetz fixed point theorem leads to $\chi(F) = \chi(X)$. The fixed-point set may consist of types 1), 3) and 4) fixed points in Theorem 2.5, actually there are no type 1) fixed points by the assumption that the action is nontrivial and Theorem 2.6. Let a_3, a_4 be the numbers of groups of fixed points of types 3) and 4), respectively. Then we have

$$\begin{cases} -32n = -8a_3 - 4a_4, \\ 12n = 3a_3 + 4a_4. \end{cases} \tag{1}$$

Here we use the fact that $\text{def}_{(3)} = -8$ and $\text{def}_{(4)} = -4$. The solutions for a_3, a_4 are $a_3 = 4n$ and $a_4 = 0$. Then the fixed-point set consists of 4n groups of type 3) fixed points in Theorem 2.5, then the theorem follows. □

If a set of ordered pairs of nonzero elements is given which satisfies REP, GSM, and TOR conditions in [7], then it can be realized as the fixed point data of some locally linear, pseudofree G action on a closed, oriented, simply connected, topological 4-manifold. By the consideration of Edmonds and Ewing, we can see that there exists a locally linear, pseudofree topological $\mathbb{Z}_5$-action on $E(n)$ with the fixed-point set consisting entirely of the fixed points in Theorem 1.2. Note that the fixed-point set consists of $12n$ fixed points. Divide them into n groups, and assign the points in each group with local representations $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$, $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$, and $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$, evaluated at $k = 1, 2, 3, 4$. Now, we get a set of $12n$ ordered pairs of nonzero elements, denoted by $\mathscr{D}$. In order to realize the set $\mathscr{D}$ as the fixed-point data of some locally linear, pseudofree $\mathbb{Z}_5$-action on $E(n)$, we need to check the REP, GSM and TOR conditions. For $p = 5$, GSM condition is the only condition needed for the realization of the fixed-point data by a homologically trivial action ([7], Corollary 3.2 and [3]). The GSM condition becomes $n \cdot \text{def}_{(3)} = \sigma(g, X), \forall g \in \mathbb{Z}_5$. We use the fact that $\sigma(g, X) = \sigma(X) = -8n$ for any $g \in \mathbb{Z}_5$ and $\text{def}_{(3)} = -8$ to verify the GSM condition. Consequently, there is a locally linear topological action of $\mathbb{Z}_5$ on $X = E(n)$ with the fixed-point set consisting of fixed points in Theorem 1.2 entirely.

The cyclic actions of the other orders on elliptic surface can be considered similarly. The fixed-point set may be determined, and sometimes more assumptions may be needed.

Example 3.3 If $X = E(n)$ admits a nontrivial, pseudofree action of $G = \mathbb{Z}_7$, such that the symplectic structure is preserved under the action and the induced action on $H^2(X; \mathbb{Q})$ is trivial, furthermore suppose that X is spin (or equivalently, n is even). The fixed-point set may consist of types 2), 3) and 4) fixed points in Theorem 2.5, and let a_2, a_3, a_4 be the numbers of groups of fixed points of types 2), 3) and 4), respectively. Similarly, we have

$$\begin{cases} -48n = -8a_2 + 2a_3 - 16a_4, \\ 12n = 2a_2 + 3a_3 + 4a_4, \end{cases} \tag{2}$$

here we use the fact that $\text{def}_{(2)} = -8$, $\text{def}_{(3)} = 2$ and $\text{def}_{(4)} = -16$. By a calculation we have $a_3 = 0$, $a_2 + 2a_4 = 6n$. To obtain a_2 and a_4, we consider the Spin-number in Theorem 2.4. In our case, the fixed-point set consists of a_2 groups of fixed points and a_4 groups of fixed points in Theorem 2.5. Recall that

$$\text{Spin}(g, X) = -\sum_{m \in F} \epsilon(g, X) \cdot \frac{1}{4} \csc(\frac{a_m \pi}{7}) \cdot \csc(\frac{b_m \pi}{7}),$$

for each $g \in G$.

Let ν_2, ν_4 be the total contribution to the Spin-number Spin(g,X) of a group of fixed points of types 2) and 4), respectively. All the possible values of them (in approximations) are listed below, taken at $k = 1, 2, 3$, respectively.

- $\nu_2 = -1,\ -1,\ -1,$
- $\nu_4 = -0.44504,\ 1.80194,\ -1.24698.$

Now, the Spin-number can be written as

$$\mathrm{Spin}(g, X) = -a_2 + \text{the total contribution of fixed points of type 4)}.$$

Denote by δ the total contribution of fixed points of type 4), and suppose that x_1, x_2, x_3 groups of fixed points of type 4) contribute $-0.44504,\ 1.80194,\ -1.24698$ respectively, then we have

$$\begin{cases} \delta = 0.44504x_1 + 1.80194x_2 - 1.24698x_3, \\ a_4 = x_1 + x_2 + x_3. \end{cases} \tag{3}$$

The Spin-number can be written as $\mathrm{Spin}(g, X) = \sum_{i=0}^{6} d_i \mu_7^i$, so

$$(a_2 + d_0 - \delta) + d_1\mu_7 + d_2\mu_7^2 + d_3\mu_7^3 + d_4\mu_7^4 + d_5\mu_7^5 + d_6\mu_7^6 = 0.$$

Then we have $a_2 + d_0 - \delta = d_k$, where $k = 1, \ldots, 6$. Together with the relation $\sum_{i=0}^{6} d_i = -\frac{\sigma}{8} = n$, we have

$$7d_0 + 6a_2 - 6\delta = n.$$

If we let $x_1 = x_2 = x_3$ in (3), then $\delta = \frac{a_4}{3}$. Further if $d_0 = 0$, one can get $6a_2 - 2a_4 = 6n$. Note that $a_2 + 2a_4 = 6n$, so $a_2 = n$ and $a_4 = \frac{5n}{2}$. Precisely, the fixed-point set of the $\mathbb{Z}_7$-action on $E(n)$ consists of n groups of fixed points of type 2) and $\frac{5n}{2}$ groups of fixed points of type 4) in Theorem 2.5. $\square$

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