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A Lower Bound for the Heegaard Genera of Annulus Sum

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Abstract Let M_i, $i=1,2$, be a compact orientable 3-manifold, and A_i an incompressible annulus on a component F_i of ∂M_i. Suppose A_1 is separating on F_1 and A_2 is non-separating on F_2. Let M be the annulus sum of M_1 and M_2 along A_1 and A_2. In the present paper, we give a lower bound for the genus of the annulus sum M in the condition of the Heegaard distances of the submanifolds M_1 and M_2.

Keywords genus; distance; annulus.

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1. Introduction

Let M_i be a compact connected orientable bordered 3-manifold, and A_i an incompressible annulus on ∂M_i, $i=1,2$. Let $h: A_1 \to A_2$ be a homeomorphism. The manifold M obtained by gluing M_1 and M_2 along A_1 and A_2 via h is called an *annulus sum* of M_1 and M_2 along A_1 and A_2, and is denoted by $M_1 \cup_h M_2$ or $M_1 \cup_{A_1=A_2} M_2$.

Let $V_i \cup_{S_i} W_i$ be a Heegaard splitting of M_i for $i=1,2$, and $M = M_1 \cup_{A_1=A_2} M_2$. Then from Schultens [13], we know that M has a natural Heegaard splitting $V \cup_S W$ induced from $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ with genus $g(S) = g(S_1) + g(S_2)$. So we always have $g(M) \le g(M_1) + g(M_2)$.

In the present paper, we suppose A_1 is separating on F_1 and A_2 is non-separating on F_2, and we give a lower bound for the genus of the annulus sum M in the condition of the Heegaard distances of M_1 and M_2. The main results are as follows:

Theorem 1 *Let M_i, $i=1,2$, be a compact orientable 3-manifold, and A_i an incompressible annulus on a component F_i of ∂M_i. Suppose that A_1 is separating on F_1 and A_2 is non-separating on F_2, and $M = M_1 \cup_{A_1=A_2} M_2$. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) > 2t_i + 2g(F_{3-i})$ where $(g(M_i) - g(F_{3-i})) \le t_i \le g(M_i)$, $i=1,2$. Then $g(M) \ge t_1 + t_2$.*

Furthermore, we have:

Corollary 2 *Let M_i, $i=1,2$, be a compact orientable 3-manifold, and A_i an incompressible*

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annulus on a component F_i of ∂M_i. Suppose that A_1 is separating on F_1 and A_2 is non-separating on F_2, and $M = M_1 \cup_{A_1=A_2} M_2$. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $d(S_i) > 2g(M_i) + 2g(F_{3-i})$ for $i = 1, 2$, then $g(M) = g(M_1) + g(M_2)$.

The paper is organized as follows. In Section 2, we review some preliminaries which will be used later. In Section 3, we give the proof of the main results.

2. Preliminaries

In this section, we will review some fundamental facts on surfaces in 3-manifolds. Definitions and terms which have not been defined are all standard, see [4].

A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V and W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M{=}V \cup W$. S is called a Heegaard surface of M. $V \cup_S W$ is said to be weakly reducible if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ with $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, $V \cup_S W$ is strongly irreducible.

A properly embedded surface is essential if it is incompressible and not ∂-parallel.

Let P be a properly embedded separating surface in a 3-manifold M which cuts M into two 3-manifolds M_1 and M_2. Then P is bicompressible if P has compressing disks in both M_1 and M_2. P is strongly irreducible if it is bicompressible and each compressing disk in M_1 meets each compressing disk in M_2.

Now let P be a closed bicompressible surface in an irreducible 3-manifold M. By maximally compressing P in both sides of P and removing the possible 2-sphere components, we denote the resulting surfaces by P_+ and P_-. Let H_1^P denote the closure of the region that lies between P and P_+ and similarly define H_2^P to denote the closure of the region that lies between P and P_-. Then H_1^P and H_2^P are compression bodies. If P is strongly irreducible in M, then the Heegaard splitting $H_1^P \cup_P H_2^P$ is strongly irreducible. Two strongly irreducible surfaces P and Q are said to be *well-separated* in M if $H_1^P \cup_P H_2^P$ is disjoint from $H_1^Q \cup_Q H_2^Q$ by isotopy.

Let $M = V \cup_S W$ be a Heegaard splitting, α and β be two essential simple closed curves in S. The distance $d(\alpha, \beta)$ of α and β is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$ in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$, for $1 \leq i \leq n$. The distance of the Heegaard splitting $V \cup_S W$ is defined to be $d(S) = \min\{d(\alpha, \beta)\}$, where α bounds an essential disk in V and β bounds an essential disk in W. $d(S)$ was first defined by Hempel [3].

Scharlemann and Thompson [11] showed that any irreducible and ∂-irreducible Heegaard splitting $M = V \cup_S W$ has an untelescoping as

$$V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m),$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with $F_i = \partial_- W_i \cap \partial_- V_{i+1}$, $1 \leq i \leq m-1$, $\partial_- V_1 = \partial_- V$, $\partial_- W_m = \partial_- W$, and for each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product. It is easy to see that when $m \geq 2$, $g(S) \geq g(S_i) + 1 \geq g(F_i) + 2$ for each i. From $V_1 \cup_{S_1} W_1$, $\cdots$, $V_m \cup_{S_m} W_m$, we can get a Heegaard splitting of M by a process called amalgamation [14].

The following are some basic facts and results on Heegaard splittings.

Lemma 1 ([7]) *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of essential annuli properly embedded in V. Then there is an essential disk D in V with $D \cap \mathcal{A} = \emptyset$.*

Lemma 2 ([2, 9]) *Let $V \cup_S W$ be a Heegaard splitting of M and F be an properly embedded incompressible surface (maybe not connected) in M. Then any component of F is parallel to ∂M or $d(S) \leq 2 - \chi(F)$.*

The following Lemma is a well known fact [13].

Lemma 3 *An incompressible surface F in a compression body V cuts V into compression bodies.*

Lemma 4 ([12]) *Let P and Q be strongly irreducible connected closed separating surfaces in a 3-manifold M. Then either*

(1) P and Q are well-separated;

(2) P and Q are isotopic, or

(3) $d(P) \leq 2g(Q)$.

Lemma 5 ([10]) *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of incompressible annuli properly embedded in V. If U is a component of $V \backslash \mathcal{A}$ with $U \cap \partial_- V \neq \emptyset$, then $\chi(U \cap \partial_- V) \geq \chi(U \cap \partial_+ V)$.*

Lemma 6 ([6]) *Let N be a compact orientable 3-manifold which is not a compression body, $F = \partial N$. Suppose Q is a properly embedded connected separating surface in N with ∂Q essential in F, and Q cuts N into two compression bodies N_1 and N_2 with $Q = \partial_+ N_1 \cap \partial_+ N_2$ and $F \cap N_2$ is a collection of annuli. If Q is compressible in both N_1 and N_2, and Q can be compressed to Q^* in N_1 such that any component of Q^* is ∂-parallel in N, then N has a Heegaard splitting $V \cup_S W$ with $d(S) \leq 2$ and $g(S) = 1 - \frac{1}{2}\chi(Q)$.*

3. Proof of the main results

In $M = M_1 \cup_h M_2$, let $A = A_2 = h(A_1)$ and F_i be the component of ∂M_i in which A_i lies, $i = 1, 2$. We denote the two components of $F_1 - intA$ by F_1^1 and F_1^2, and let $F_3 = F_1^1 \cup (F_2 - intA) \cup F_1^2$. Then F_3 is a boundary component of M. Let $I = [0, 1]$ and $F_i \times I$ be a regular neighborhood of F_i in M_i with $F_i = F_i \times \{0\}$. We denote by F^i the surface $F_i \times \{1\}$. Let $M^i = M_i - F_i \times [0, 1)$ for $i = 1, 2$, and $M^0 = F_1 \times I \cup_A F_2 \times I$. Then $M = M^1 \cup_{F^1} M^0 \cup_{F^2} M^2$.

Note that M^0 contains three boundary components F^1, F^2 and F_3. By [8, Lemma 2.3], M^0 contains two essential closed surfaces up to isotopy, we denote them by F_1^* and F_2^*, then $F_1^* = X_1 \cup X_2 \cup X_3$ such that X_1 and X_3 are isotopic to F_1^1, and X_2 is a copy of $F_2 - intA$. And $F_2^* = X_1 \cup X_2 \cup X_3$ such that X_1 and X_3 are isotopic to F_1^2, and X_2 is a copy of $F_2 - intA$.

Now we come to the proof of Theorem 1.

Proof of Theorem 1 Let us suppose for a contradiction that $g(M) < t_1 + t_2$. Then there exists a minimal Heegaard splitting $V \cup_S W$ of M with $g(S) \leq t_1 + t_2 - 1$.

Now if $V \cup_S W$ is strongly irreducible, $S \cap A \neq \emptyset$ since F^1 is essential in M and there is no closed essential surface in a compression body. By [13, Lemma 6], we may assume that each component of $S \cap A$ is essential in both S and A, and $|S \cap A|$ is minimal. Since A is an essential annulus in M, by Lemma 1, we assume that all components of $S \backslash A$ are incompressible except exactly one bicompressible component in $M \backslash A$.

Claim 1 $\chi(S \cap M_1) \leq -2t_1$.

Proof Now if $S \cap M_1$ is incompressible in M_1, it is essential in M_1. Otherwise, any component of $S \cap M_1$ is ∂-parallel in M_1, which means that M_1 is a compression body, a contradiction to $d(S_1) > 2t_1 + 2g(F_2)$. By Lemma 2, $2 - \chi(S \cap M_1) \geq d(S_1) > 2t_1 + 2g(F_2)$, thus $\chi(S \cap M_1) < 2 - 2t_1 - 2g(F_2) \leq -2t_1$.

Next we assume $S \cap M_1$ is bicompressible. We denote the bicompressible component of $S \cap M_1$ by P. In fact, P is strongly irreducible in M_1. $\chi(P) \leq -2$. If not, P is either a disk, an annulus, a pair of pants, or a once punctured torus, in each case we conclude that a component of ∂P bounds a disk in M_1, therefore A is compressible in M_1, a contradiction. If there exists an incompressible component Q of $S \cap M_1$ which is essential in M_1, by Lemma 2, $2 - \chi(Q) \geq d(S_1) > 2t_1 + 2g(F_2)$, then $\chi(S \cap M_1) \leq \chi(Q) + \chi(P) \leq -2t_1 - 2g(F_2) < -2t_1$. Hence in the following we may assume that the incompressible components of $S \cap M_1$ are all ∂-parallel in M_1. Let P^V be the surface obtained by maximally compressing P in V and removing all possible 2-sphere components. Since P is strongly irreducible, P^V is incompressible in M_1. Now if P^V is essential in M_1, by Lemma 2, $2 - \chi(P^V) \geq d(S_1) > 2t_1 + 2g(F_2)$, then $\chi(S \cap M_1) \leq \chi(P) \leq \chi(P^V) - 2 \leq -2t_1 - 2g(F_2) < -2t_1$. Then we may assume that each component of P^V is ∂-parallel in M_1.

Since A is an essential annulus in M and by Lemma 3, each component of $V \cap M_1$ and $W \cap M_1$ is a compression body. Let U_1 be the component of $V \cap M_1$ containing P and U_2 be the component of $W \cap M_1$ containing P. Then by parallelism $U_1 \cup_P U_2 \cong M_1$ and $\partial_+ U_1 \cap \partial_+ U_2 = P$. Since M_1 is not a compression body and A is an annulus, by Lemma 6, there exists a Heegaard surface S^1 of M_1 with $d(S^1) \leq 2$ and $g(S^1) \leq 1 - \frac{1}{2}\chi(P)$. Now $d(S_1) > 2t_1 + 2g(F_2) \geq 2g(M_1)$, then by [5, Lemma 3.3], S_1 is the unique minimal Heegaard surface of M_1. But $d(S^1) \leq 2$, hence S^1 is not isotopic to S_1. Then we have $g(S^1) \geq g(M_1) + 1$. Hence $\chi(S \cap M_1) \leq \chi(P) \leq 2 - 2g(S^1) \leq -2g(M_1) \leq -2t_1$.

This completes the proof of Claim 1. □

Then by Claim 1, we have $\chi(S \cap M_1) \leq -2t_1$, $\chi(S \cap M_2) < -2t_2$. Then $2g(S) = 2 - \chi(S \cap M_1) - \chi(S \cap M_2) > 2t_1 + 2t_2 + 2$, a contradiction.

Hence $V \cup_S W$ is weakly reducible, then $V \cup_S W$ has an untelescoping as

$$V \cup_S W = (V_1^{'} \cup_{S_1^{'}} W_1^{'}) \cup_{H_1} \cdots \cup_{H_{m-1}} (V_m^{'} \cup_{S_m^{'}} W_m^{'}),$$

where $m \geq 2$, and for each i, each component of H_i is a closed essential surface in M. Let $\mathcal{F} = \{H_1, \ldots, H_{m-1}\}$.

Claim 2 Let H_i be a component of $\mathcal{F}$. Suppose H_i is a boundary component of $N_i = V_i^{'} \cup_{S_i^{'}} W_i^{'}$ in the untelescoping. If $H_i \cap M_1$ is essential in M_1, then $\chi(S_i^{'} \cap M_1) < 2 - 2t_1 - 2g(F_2)$.

Proof Since $H_i \cap M_1$ is essential in M_1, let Q^* be an essential component of $H_i \cap M_1$. By Lemma 2, $2-\chi(Q^*) \geq d(S_1) > 2t_1+2g(F_2)$. If we denote the component of $V_i^{'} \cap M_1$ or $W_i^{'} \cap M_1$ which contains Q^* as part of boundary component by U, since A is an essential annulus in M, by Lemma 5, we have $\chi(S_i^{'} \cap M_1) \leq \chi(U \cap (S_i^{'} \cap M_1)) \leq \chi(U \cap Q^*) = \chi(Q^*) < 2-2t_1-2g(F_2)$.

This completes the proof of Claim 2. □

Claim 3 For any $i \in \{2, \ldots, m-1\}$, there are no two components H_{i-1}, H_i in $\mathcal{F}$ so that $H_{i-1} \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is essential in M_2 whether $H_{i-1} \cap M_1$ and $H_i \cap M_2$ are with boundary or not.

Proof Suppose there exist two components of $\mathcal{F}$ so that $H_{i-1} \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is essential in M_2. Suppose $V_i^{'} \cup_{S_i^{'}} W_i^{'}$ is the Heegaard splitting in the untelescoping between them. Then by Claim 2, we have $\chi(S_i^{'} \cap M_1) < 2-2t_1-2g(F_2)$, and $\chi(S_i^{'} \cap M_2) < 2-2t_2-2g(F_1)$. Hence $2g(S) \geq 4-\chi(S_i^{'}) > 2t_1+2t_2+2g(F_1)+2g(F_2)$, a contradiction.

This completes the proof of Claim 3. □

We now divide the proof into the following four cases to discuss.

Case 1 Any component of $\mathcal{F}$ is not ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} = \emptyset$.

In this case, by Claim 3 and the assumption, we may assume that any component of $\mathcal{F}$ is contained in M_1. Let H be an outermost component of $\mathcal{F}$ in M_1, H is essential in M_1.

Suppose $A \subset N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. A is essential in M, so is in N_j. Since H is essential in M_1, by Claim 2, we have $\chi(S_j^{'} \cap M_1) < 2-2t_1-2g(F_2)$. Now $N_j \cap M_2 = M_2$, by Claim 1, we have $\chi(S_j^{'} \cap M_2) \leq -2t_2$, then $2g(S) \geq 4-\chi(S_j^{'}) > 2t_1+2t_2+2g(F_2)$, a contradiction.

Case 2 Any component of $\mathcal{F}$ is not ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} \neq \emptyset$.

In this case, we may assume that any component of $\mathcal{F} \cap A$ is essential in both A and $\mathcal{F}$, and $|\mathcal{F} \cap A|$ is minimal. There are three subcases.

Subcase 2.1 The outermost component H of $\mathcal{F}$ with $H \cap A \neq \emptyset$ is essential in M_1 but ∂-parallel in M_2.

By Claim 3, we may assume that each component of $\mathcal{F} \cap M_1$ with boundary is essential in M_1 and each component of $\mathcal{F} \cap M_2$ with boundary is ∂-parallel in M_2. Among the surfaces of $\mathcal{F} \cap M_2$, let B be the innermost one, that is, B cuts M_2 into two pieces M_2' and M_2'', where $M_2' \cong M_2$ and $M_2'' \cong B \times I$, and the interior of M_2' contains no component of $\mathcal{F} \cap M_2$. B lies in a component, say H_r, of $\mathcal{F}$. Hence $H_r \cap M_1$ is essential in M_1 and $H_r \cap M_2$ is ∂-parallel in M_2, see Figure 1(a).

We may assume that $M_2^{'}$ is contained in the submanifold $N_r = V_r^{'} \cup_{S_r^{'}} W_r^{'}$ of the untelescoping. Since $H_r \cap M_1$ is essential in M_1, by Claim 2, $\chi(S_r^{'} \cap M_1) < 2-2t_1-2g(F_2)$. Note that $N_r \cap M_2 \cong M_2$, by Claim 1, we have $\chi(S_r^{'} \cap M_2) \leq -2t_2$, then $2g(S) \geq 4-\chi(S_r^{'}) > 2t_1+2t_2+2g(F_2)$, a contradiction.

Subcase 2.2 The outermost component H of $\mathcal{F}$ with $H \cap A \neq \emptyset$ is essential in M_2 but ∂-parallel in M_1.

There are two sub-subcases.

Sub-subcase 2.2.1 Each component of $H \cap M_1$ is parallel to the same one of F_1^1 or F_1^2, say F_1^1, in M_1.

We denote the Heegaard splitting in the untelescoping between F_3 and H by $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. See Figure 1($b$). Note that $N_j \cap M_1 \cong M_1$, by Claim 1, we have $\chi(S_j^{'} \cap M_1) \leq -2t_1$. Since $H \cap M_2$ is essential in M_2, by Claim 2, $\chi(S_j^{'} \cap M_2) < 2 - 2t_2 - 2g(F_1)$. Then $2g(S) \geq 4 - \chi(S_j^{'}) > 2t_1 + 2t_2 + 2g(F_1)$, a contradiction.

Sub-subcase 2.2.2 At least one component of $H \cap M_1$ is parallel to F_1^1 and at least one component of $H \cap M_1$ is parallel to F_1^2 in M_1.

By Claim 3, we may assume that each component of $\mathcal{F} \cap M_1$ with boundary is ∂-parallel in M_1. Among the surfaces of $\mathcal{F} \cap M_1$ which is parallel to F_1^i, let B_i be the innermost one, $i = 1, 2$. Then B_1 and B_2 cut M_1 into three pieces $M_1^{'}$, $M_1^{''}$ and $M_1^{'''}$ with $M_1^{'} \cong B_1 \times I$, $M_1^{''} \cong M_1$ and $M_1^{'''} \cong B_2 \times I$, and the interior of $M_1^{''}$ contains no component of $\mathcal{F} \cap M_1$. B_2 lies in a component, say H_j, of $\mathcal{F}$. Hence by Claim 3, we have that $H_j \cap M_1$ is ∂-parallel in M_1 and $H_j \cap M_2$ is essential in M_2. We may assume that $M_1^{''}$ is contained in the submanifold $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$ of the untelescoping, see Figure 1(c). Since $H_j \cap M_2$ is essential in M_2, by Claim 2, $\chi(S_j^{'} \cap M_2) < 2 - 2t_2 - 2g(F_1)$. Note that $N_j \cap M_1 \cong M_1$, by Claim 1, we have $\chi(S_j^{'} \cap M_1) \leq -2t_1$, then $2g(S) \geq 4 - \chi(S_j^{'}) > 2t_1 + 2t_2 + 2g(F_1)$, a contradiction.

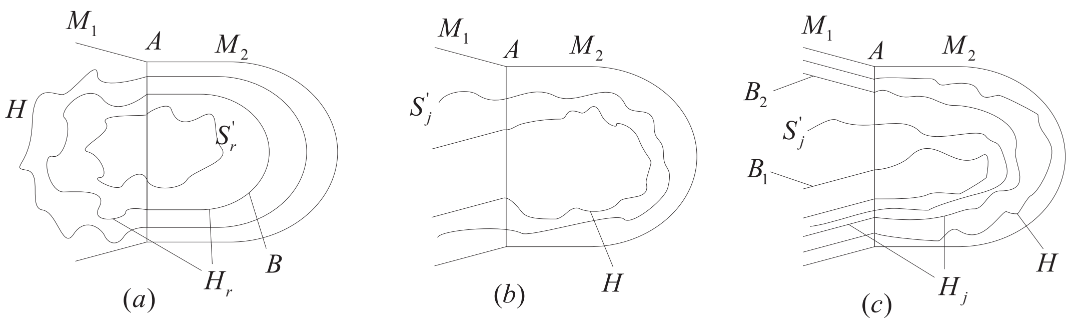

Figure 1 Surfaces intersecting A

Subcase 2.3 The outermost component H of $\mathcal{F}$ with $H \cap A \neq \emptyset$ is isotopic to F_1^* or F_2^*, say, F_1^*.

We denote the Heegaard splitting in the untelescoping between F_3 and F_1^* by $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. Let $S_j^1 = S_j^{'} \cap M_1$ and $S_j^2 = S_j^{'} \cap M_2$. Now if N_j has some other boundary component H^*, then by assumption $H^* \cap A = \emptyset$, i.e., H^* is a closed essential surface in M_1 or M_2. Now $N_j \cap M_2 \cong (F_2 - \operatorname{int} A) \times I$, hence $H^* \subset M_1$. Since H^* is an essential surface in M_1, by Claim 2, we have $\chi(S_j^1) < 2 - 2t_1 - 2g(F_2)$. If N_j has no other boundary component, then $N_j \cap M_1 \cong M_1$. By Claim 1, we have $\chi(S_j^1) \leq -2t_1$.

Claim 4 In either case, $\chi(S_j^2) \leq \chi(F_2)$.

Proof In either case, $N_j \cap M_2 \cong (F_2 - \operatorname{int} A) \times I$. Now if S_j^2 is incompressible in $F_2 \times I$, since the incompressible and ∂-incompressible surface in a trivial compression body is just spanning annulus, by [9, Lemma 2.3], any component of S_j^2 is parallel to $F_2\backslash A_2$ in M_2, hence $\chi(S_j^2) \leq \chi(F_2)$.

Now if S_j^2 is bicompressible in $F_2 \times I$, by maximally compressing it in $V_j^{'}$, we obtain a surface S_j^{2*}. Then by [9, Lemma 2.3], any component of S_j^{2*} is parallel to $F_2\backslash A_2$ in M_2, hence $\chi(S_j^2) \leq \chi(S_j^{2*}) - 2 \leq \chi(F_2) - 2 < \chi(F_2)$.

This completes the proof of Claim 4. □

Hence whether N_j has some other boundary component or not, we have $\chi(S_j^{'}) = \chi(S_j^1) + \chi(S_j^2) \leq 2 - 2t_1 - 2g(F_2)$.

We denote the Heegaard splitting in the untelescoping on the other side of F_1^* which has F_1^* as a boundary component by $N_r = V_r^{'} \cup_{S_r^{'}} W_r^{'}$. Let $S_r^i = S_r^{'} \cap M_i$, $i = 1, 2$.

There are three sub-subcases.

Sub-subcase 2.3.1 N_r has another boundary component $H^{'}$ of $\mathcal{F}$ with $H^{'} \cap M_1$ essential in M_1.

In this case, if $H^{'} \cap M_2 = \emptyset$, then $H^{'} \subset (F_1^1 \times I)$, which means that a compression body contains a closed essential surface, a contradiction. Hence $H^{'} \cap M_2 \neq \emptyset$, then all components of $H^{'} \cap M_2$ are ∂-parallel in M_2, and furthermore, by Claim 3, we may assume that each component of $(\mathcal{F} - \{F_1^*\}) \cap M_1$ with boundary is essential in M_1 and each component of $\mathcal{F} \cap M_2$ with boundary is ∂-parallel in M_2.

The following arguments are in some sense similar to those of subcase 2.1. Take the innermost component B of $\mathcal{F} \cap M_2$, that is, B cuts M_2 into two pieces M_2' and M_2'', where $M_2' \cong M_2$ and $M_2'' \cong B \times I$, and the interior of M_2' contains no component of $\mathcal{F} \cap M_2$. B lies in a component, say H_i, of $\mathcal{F}$. Hence $H_i \cap M_1$ is essential in M_1 and $H_i \cap M_2$ is ∂-parallel in M_2. We may assume that $M_2^{'}$ is contained in the submanifold $N_i = V_i^{'} \cup_{S_i^{'}} W_i^{'}$ of the untelescoping. Since $H_i \cap M_1$ is essential in M_1, by Claim 2, $\chi(S_i^{'} \cap M_1) < 2 - 2t_1 - 2g(F_2)$. Note that $N_i \cap M_2 \cong M_2$, by Claim 1, $\chi(S_i^{'} \cap M_2) \leq -2t_2$, then $2g(S) \geq 4 - \chi(S_i^{'}) > 2t_1 + 2t_2 + 2g(F_2)$, a contradiction.

Sub-subcase 2.3.2 N_r has another boundary component $H^{'}$ of $\mathcal{F}$ with $H^{'} \cap M_2$ essential in M_2.

In this case, $H^{'} \cap M_2$ is essential in M_2. By Claim 2, we have that $\chi(S_r^2) < 2 - 2t_2 - 2g(F_1)$. Whether $H^{'} \cap M_1 = \emptyset$ or not, since $S_2^{'}$ is separating in N_2, $|S_2^{'} \cap A|$ is even while $|\partial F_1^1| = 1$. This means that $S_2^{'} \cap (F_1 \times I)$ has at least two components. Then by Claim 4, we have that $\chi(S_r^1) \leq 2\chi(F_1^1)$. Hence $2g(S) \geq 2 - \chi(S_r^{'}) - \chi(S_j^{'}) + \chi(F_1^*) > 2t_1 + 2t_2 + 2g(F_1)$, a contradiction.

Sub-subcase 2.3.3 N_r has no other boundary component.

In this case, $N_r \cap M_2 \cong M_2$. By Claim 1, we have $\chi(S_r^1) \leq 2\chi(F_1^1)$, $\chi(S_r^2) \leq -2t_2$. Hence $2g(S) = 2 - \chi(S_r^{'}) - \chi(S_j^{'}) + \chi(F_1^*) \geq 2t_1 + 2t_2 + 2$, a contradiction.

Case 3 There exists one component of $\mathcal{F}$ which is ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} = \emptyset$.

In this case, without loss of generality, we may assume that $F^1 \subset \mathcal{F}$. Now whether there exists some component of $\mathcal{F}$ in $intM^1$ or not, by amalgamating the Heegaard splittings in the untelescoping contained in M^1, we get a generalized Heegaard splitting $M^1 = V_1^* \cup_{S_1^*} W_1^*$ with

$g(S_1^*) \geq g(M_1)$.

If there is no other component of $\mathcal{F}$ in M_2, we denote the Heegaard splitting of $M^0 \cup_{F^2} M^2$ in the untelescoping by $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. Since $S_j^{'}$ is a Heegaard surface of $M^0 \cup_{F^2} M^2$ while S_2 is a Heegaard surface of M_2, $S_j^{'}$ is not isotopic to S_2, and furthermore, they are not well-separated. Then by Lemma 4, we have $d(S_2) \leq 2g(S_j^{'})$, hence $g(S_j^{'}) > t_2 + g(F_1)$. Then we have $g(S) \geq g(S_1^*) + g(S_j^{'}) - g(F_1) > g(M_1) + t_2 \geq t_1 + t_2$, a contradiction. Hence there is some other component of $\mathcal{F}$ in M_2, let F_* be the outermost one. If F_* is essential in M_2, we denote the Heegaard splitting in the untelescoping between F^1, F_* and F_3 by $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. Now by Claim 2, we have $\chi(S_j^{'} \cap M_2) < 2 - 2t_2 - 2g(F_1)$. Since $\chi(S_j^{'} \cap M_1) \leq 0$, we have $g(S) \geq g(S_1^*) + g(S_j^{'}) - g(F_1) + 1 > g(M_1) + t_2 \geq t_1 + t_2$, a contradiction. Hence F_* is ∂-parallel in M_2.

Then we get a generalized Heegaard splitting as: $V \cup_S W = (V_1^{'} \cup_{S_1^{'}} W_1^{'}) \cup_{H_1} (V_2^{'} \cup_{S_2^{'}} W_2^{'}) \cup_{H_2} (V_3^{'} \cup_{S_3^{'}} W_3^{'})$, and H_1 is isotopic to F^1, H_2 is isotopic to F^2. We may further assume that $V_1^{'} \cup_{S_1^{'}} W_1^{'}$ is a Heegaard splitting of M^1, $V_2^{'} \cup_{S_2^{'}} W_2^{'}$ is a Heegaard splitting of M^0, and $V_3^{'} \cup_{S_3^{'}} W_3^{'}$ is a Heegaard splitting of M^2. Since A is separating on F_1 and non-separating on F_2, M^0 contains only three boundary components F^1, F^2 and F_3. Note that $g(F_3) = g(F_1) + g(F_2) - 1$, hence $g(S_2^{'}) \geq g(M^0) \geq g(F_1) + g(F_2)$. Then we have $g(S) = g(S_1^{'}) + g(S_2^{'}) + g(S_3^{'}) - g(H_1) - g(H_2) \geq g(M_1) + g(M_2) \geq t_1 + t_2$, a contradiction.

Case 4 There exists one component of $\mathcal{F}$ which is ∂-parallel in M_1 or M_2, and $A \cap \mathcal{F} \neq \emptyset$.

Now there are two subcases.

Subcase 4.1 $F^2 \subset \mathcal{F}$.

let H be a component of $\mathcal{F}$ with $H \cap A \neq \emptyset$. If $H \cap M_1$ is essential in M_1 and $H \cap (F_2 \times I)$ is ∂-parallel in $F_2 \times I$, by Lemma 2, $2 - \chi(H \cap M_1) \geq d(S_1) > 2t_1 + 2g(F_2)$, $\chi(H \cap (F_2 \times I)) \leq \chi(F_2)$, then $g(S) \geq g(M_2) + g(H) + 1 - g(F_2) > t_1 + g(M_2)$, a contradiction.

Hence if $H \cap A \neq \emptyset$, $H \cap M_1$ is ∂-parallel in M_1 and $H \cap (F_2 \times I)$ is ∂-parallel in $F_2 \times I$. Then H can be isotoped to be an essential closed surface in M^0, hence H is isotopic to either F_1^* or F_2^*. We may assume that H is isotopic to F_1^*.

If there is no other component of $\mathcal{F}$ in M_1, we denote the Heegaard splitting in the untelescoping between F_1^* and F_3 by $N_1 = V_1^{'} \cup_{S_1^{'}} W_1^{'}$. Note that $N_1 \cap M_1 \cong M_1$ and $N_1 \cap M_2 \cong F_2 \times I$. By Claim 1, we have $\chi(S_1^{'} \cap M_1) \leq -2t_1$, $\chi(S_1^{'} \cap (F_2 \times I)) \leq \chi(F_2)$. Then $g(S) \geq g(M_2) + g(S_1^{'}) + 1 - g(F_2) \geq t_1 + g(M_2)$, a contradiction.

Hence one component of $\mathcal{F}$ must be parallel to F^1 in M_1. Then by the same arguments as the last paragraph of case 3, we get a contradiction.

Subcase 4.2 $F^1 \subset \mathcal{F}$.

Let $\mathcal{H} = \{\mathcal{H} : \mathcal{H} \subset \mathcal{F}$ and $H \cap M_2$ is essential in $M_2\}$. If some component $H^{'}$ of $\mathcal{H}$ and F^1 cobound a Heegaard splitting in the untelescoping, we denote the Heegaard splitting between $H^{'}$ and F^1 by $N_j = V_j^{'} \cup_{S_j^{'}} W_j^{'}$. Since $H^{'} \cap M_2$ is essential in M_2, by Claim 2, we have $\chi(S_j^{'} \cap M_2) < 2 - 2t_2 - 2g(F_1)$, $\chi(S_j^{'} \cap (F_1 \times I)) \leq 0$, then we have $g(S) \geq g(M_1) + g(S_j^{'}) - g(F_1) > g(M_1) + t_2$, a contradiction.

Hence the outermost component with $H \cap A \neq \emptyset$ must be ∂-parallel in M_2. We may assume that H is isotopic to F_1^*. Let $N_1 = V_1' \cup_{S_1'} W_1'$ be the Heegaard splitting bounded by F^1, F_1^* and F_3 in the untelescoping. Then $g(N_1) \geq \min\{g(F_1)+g(F_1^*), g(F_1)+g(F_3), g(F_1^*)+g(F_3)\}$. Note that $g(F_3) = g(F_1)+g(F_2)-1$ and $g(F_1^*) = g(F_2)+2g(F_1^1)-1$, hence $g(S_1') \geq g(N_1) \geq g(F_1)+g(F_2)$.

If there is no other component of $\mathcal{F}$, we denote the Heegaard splitting in the untelescoping bounded by F_1^* by $N_j = V_j' \cup_{S_j'} W_j'$. A is essential in M, so is in N_j. Note that $N_j \cap M_1 \cong F_1^1 \times I$ and $N_j \cap M_2 \cong M_2$. By Claim 1, we have $\chi(S_j' \cap M_2) \leq -2t_2$, and by Claim 4, $\chi(S_2' \cap (F_1 \times I)) \leq 2\chi(F_1^1)$. Then we have $g(S) \geq g(M_1)+g(S_1')+g(S_j')-g(F_1)-g(F_1^*) \geq g(M_1)+t_2$, a contradiction.

Hence there is some other component F^* of $\mathcal{F}$. If $F^* \cap M_2$ is essential in M_2, we denote the Heegaard splitting in the untelescoping between F_1^* and F^* by $N_2 = V_2' \cup_{S_2'} W_2'$. Then by Claim 2, we have $\chi(S_2' \cap M_2) < 2 - 2t_2 - 2g(F_1)$. By Claim 4, we have $\chi(S_2' \cap (F_1 \times I)) \leq 2\chi(F_1^1)$. Then $g(S) \geq g(M_1) + g(S_1') + g(S_2') - g(F_1) - g(F_1^*) + 1 > g(M_1) + t_2$, a contradiction.

Hence one component of $\mathcal{F}$ must be parallel to F^2 in M_2. Then by the same arguments as the last paragraph of Case 3, we get a contradiction.

Therefore, the required equation holds. This finishes the proof of Theorem 1. □

We now come to the proof of Corollary 2.

Proof of Corollary 2 Now let $t_i = g(M_i)$, $i = 1, 2$. Then by the results of Theorem 1, we have $g(M) \geq g(M_1) + g(M_2)$. Since M is the annulus sum of M_1 and M_2, by the result of Schultens [13], we have $g(M) \leq g(M_1) + g(M_2)$. Hence $g(M) = g(M_1) + g(M_2)$.

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