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Existence of Positive Solutions for Systems of Nonlinear Second-Order Differential Equations on the Half Line in a Banach Space

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Abstract In this paper, the cone theory and Mönch fixed point theorem combined with the monotone iterative technique are used to investigate the positive solutions for a class of systems of nonlinear singular differential equations with multi-point boundary value conditions on the half line in a Banach space. The conditions for the existence of positive solutions are formulated. In addition, an explicit iterative approximation of the solution is also derived.

Keywords systems of singular differential equations; cone and ordering; positive solutions; Mönch fixed point theorem; measure of non-compactness.

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1. Introduction

In recent years, the theory of ordinary differential equations in Banach space has become a new important branch of investigation (see, for example, [1–4] and references therein). In a recent paper, Liu [14] investigated the existence of solutions of the following second-order two-point boundary value problems (BVP for short) on infinite intervals in a Banach space E:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)), \quad t \in J, \\ x(0) = x_0, \quad x'(\infty) = y_\infty, \end{cases}$$

where $f \in C[J \times E \times E, E]$, $J = [0, +\infty)$, $x'(\infty) = \lim_{t\to\infty} x'(t)$. The main tool used is the Sadovskii's fixed point theorem. On the other hand, the multi-point boundary value problems arising from applied mathematics and physics have been studied extensively in the literature. There are many excellent results about the existence of positive solutions for multi-point boundary value problems in scalar case (see, for instance, [5–11] and references therein). However, such

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results are fewer in Banach spaces [12, 13, 16]. In [16], we investigated the positive solutions for the following multi-point boundary value problems in a Banach space E

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in J_+, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(\infty) = y_\infty, \end{cases}$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, $\alpha_i \in [0, +\infty)$, $\xi_i \in (0, +\infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $\sum_{i=1}^{m-2} \alpha_i \xi_i / (1 - \sum_{i=1}^{m-2} \alpha_i) > 1$.

It seems that there are few results available for systems of second-order differential equations with multi-point in Banach spaces. In this paper, we consider the following singular m-point boundary value problem on the half line in a Banach space E:

$$\begin{cases} x''(t) + f(t, x(t), x'(t), y(t), y'(t)) = 0, \\ y''(t) + g(t, x(t), x'(t), y(t), y'(t)) = 0, \quad t \in J_+, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(\infty) = x_\infty, \\ y(0) = \sum_{i=1}^{m-2} \beta_i y(\xi_i), \quad y'(\infty) = y_\infty, \end{cases} \tag{1}$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, $\alpha_i, \beta_i \in [0, +\infty)$, $\xi_i \in (0, +\infty)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $0 < \sum_{i=1}^{m-2} \beta_i < 1$. Nonlinear terms $f(t, x_0, x_1, y_0, y_1)$ and $g(t, x_0, x_1, y_0, y_1)$ permit singularities at $t = 0$, x_i, $y_i = \theta$ $(i = 0, 1)$ where θ denotes the zero element of Banach space E. By singularity, we mean that $\|f(t, x_0, x_1, y_0, y_1)\| \to \infty$ as $t \to 0^+$ or $x_i, y_i \to \theta$ $(i = 0, 1)$.

Recently, using Shauder fixed point theorem, Guo [15] obtained the existence of positive solutions for a class of nth-order nonlinear impulsive singular integro-differential equations in a Banach space. Motivated by Guo's work, in this paper, we shall use the cone theory and the Mönch fixed point theorem combined with a monotone iterative technique to investigate the positive solutions BVP (1). The main features are as follows: Firstly, compared with [14], the problem we discussed here is systems of multi-point boundary value problem and nonlinear terms permit singularity not only at $t = 0$ but also at x_i, $y_i = \theta$ $(i = 0, 1)$. Secondly, the construction of nonempty convex closed set is completely different from that in [15] and [16] since the problems considered here are multi-point boundary value problems for systems. It is worth pointing out that by employing the new constructed nonempty convex closed set, we relax the restriction on the coefficients a_i and ξ_i, i.e., we delete the condition that $\sum_{i=1}^{m-2} \alpha_i \xi_i / (1 - \sum_{i=1}^{m-2} \alpha_i) > 1$. Furthermore, the relative compact conditions we used are weaker. Finally, an iterative sequence for the solution under some normal type conditions is established which makes it convenient in applications.

2. Preliminaries and several lemmas

Let

$$FC[J, E] = \{x \in C[J, E] : \sup_{t \in J} \frac{\|x(t)\|}{t + 1} < \infty\},$$

and

$$DC^1[J,E] = \{x \in C^1[J,E] : \sup_{t\in J} \frac{\|x(t)\|}{t+1} < \infty \text{ and } \sup_{t\in J} \|x'(t)\| < \infty\}.$$

Evidently, $C^1[J,E] \subset C[J,E]$, $DC^1[J,E] \subset FC[J,E]$. It is easy to see that $FC[J,E]$ is a Banach space with norm

$$\|x\|_F = \sup_{t\in J} \frac{\|x(t)\|}{t+1},$$

and $DC^1[J,E]$ is also a Banach space with norm

$$\|x\|_D = \max\{\|x\|_F, \|x'\|_C\},$$

where

$$\|x'\|_C = \sup_{t\in J} \|x'(t)\|.$$

Let $X = DC^1[J,E] \times DC^1[J,E]$ with norm $\|(x,y)\|_X = \max\{\|x\|_D, \|y\|_D\}$, $\forall\ (x,y) \in X$. Then $(X, \|\cdot,\cdot\|_X)$ is also a Banach space. The basic space in this paper is $(X, \|\cdot,\cdot\|_X)$.

Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x \le y$. If $x \le y$ and $x \ne y$, we write $x < y$. Let $P_+ = P\backslash\{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [4].

In what follows, we always assume that $x_\infty \ge x_0^*$, $y_\infty \ge y_0^*$, x_0^*, $y_0^* \in P_+$. Let $P_{0\lambda} = \{x \in P : x \ge \lambda x_0^*\}$, $P_{1\lambda} = \{y \in P : y \ge \lambda y_0^*\}$ $(\lambda > 0)$. Obviously, $P_{0\lambda}, P_{1\lambda} \subset P_+$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_{01}$, $P_1 = P_{11}$, i.e., $P_0 = \{x \in P : x \ge x_0^*\}$, $P_1 = \{y \in P : y \ge y_0^*\}$. Let $P(F) = \{x \in FC[J,E] : x(t) \ge \theta, \forall\ t \in J\}$, and $P(D) = \{x \in DC^1[J,E] : x(t) \ge \theta, x'(t) \ge \theta, \forall t \in J\}$. Clearly, $P(F)$, $P(D)$ are cones in $FC[J,E]$ and $DC^1[J,E]$, respectively. A map $(x,y) \in DC^1[J,E] \cap C^2[J'_+,E]$ is called a positive solution of BVP (1) if $(x,y) \in P(D) \times P(D)$ and $(x(t),y(t))$ satisfies BVP (1).

Let α, α_F, α_D, α_X denote Kuratowski measure of non-compactness in E, $FC[J,E]$, $DC^1[J,E]$ and X, respectively. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [1–4]. For notational simplicity, denote

$$D_0 = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i} \sum_{i=1}^{m-2} \alpha_i \xi_i, \quad D_1 = \frac{1}{1-\sum_{i=1}^{m-2}\beta_i} \sum_{i=1}^{m-2} \beta_i \xi_i,$$
$$\lambda_0^* = \min\{D_0, 1\}, \quad \lambda_1^* = \min\{D_1, 1\}. \tag{2}$$

Throughout this paper, we make the following assumptions.

(H_1) $f, g \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P_{1\lambda} \times P_{1\lambda}, P]$ for any $\lambda > 0$ and there exist a_i, b_i, $c_i \in L[J_+, J]$ and $h_i \in C[J_+ \times J_+ \times J_+ \times J_+, J]$ $(i = 0,1)$ such that

$$\|f(t,x_0,x_1,y_0,y_1)\| \le a_0(t) + b_0(t)h_0(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$
$$\forall\ t \in J_+, x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*}, \quad i = 0,1,$$
$$\|g(t,x_0,x_1,y_0,y_1)\| \le a_1(t) + b_1(t)h_1(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$
$$\forall\ t \in J_+, x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*}, \quad i = 0,1,$$

and

$$\frac{\|f(t,x_0,x_1,y_0,y_1)\|}{c_0(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|)}\to 0,\quad \frac{\|g(t,x_0,x_1,y_0,y_1)\|}{c_1(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|)}\to 0$$
$$\text{as } x_i\in P_{0\lambda_0^*}, y_i\in P_{1\lambda_1^*}\ (i=0,1), \|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|\to\infty,$$

uniformly for $t\in J_+$, and

$$\int_0^\infty a_i(t)\mathrm{d}t=a_i^*<\infty,\ \int_0^\infty b_i(t)\mathrm{d}t=b_i^*<\infty,\ \int_0^\infty c_i(t)(1+t)\mathrm{d}t=c_i^*<\infty,\quad i=0,1.$$

(H_2) For any $t\in J_+$ and countable bounded set $V_i\subset DC^1[J,P_{0\lambda_0^*}]$, $W_i\subset DC^1[J,P_{1\lambda_1^*}]$ $(i=0,1)$, there exist $L_i(t),K_i(t)\in L[J,J]$ $(i=0,1)$ such that

$$\alpha(f(t,V_0(t),V_1(t),W_0(t),W_1(t)))\le\sum_{i=0}^{1}L_{0i}(t)\alpha(V_i(t))+K_{0i}(t)\alpha(W_i(t)),$$

$$\alpha(g(t,V_0(t),V_1(t),W_0(t),W_1(t)))\le\sum_{i=0}^{1}L_{1i}(t)\alpha(V_i(t))+K_{1i}(t)\alpha(W_i(t))$$

with

$$(D_i+1)\int_0^{+\infty}[(L_{i0}(s)+K_{i0}(s))(1+s)+L_{i1}(s)+K_{i1}(s)]\mathrm{d}s<\frac{1}{2},\quad i=0,1.$$

(H_3) $t\in J_+,\lambda_0^*x_0^*\le x_i\le\overline{x}_i,\lambda_1^*y_0^*\le y_i\le\overline{y}_i$ $(i=0,1)$ imply

$$f(t,x_0,x_1,y_0,y_1)\le f(t,\overline{x}_0,\overline{x}_1,\overline{y}_0,\overline{y}_1),\quad g(t,x_0,x_1,y_0,y_1)\le g(t,\overline{x}_0,\overline{x}_1,\overline{y}_0,\overline{y}_1).$$

Hereafter, we write $Q_1=\{x\in DC^1[J,P]:x^{(i)}(t)\ge\lambda_0^*x_0^*,\ \forall t\in J,i=0,1\}$, $Q_2=\{y\in DC^1[J,P]:y^{(i)}(t)\ge\lambda_1^*y_0^*,\forall\ t\in J,i=0,1\}$, and $Q=Q_1\times Q_2$. Evidently, Q_1,Q_2 and Q are closed convex set in $DC^1[J,E]$ and X, respectively.

We shall reduce BVP (1) to a system of integral equations in E. To this end, we first consider operator A defined by

$$A(x,y)(t)=(A_1(x,y)(t),A_2(x,y)(t)),\tag{3}$$

where

$$\begin{aligned}&A_1(x,y)(t)\\&\quad=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\&\qquad\int_0^t\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s+tx_\infty,\end{aligned}\tag{4}$$

and

$$\begin{aligned}&A_2(x,y)(t)\\&\quad=\frac{1}{1-\sum_{i=1}^{m-2}\beta_i}\Big[\Big(\sum_{i=1}^{m-2}\beta_i\xi_i\Big)y_\infty+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\int_s^{+\infty}g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\&\qquad\int_0^t\int_s^{+\infty}g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s+ty_\infty.\end{aligned}\tag{5}$$

Lemma 1 *If condition* (H_1) *is satisfied, then operator* A *defined by (3) is a continuous operator from* Q *into* Q.

Proof Let

$$\varepsilon_0 = \min\Big\{\frac{1}{8c_0^*\Big(1+\frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)}, \frac{1}{8c_1^*\Big(1+\frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\beta_i}\Big)}\Big\}, \tag{6}$$

and

$$r = \min\Big\{\frac{\lambda_0^*\|x_0^*\|}{N}, \frac{\lambda_1^*\|y_0^*\|}{N}\Big\} > 0. \tag{7}$$

By (H_1), there exists an $R > r$ such that

$$\|f(t,x_0,x_1,y_0,y_1)\| \le \varepsilon_0 c_0(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|),\ \forall\ t\in J_+,$$
$$x_i \in P_{0\lambda_0^*}, y_i\in P_{1\lambda_1^*},\ i=0,1, \|x_0\|+\|x_1\|+\|y_0\|+\|y_1\| > R,$$

and

$$\|f(t,x_0,x_1,y_0,y_1)\| \le a_0(t) + M_0 b_0(t),\ \forall\ t\in J_+,$$
$$x_i \in P_{0\lambda_0^*}, y_i\in P_{1\lambda_1^*},\ i=0,1, \|x_0\|+\|x_1\|+\|y_0\|+\|y_1\| \le R,$$

where

$$M_0 = \max\{h_0(u_0,u_1,v_0,v_1) : r\le u_i, v_i\le R,\ i=0,1\}.$$

Hence

$$\|f(t,x_0,x_1,y_0,y_1)\| \le \varepsilon_0 c_0(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|) + a_0(t) + M_0 b_0(t),$$
$$\forall\ t\in J_+, x_i\in P_{0\lambda_0^*}, y_i\in P_{1\lambda_1^*}, i=0,1. \tag{8}$$

Let $(x,y)\in Q$. By (8) we have

$$\begin{aligned}
&\|f(t,x(t),x'(t),y(t),y'(t))\| \\
&\le \varepsilon_0 c_0(t)(1+t)\Big(\frac{\|x(t)\|}{t+1}+\frac{\|x'(t)\|}{t+1}+\frac{\|y(t)\|}{t+1}+\frac{\|y'(t)\|}{t+1}\Big) + a_0(t) + M_0 b_0(t) \\
&\le \varepsilon_0 c_0(t)(1+t)(\|x\|_F+\|x'\|_C+\|y\|_F+\|y'\|_C) + a_0(t) + M_0 b_0(t) \\
&\le 2\varepsilon_0 c_0(t)(1+t)(\|x\|_D+\|y\|_D) + a_0(t) + M_0 b_0(t) \\
&\le 4\varepsilon_0 c_0(t)(1+t)\|(x,y)\|_X + a_0(t) + M_0 b_0(t),\ \forall\ t\in J_+,
\end{aligned} \tag{9}$$

which together with condition (H_2) implies the convergence of the infinite integral

$$\int_0^\infty \|f(s,x(s),x'(s),y(s),y'(s))\|\mathrm{d}s. \tag{10}$$

Thus, we have

$$\begin{aligned}
&\Big\|\int_0^t\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big\| \\
&\le \int_0^t\int_s^{+\infty} \|f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\|\mathrm{d}\tau\mathrm{d}s \\
&\le t\int_0^\infty \|f(s,x(s),x'(s),y(s),y'(s))\|\mathrm{d}s.\ \forall\ t\in J_+.
\end{aligned} \tag{11}$$

This together with (4) and (H_1) means that

$$\begin{aligned}\|(A_1(x,y)(t)\| \leq & \int_0^t \int_s^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau \mathrm{d}s + t\|x_\infty\| + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\|x_\infty\| + \\ & \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_{m-2}} \int_s^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau \mathrm{d}s \\ \leq & t\Big(\int_0^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau + \|x_\infty\|\Big) + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\|x_\infty\| + \\ & \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}\Big(\int_0^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau\Big).\end{aligned}$$

Therefore, by (6) and (9), we get

$$\begin{aligned}\frac{\|A_1(x,y)(t)\|}{1+t} \leq & \int_0^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau + \|x_\infty\| + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\|x_\infty\| + \\ & \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}\Big(\int_0^{+\infty} \|f(\tau, x(\tau), x'(\tau), y(\tau), y'(\tau))\| \mathrm{d}\tau\Big) \\ \leq & \Big(1 + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}\Big)[4\varepsilon_0 c_0^*\|(x,y)\|_X + a_0^* + M_0 b_0^*] + \\ & \Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)\|x_\infty\| \\ \leq & \frac{1}{2}\|(x,y)\|_X + \Big(1 + \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}\Big)(a_0^* + M_0 b_0^*) + \\ & \Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)\|x_\infty\|. \end{aligned} \tag{12}$$

Differentiating (4), we find

$$A_1'(x,y)(t) = \int_t^{+\infty} f(s, x(s), x'(s), y(s), y'(s)) \mathrm{d}s + x_\infty. \tag{13}$$

Hence,

$$\begin{aligned}\|A_1'(x,y)(t)\| & \leq \int_0^{+\infty} \|f(s, x(s), x'(s), y(s), y'(s))\| \mathrm{d}s + \|x_\infty\| \\ & \leq 4\varepsilon_0 c_0^*\|(x,y)\|_X + a_0^* + M_0 b_0^* + \|x_\infty\| \\ & \leq \frac{1}{2}\|(x,y)\|_X + a_0^* + M_0 b_0^* + \|x_\infty\|, \quad \forall\ t \in J.\end{aligned} \tag{14}$$

By (12) and (14), we have

$$\|A_1(x,y)\|_D \leq \frac{1}{2}\|(x,y)\|_X + \Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)(a_0^* + M_0 b_0^*) + \Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)\|x_\infty\|. \tag{15}$$

So, $A_1(x,y) \in DC^1[J,E]$. On the other hand, it can be easily seen that

$$A_1(x,y)(t) \geq \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}x_\infty \geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*, \quad A_1'(x,y)(t) \geq x_\infty \geq x_0^* \geq \lambda_0^* x_0^*,\ \forall\ t \in J.$$

That is, $A_1(x,y) \in Q_1$. In the same way, one has

$$\|A_2(x,y)\|_D \le \frac{1}{2}\|(x,y)\|_X + \Big(1 + \frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\beta_i}\Big)(a_1^* + M_1 b_1^*) + \Big(1 + \frac{\sum_{i=1}^{m-2}\beta_i\xi_i}{1-\sum_{i=1}^{m-2}\beta_i}\Big)\|y_\infty\|, \quad (16)$$

and

$$A_2(x,y)(t) \ge \frac{\sum_{i=1}^{m-2}\beta_i\xi_i}{1-\sum_{i=1}^{m-2}\beta_i} y_\infty \ge \lambda_1^* y_\infty \ge \lambda_1^* y_0^*, \quad A_2'(x,y)(t) \ge y_\infty \ge y_0^* \ge \lambda_1^* y_0^*, \ \forall\, t \in J,$$

where $M_1 = \max\{h_1(u_0,u_1,v_0,v_1) : r \le u_i, v_i \le R \ (i=0,1)\}$. Thus, we have proved that A maps Q into Q and we have

$$\|A(x,y)\|_X \le \frac{1}{2}\|(x,y)\|_X + \gamma, \quad (17)$$

where

$$\gamma = \max\Big\{\Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)(a_0^* + M_0 b_0^*) + \Big(1 + \frac{\sum_{i=1}^{m-2}\alpha_i\xi_i}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)\|x_\infty\|,$$
$$\Big(1 + \frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\beta_i}\Big)(a_1^* + M_1 b_1^*) + \Big(1 + \frac{\sum_{i=1}^{m-2}\beta_i\xi_i}{1-\sum_{i=1}^{m-2}\beta_i}\Big)\|y_\infty\|\Big\}. \quad (18)$$

Finally, we show that A is continuous. Let $(x_m,y_m),(\overline{x},\overline{y}) \in Q, \|(x_m,y_m)-(\overline{x},\overline{y})\|_X \to 0 \ (m\to\infty)$. Then $\{(x_m,y_m)\}$ is a bounded subset of Q. Thus, there exists $r>0$ such that $\sup_m \|(x_m,y_m)\|_X < r$ for $m \ge 1$ and $\|(\overline{x},\overline{y})\|_X \le r+1$. Similarly to (12) and (14), it is easy to see that

$$\begin{aligned}&\|A_1(x_m,y_m) - A_1(\overline{x},\overline{y})\|_X\\ &\le \int_0^{+\infty}\|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) - f(s,\overline{x}(s),\overline{x'}(s),\overline{y}(s),\overline{y'}(s))\|\mathrm{d}s +\\ &\quad \frac{\sum_{i=1}^{m-2}\alpha_i\xi_{m-2}}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^{+\infty}\|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) -\\ &\quad f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\|\mathrm{d}s. \end{aligned}\quad (19)$$

Clearly,

$$f(t,x_m(t),x_m'(t),y_m(t),y_m'(t)) \to f(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t)) \text{ as } m\to\infty, \ \forall\, t \in J_+. \quad (20)$$

By (9), we get

$$\begin{aligned}&\|f(t,x_m(t),x_m'(t),y_m(t),y_m'(t)) - f(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t))\|\\ &\le 8\varepsilon_0 c_0(t)(1+t)r + 2a_0(t) + 2M_0 b_0(t)\\ &= \sigma_0(t) \in L[J,J], \quad m=1,2,3,\ldots, \ \forall\, t\in J_+.\end{aligned} \quad (21)$$

Lebesgue dominated convergence theorem together with (20) and (21) guarantees that

$$\lim_{m\to\infty}\int_0^\infty \|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) - f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\|\mathrm{d}s = 0. \quad (22)$$

It follows from (19) and (22) that $\|A_1(x_m,y_m) - A_1(\overline{x},\overline{y})\|_D \to 0$ as $m\to\infty$. By the same method, we have $\|A_2(x_m,y_m) - A_2(\overline{x},\overline{y})\|_D \to 0$ as $m\to\infty$. Therefore, the continuity of A is proved. □

Lemma 2 *If condition* (H_1) *is satisfied, then* $(x,y)\in Q\cap(C^2[J_+,E]\times C^2[J_+,E])$ *is a solution of BVP (1) if and only if* $(x,y)\in Q$ *is a fixed point of operator* A.

Proof Suppose that $(x,y)\in Q\cap(C^2[J_+,E]\times C^2[J_+,E])$ is a solution of BVP (1). For $t\in J$, integrating (1) from t to $+\infty$, we have

$$x'(t)=x_\infty+\int_t^{+\infty} f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s, \tag{23}$$

$$y'(t)=y_\infty+\int_t^{+\infty} g(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s. \tag{24}$$

Integrating (23) and (24) from 0 to t, we get

$$x(t)=x(0)+tx_\infty+\int_0^t\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s, \tag{25}$$

$$y(t)=y(0)+ty_\infty+\int_0^t\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s. \tag{26}$$

Thus, we obtain

$$x(\xi_i)=x(0)+\xi_i x_\infty+\int_0^{\xi_i}\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s,$$

and

$$y(\xi_i)=y(0)+\xi_i y_\infty+\int_0^{\xi_i}\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s,$$

which together with the boundary value condition implies that

$$x(0)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi}\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big], \tag{27}$$

and

$$y(0)=\frac{1}{1-\sum_{i=1}^{m-2}\beta_i}\Big[\Big(\sum_{i=1}^{m-2}\beta_i\xi_i\Big)y_\infty+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi}\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]. \tag{28}$$

Substituting (27), (28) into (25) and (26), respectively, we have

$$\begin{aligned}x(t)=&\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\&\int_0^t\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s+tx_\infty,\end{aligned}$$

and

$$\begin{aligned}y(t)=&\frac{1}{1-\sum_{i=1}^{m-2}\beta_i}\Big[\Big(\sum_{i=1}^{m-2}\beta_i\xi_i\Big)y_\infty+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\&\int_0^t\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s+ty_\infty.\end{aligned}$$

Integrals $\int_0^t\int_s^{+\infty} f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s$ and $\int_0^t\int_s^{+\infty} g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s$ are obviously convergent. Therefore, (x,y) is a fixed point of operator A.

Conversely, if (x,y) is fixed point of operator A, then direct differentiation gives the proof. □

Lemma 3 *Let* (H_1) *be satisfied,* $V\subset Q$ *be a bounded set. Then* $\frac{(A_iV)(t)}{1+t}$ *and* $(A_i'V)(t)$ *are equicontinuous on any finite subinterval of* J *and for any* $\varepsilon>0$, *there exists* $N_i>0$ *such that*

$$\Big\|\frac{A_i(x,y)(t_1)}{1+t_1}-\frac{A_i(x,y)(t_2)}{1+t_2}\Big\|<\varepsilon,\quad \|A_i'(x,y)(t_1)-A_i'(x,y)(t_2)\|<\varepsilon$$

uniformly with respect to $(x,y)\in V$ *as* $t_1,t_2\geq N_i$ $(i=1,2)$.

Proof We only give the proof for operator A_1, and the proof for operator A_2 can be given in a similar way. From (4), we find

$$\begin{aligned}
&A_1(x,y)(t)\\
&=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\
&\quad\int_0^t\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s+tx_\infty\\
&=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\
&\quad tx_\infty+t\int_t^{+\infty}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s+\int_0^t sf(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s. \qquad (29)
\end{aligned}$$

For $(x,y)\in V, t_2>t_1$, we have by (29)

$$\begin{aligned}
&\Big\|\frac{A_1(x,y)(t_1)}{1+t_1}-\frac{A_1(x,y)(t_2)}{1+t_2}\Big\|\\
&\leq\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\cdot\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)\|x_\infty\|+\\
&\quad\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\Big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\Big|\cdot\|x_\infty\|+\\
&\quad\Big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\Big|\cdot\Big\|\int_0^{+\infty}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|+\\
&\quad\Big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\Big|\cdot\Big\|\int_0^{t_1}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|+\\
&\quad\frac{t_2}{1+t_2}\Big\|\int_{t_1}^{t_2}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|+\\
&\quad\Big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\Big|\cdot\Big\|\int_0^{t_1}sf(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|+\\
&\quad\Big\|\int_{t_1}^{t_2}sf(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|. \qquad (30)
\end{aligned}$$

Then, it is easy to see by (30) and (H_1) that $\{\frac{A_1V(t)}{1+t}\}$ is equicontinuous on any finite subinterval of J.

Since $V\subset Q$ is bounded, there exists $r>0$ such that for any $(x,y)\in V, \|(x,y)\|_X\leq r$. By

(13), we get

$$\|A_1'(x,y)(t_1)-A_1'(x,y)(t_2)\| = \Big\|\int_{t_1}^{t_2} f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|$$
$$\le \int_{t_1}^{t_2}[4\varepsilon_0 rc_0(s)(1+s)+a_0(s)+M_0b_0(s)]\mathrm{d}s. \tag{31}$$

It follows from (31), (H_1) and the absolute continuity of Lebesgue integral that $\{A_1'V(t)\}$ is equicontinuous on any finite subinterval of J.

In the following, we are in position to show that for any $\varepsilon>0$, there exists $N_1>0$ such that

$$\Big\|\frac{A_1(x,y)(t_1)}{1+t_1}-\frac{A_1(x,y)(t_2)}{1+t_2}\Big\|<\varepsilon,\quad \|A_1'(x,y)(t_1)-A_1'(x,y)(t_2)\|<\varepsilon$$

uniformly with respect to $x\in V$ as $t_1,t_2\ge N$.

Combining with (30), we need only to show that for any $\varepsilon>0$, there exists sufficiently large $N>0$ such that

$$\Big\|\int_0^{t_1}\frac{s}{1+t_1}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s-\int_0^{t_2}\frac{s}{1+t_2}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\Big\|<\varepsilon$$

for all $x\in V$ as $t_1,t_2\ge N$. The rest part of the proof is very similar to Lemma 2.3 in [14], and we omit the details. □

Lemma 4 *Let* (H_1) *be satisfied,* V *be a bounded set in* $DC^1[J,E]\times DC^1[J,E]$*. Then*

$$\alpha_D(A_iV)=\max\Big\{\sup_{t\in J}\alpha\Big(\frac{(A_iV)(t)}{1+t}\Big),\ \sup_{t\in J}\alpha((A_iV)'(t))\Big\},\quad i=0,1.$$

Proof The proof is similar to that of Lemma 2.4 in [14], we omit it. □

Lemma 5 ([1, 2], Mönch Fixed-Point Theorem) *Let* Q *be a closed convex set of* E *and* $u\in Q$*. Assume that the continuous operator* $F:Q\to Q$ *has the following property:* $V\subset Q$ *countable,* $V\subset\overline{\mathrm{co}}(\{u\}\cup F(V))\Longrightarrow V$ *is relatively compact. Then* F *has a fixed point in* Q*.*

Lemma 6 *If* (H_3) *is satisfied, then for* $x,y\in Q, x^{(i)}\le y^{(i)}, t\in J$ $(i=0,1)$ *imply that* $(Ax)^{(i)}\le(Ay)^{(i)}, t\in J$ $(i=0,1)$*.*

Proof It is easy to see that this lemma follows from (4), (5), (13) and condition (H_3). The proof is obvious. □

Lemma 7 ([16]) *Let* D *and* F *be bounded sets in* E*. Then*

$$\widetilde{\alpha}(D\times F)=\max\{\alpha(D),\alpha(F)\},$$

where $\widetilde{\alpha}$ *and* α *denote the Kuratowski measure of non-compactness in* $E\times E$ *and* E*, respectively.*

Lemma 8 ([16]) *Let* P *be normal (fully regular) in* E*,* $\widetilde{P}=P\times P$*. Then* $\widetilde{P}$ *is normal (fully regular) in* $E\times E$*.*

3. Main results

Theorem 1 *If conditions (H_1) and (H_2) are satisfied, then BVP (1) has a positive solution* $(\overline{x},\overline{y}) \in (DC^1[J,E]\cap C^2[J'_+,E])\times(DC^1[J,E]\cap C^2[J'_+,E])$ *satisfying* $(\overline{x})^{(i)}(t) \geq \lambda_0^* x_0^*$, $(\overline{y})^{(i)}(t) \geq \lambda_1^* y_0^*$ *for* $t \in J$ $(i = 0,1)$.

Proof By Lemma 1, operator A defined by (3) is a continuous operator from Q into Q, and, by Lemma 2, we need only to show that A has a fixed point $(\overline{x},\overline{y})$ in Q. Choose $R > 2\gamma$ and let $Q^* = \{(x,y) \in Q : \|(x,y)\|_X \leq R\}$. Obviously, Q^* is a bounded closed convex set in space $DC^1[J,E]\times DC^1[J,E]$. It is easy to see that Q^* is not empty since $((1+t)x_\infty,(1+t)y_\infty) \in Q^*$. It follows from (17), (18) that $(x,y) \in Q^*$ implies that $A(x,y) \in Q^*$, i.e., A maps Q^* into Q^*. Let $V = \{(x_m,y_m) : m = 1,2,\ldots\} \subset Q^*$ satisfying $V \subset \overline{\text{co}}\{\{(u_0,v_0)\}\cup AV\}$ for some $(u_0,v_0) \in Q^*$. Then $\|(x_m,y_m)\|_X \leq R$. We have, by (4) and (13),

$$\begin{aligned}
&A_1(x_m,y_m)(t)\\
&\quad= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty + \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty} f(\tau,x_m(\tau),x'_m(\tau),y_m(\tau),y'_m(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+\\
&\qquad\int_0^t\int_s^{+\infty} f(\tau,x_m(\tau),x'_m(\tau),y_m(\tau),y'_m(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty,
\end{aligned} \tag{32}$$

and

$$A'_1(x_m,y_m)(t) = \int_t^{+\infty} f(s,x_m(s),x'_m(s),y_m(s),y'_m(s))\mathrm{d}s + x_\infty. \tag{33}$$

Lemma 4 implies that

$$\alpha_D(A_1V) = \max\Big\{\sup_{t\in J}\alpha((A_1V)'(t)),\ \ \sup_{t\in J}\alpha\Big(\frac{(A_1V)(t)}{1+t}\Big)\Big\}, \tag{34}$$

where $(A_1V)(t) = \{A_1(x_m,y_m)(t) : m = 1,2,3,\ldots\}$, and $(A_1V)'(t) = \{A'_1(x_m,y_m)(t) : m = 1,2,3,\ldots\}$.

By (10), we know that the infinite integral $\int_0^{+\infty}\|f(t,x(t),x'(t),y(t),y'(t))\|\mathrm{d}t$ is convergent uniformly for $m = 1,2,3,\ldots$. So, for any $\varepsilon > 0$, we can choose a sufficiently large $T > \xi_i$ $(i = 1,2,\ldots,m-2) > 0$ such that

$$\int_T^{+\infty}\|f(t,x(t),x'(t),y(t),y'(t))\|\mathrm{d}t < \varepsilon. \tag{35}$$

Then, by Guo et al. [1, Theorem 1.2.3] (29), (32), (33), (35), (H_2) and Lemma 7, we obtain

$$\begin{aligned}
&\alpha\Big(\frac{(A_1V)(t)}{1+t}\Big)\\
&\quad\leq 2\frac{D_0}{1+t}\int_0^T \alpha(\{f(s,x_m(s),x'_m(s),y_m(s),y'_m(s)) : (x_m,y_m) \in V\})\mathrm{d}s + 2\varepsilon +\\
&\qquad 2\int_0^T \frac{t}{1+t}\alpha(\{f(s,x_m(s),x'_m(s),y_m(s),y'_m(s)) : (x_m,y_m) \in V\})\mathrm{d}s + 2\varepsilon\\
&\quad\leq (2D_0+2)\int_0^{+\infty} \alpha(\{f(s,x_m(s),x'_m(s),y_m(s),y'_m(s)) : (x_m,y_m) \in V\})\mathrm{d}s + 4\varepsilon
\end{aligned}$$

$$\leq (2D_0+2)\alpha_X(V)\int_0^{+\infty}(L_{00}(s)+K_{00}(s))(1+s)+(L_{01}(s)+K_{01}(s))\mathrm{d}s+4\varepsilon, \tag{36}$$

and

$$\alpha((A_1'V)(t))\leq 2\int_0^{+\infty}\alpha(\{f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)):(x_m,y_m)\in V\})\mathrm{d}s+2\varepsilon$$
$$\leq \alpha_X(V)\int_0^{+\infty}(L_{00}(s)+K_{00}(s))(1+s)+(L_{01}(s)+K_{01}(s))\mathrm{d}s+2\varepsilon. \tag{37}$$

It follows from (34), (36) and (37) that

$$\alpha_D(A_1V)\leq (2D_0+2)\alpha_X(V)\int_0^{+\infty}(L_{00}(s)+K_{00}(s))(1+s)+(L_{01}(s)+K_{01}(s))\mathrm{d}s. \tag{38}$$

Similarly, we can show that

$$\alpha_D(A_2V)\leq (2D_1+2)\alpha_X(V)\int_0^{+\infty}(L_{10}(s)+K_{10}(s))(1+s)+(L_{11}(s)+K_{11}(s))\mathrm{d}s. \tag{39}$$

On the other hand, $\alpha_X(V)\leq\alpha_X\{\overline{co}(\{u\}\cup(AV))\}=\alpha_X(AV)$. Then, (38), (39), ($H_2$) and Lemma 7 imply $\alpha_X(V)=0$. That is, V is relatively compact in $DC^1[J,E]\times DC^1[J,E]$. Hence, the Mönch fixed point theorem guarantees that A has a fixed point $(\overline{x},\overline{y})$ in Q_1. Thus, Theorem 1 is proved. □

Theorem 2 *Let cone P be normal and conditions (H_1)–(H_3) be satisfied. Then BVP (1) has a positive solution $(\overline{x},\overline{y})\in Q\cap(C^2[J'_+,E]\times C^2[J'_+,E])$ which is minimal in the sense that $u^{(i)}(t)\geq\overline{x}^{(i)}(t), v^{(i)}(t)\geq\overline{y}^{(i)}(t)$, $t\in J$ $(i=0,1)$ for any positive solution $(u,v)\in Q\cap(C^2[J'_+,E]\times C^2[J'_+,E])$ of BVP (1). Moreover, $\|(\overline{x},\overline{y})\|_X\leq 2\gamma+\|(u_0,v_0)\|_X$, and there exists a monotone iterative sequence $\{(u_m(t),v_m(t))\}$ such that $u_m^{(i)}(t)\to\overline{x}^{(i)}(t), v_m^{(i)}(t)\to\overline{y}^{(i)}(t)$ as $m\to\infty(i=0,1)$ uniformly on J and $u_m''(t)\to\overline{x}''(t), v_m''(t)\to\overline{y}''(t)$ as $m\to\infty$ for any $t\in J'_+$, where*

$$u_0(t)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\mathrm{d}\tau\mathrm{d}s\Big]+$$
$$\int_0^t\int_s^{+\infty}f(\tau,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\mathrm{d}\tau\mathrm{d}s+tx_\infty, \tag{40}$$
$$v_0(t)=\frac{1}{1-\sum_{i=1}^{m-2}\beta_i}\Big[\Big(\sum_{i=1}^{m-2}\beta_i\xi_i\Big)y_\infty+\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\int_s^{+\infty}g(\tau,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\mathrm{d}\tau\mathrm{d}s\Big]+$$
$$\int_0^t\int_s^{+\infty}g(\tau,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\mathrm{d}\tau\mathrm{d}s+ty_\infty, \tag{41}$$

and

$$u_m(t)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)x_\infty+$$
$$\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_s^{+\infty}f(\tau,u_{m-1}(\tau),u_{m-1}'(\tau),v_{m-1}(\tau),v_{m-1}'(\tau))\mathrm{d}\tau\mathrm{d}s\Big]+$$

$$\int_0^t \int_s^{+\infty} f(\tau, u_{m-1}(\tau), u'_{m-1}(\tau), v_{m-1}(\tau), v'_{m-1}(\tau))\mathrm{d}\tau\mathrm{d}s + tx_\infty,$$
$$\forall\ t \in J,\ m = 1, 2, 3, \dots, \tag{42}$$

$$v_m(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i}\Big[\Big(\sum_{i=1}^{m-2} \beta_i \xi_i\Big) y_\infty + \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \int_s^{+\infty} g(\tau, u_{m-1}(\tau), u'_{m-1}(\tau), v_{m-1}(\tau), v'_{m-1}(\tau))\mathrm{d}\tau\mathrm{d}s\Big] + \int_0^t \int_s^{+\infty} g(\tau, u_{m-1}(\tau), u'_{m-1}(\tau), v_{m-1}(\tau), v'_{m-1}(\tau))\mathrm{d}\tau\mathrm{d}s + ty_\infty,$$
$$\forall\ t \in J,\ m = 1, 2, 3, \dots. \tag{43}$$

Proof From (40) and (41) one can see that $(u_0, v_0) \in C[J, E] \times C[J, E]$ and

$$u'_0(t) = \int_t^{+\infty} f(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\mathrm{d}s + x_\infty. \tag{44}$$

By (40) and (44), we know that $u_0^{(i)} \geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*$ $(i = 0, 1)$ and

$$\begin{aligned}
&\|u_0(t)\| \\
&\leq t\Big(\int_0^{+\infty} \|f(\tau, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\|\mathrm{d}\tau + \|x_\infty\|\Big) + \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i}\|x_\infty\| + \\
&\quad \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i \xi_{m-2} \Big(\int_0^{+\infty} \|f(\tau, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\|\mathrm{d}\tau\Big) \\
&\leq t\Big[\int_0^{+\infty} a_0(s) + b_0(s) h_0(\|\lambda_0^* x_0^*\|, \|\lambda_0^* x_0^*\|, \|\lambda_1^* y_0^*\|, \|\lambda_1^* y_0^*\|)\mathrm{d}s + \|x_\infty\|\Big] + \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i}\|x_\infty\| + \\
&\quad \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \sum_{i=1}^{m-2} \alpha_i \xi_{m-2} \Big(\int_0^{+\infty} a_0(s) + b_0(s) h_0(\|\lambda_0^* x_0^*\|, \|\lambda_0^* x_0^*\|, \|\lambda_1^* y_0^*\|, \|\lambda_1^* y_0^*\|)\mathrm{d}s\Big),
\end{aligned}$$

$$\begin{aligned}
\|u'_0(t)\| &\leq \int_t^{+\infty} \|f(\tau, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\|\mathrm{d}\tau + \|x_\infty\| \\
&\leq \int_0^{+\infty} a_0(s) + b_0(s) h_0(\|\lambda_0^* x_0^*\|, \|\lambda_0^* x_0^*\|, \|\lambda_1^* y_0^*\|, \|\lambda_1^* y_0^*\|)\mathrm{d}s + \|x_\infty\|,
\end{aligned}$$

which imply that $\|u_0\|_D < \infty$. Similarly, we have $\|v_0\|_D < \infty$. Thus, $(u_0, v_0) \in DC^1[J, E] \times DC^1[J, E]$. It follows from (4) and (42) that

$$(u_m, v_m)(t) = A(u_{m-1}, v_{m-1})(t), \quad \forall t \in J,\ m = 1, 2, 3, \dots. \tag{45}$$

By Lemma 1, we get $(u_m, v_m) \in Q$ and

$$\|(u_m, v_m)\|_X = \|A(u_{m-1}, v_{m_1})\|_X \leq \frac{1}{2}\|(u_{m-1}, v_{m-1})\|_X + \gamma. \tag{46}$$

By (H_3) and (45), we have

$$u_1(t) = A_1(u_0(t), v_0(t)) \geq A_1(\lambda_0^* x_0^*, \lambda_1^* y_0^*) = u_0(t), \quad \forall t \in J, \tag{47}$$

and

$$v_1(t) = A_2(u_0(t), v_0(t)) \geq A_2(\lambda_0^* x_0^*, \lambda_1^* y_0^*) = v_0(t), \quad \forall t \in J. \tag{48}$$

From Lemma 6, (45)–(48), it is easy to see by induction that

$$(\lambda_0^* x_0^*, \lambda_1^* y_0^*) \leq (u_0^{(i)}(t), v_0^{(i)}(t)) \leq (u_1^{(i)}(t), v_1^{(i)}(t)) \leq \cdots \leq (u_m^{(i)}(t), v_m^{(i)}(t)) \leq \cdots, \\ \forall t \in J, \ i = 0, 1, \tag{49}$$

and

$$\begin{aligned} \|(u_m, v_m)\|_X &\leq \gamma + \frac{1}{2}\gamma + \cdots + \Big(\frac{1}{2}\Big)^{m-1}\gamma + \Big(\frac{1}{2}\Big)^m \|(u_0, v_0)\|_X \\ &\leq 2\gamma + \|(u_0, v_0)\|_X, \quad m = 1, 2, 3, \ldots . \end{aligned} \tag{50}$$

Let $K = \{(x, y) \in Q : \|(x, y)\|_X \leq 2\gamma + \|(u_0, v_0)\|_X\}$. Then, K is a bounded closed convex set in space $DC^1[J, E] \times DC^1[J, E]$ and operator A maps K into K. Clearly, K is not empty since $(u_0, v_0) \in K$. Let $W = \{(u_m, v_m) : m = 0, 1, 2, \ldots\}$, $AW = \{A(u_m, v_m) : m = 0, 1, 2, \ldots\}$. Obviously, $W \subset K$ and $W = \{(u_0, v_0)\} \cup A(W)$. Similarly to the above proof of Theorem 1, we can obtain $\alpha_X(AW) = 0$, i.e., W is relatively compact in $DC^1[J, E] \times DC^1[J, E]$. So, there exists a $(\overline{x}, \overline{y}) \in DC^1[J, E] \times DC^1[J, E]$ and a subsequence $\{(u_{m_j}, v_{m_j}) : j = 1, 2, 3, \ldots\} \subset W$ such that $\{(u_{m_j}, v_{m_j})(t) : j = 1, 2, 3, \ldots\}$ converges to $(\overline{x}^{(i)}(t), \overline{y}^{(i)}(t))$ uniformly on J $(i = 0, 1)$. Since P is normal and $\{(u_m^{(i)}(t), v_m^{(i)}(t)) : m = 1, 2, 3, \ldots\}$ is nondecreasing, by Lemma 8 it is easy to see that the entire sequence $\{(u_m^{(i)}(t), v_m^{(i)}(t)) : m = 1, 2, 3, \ldots\}$ converges to $(\overline{x}^{(i)}(t), \overline{y}^{(i)}(t))$ uniformly on J $(i = 0, 1)$. Considering the fact that $(u_m, v_m) \in K$ and K is a closed convex set in space $DC^1[J, E] \times DC^1[J, E]$, we have $(\overline{x}, \overline{y}) \in K$. It is clear that

$$f(s, u_m(s), u_m'(s), v_m(s), v_m'(s)) \to f(s, \overline{x}(s), \overline{x}'(s), \overline{y}(s), \overline{y}'(s)), \quad \text{as } m \to \infty, \ \forall\, s \in J_+. \tag{51}$$

By (H_1) and (50), we have

$$\begin{aligned} &\|f(s, u_m(s), u_m'(s), v_m(s), v_m'(s)) - f(s, \overline{x}(s), \overline{x}'(s), \overline{y}(s), \overline{y}'(s))\| \\ &\quad \leq 8\varepsilon_0 c_0(s)(1 + s)\|(u_m, v_m)\|_X + 2a_0(s) + 2M_0 b_0(s) \\ &\quad \leq 8\varepsilon_0 c_0(s)(1 + s)(2\gamma + \|(u_0, v_0)\|_X) + 2a_0(s) + 2M_0 b_0(s). \end{aligned} \tag{52}$$

Noticing (51) and (52) and taking limit as $m \to \infty$ in (42), we obtain

$$\begin{aligned} \overline{x}(t) =& \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}\Big[\Big(\sum_{i=1}^{m-2} \alpha_i \xi_i\Big)x_\infty + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_s^{+\infty} f(\tau, \overline{x}(\tau), \overline{x}'(\tau), \overline{y}(\tau), \overline{y}'(\tau))\mathrm{d}\tau\mathrm{d}s\Big] + \\ & \int_0^t \int_s^{+\infty} f(\tau, \overline{x}(\tau), \overline{x}'(\tau), \overline{y}(\tau), \overline{y}'(\tau))\mathrm{d}\tau\mathrm{d}s + t x_\infty. \end{aligned} \tag{53}$$

In the same way, taking limit $m \to \infty$ in (43), we get

$$\begin{aligned} \overline{y}(t) =& \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i}\Big[\Big(\sum_{i=1}^{m-2} \beta_i \xi_i\Big)y_\infty + \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} \int_s^{+\infty} g(\tau, \overline{x}(\tau), \overline{x}'(\tau), \overline{y}(\tau), \overline{y}'(\tau))\mathrm{d}\tau\mathrm{d}s\Big] + \\ & \int_0^t \int_s^{+\infty} g(\tau, \overline{x}(\tau), \overline{x}'(\tau), \overline{y}(\tau), \overline{y}'(\tau))\mathrm{d}\tau\mathrm{d}s + t y_\infty, \end{aligned} \tag{54}$$

which together with (53) and Lemma 2 shows that $(\overline{x}, \overline{y}) \in K \cap C^2[J_+, E] \times C^2[J_+, E]$ and $(\overline{x}(t), \overline{y}(t))$ is a positive solution of BVP (1). Differentiating (42) twice, we have

$$u''_m(t) = -f(t, u_{m-1}(t), u'_{m-1}(t), v_{m-1}(t), v'_{m-1}(t)), \quad \forall t \in J'_+, \ m = 1, 2, 3, \dots .$$

Hence, by (51), we obtain

$$\lim_{m\to\infty} u''_m(t) = -f(t, \overline{x}(t), \overline{x}'(t), \overline{y}(t), \overline{y}'(t)) = \overline{x}''(t), \quad \forall t \in J'_+.$$

Similarly, one has

$$\lim_{m\to\infty} v''_m(t) = -g(t, \overline{x}(t), \overline{x}'(t), \overline{y}(t), \overline{y}'(t)) = \overline{y}''(t), \quad \forall t \in J'_+.$$

Let $(m(t), n(t))$ be any positive solution of BVP (1). By Lemma 2, we have $(m, n) \in Q$ and $(m(t), n(t)) = A(m, n)(t)$, for $t \in J$. It is clear that $m^{(i)}(t) \geq \lambda_0^* x_0^* > \theta, n^{(i)}(t) \geq \lambda_1^* y_0^* > \theta$ for any $t \in J$ $(i = 0, 1)$. So, by Lemma 6, we know that $m^{(i)}(t) \geq u_0^{(i)}(t), n^{(i)}(t) \geq v_0^{(i)}(t)$ for any $t \in J$ $(i = 0, 1)$. Assume that $m^{(i)}(t) \geq u_{m-1}^{(i)}(t), n^{(i)}(t) \geq v_{m-1}^{(i)}(t)$ for $t \in J, m \geq 1$ $(i = 0, 1)$. Then, we have from Lemma 6 that $(A_1^{(i)}(m, n)(t), A_2^{(i)}(m, n)(t)) \geq (A_1^{(i)}(u_{m-1}, v_{m-1}))(t)$, $A_2^{(i)}(u_{m-1}, v_{m-1}))(t))$ for $t \in J$ $(i = 0, 1)$, i.e., $(m^{(i)}(t), n^{(i)}(t)) \geq (u_m^{(i)}(t), v_m^{(i)}(t))$ for $t \in J$ $(i = 0, 1)$. Hence, by induction, we get

$$m^{(i)}(t) \geq \overline{x}_m^{(i)}(t), n^{(i)}(t) \geq \overline{y}_m^{(i)}(t), \quad \forall t \in J, \ i = 0, 1; \ m = 0, 1, 2, \dots . \tag{55}$$

Now, taking limits in (55) gives $m^{(i)}(t) \geq \overline{x}^{(i)}(t), n^{(i)}(t) \geq \overline{y}^{(i)}(t)$ for $t \in J$ $(i = 0, 1)$. The proof is completed. □

Theorem 3 *Let cone P be fully regular and conditions (H_1) and (H_3) be satisfied. Then the conclusion of Theorem 2 holds.*

Proof The proof is almost the same as that of Theorem 2. The only difference is that, instead of using condition (H_2), the conclusion $\alpha_X(W) = 0$ is implied directly by (49) and (50), the full regularity of P and Lemma 8.

4. An example

Consider the infinite system of scalar singular second order three-point boundary value problems:

$$\begin{cases} -x''_n(t) = \dfrac{1}{3n^2\sqrt{t}(1+t)}\Big(2 + x_n(t) + y_n(t) + x'_{2n}(t) + y'_{3n}(t) + \dfrac{1}{2n^2 x_n(t)} + \dfrac{1}{8n^3 x'_{2n}(t)}\Big)^{\frac{1}{3}} + \\ \qquad\qquad \dfrac{1}{3e^{2t}(1+t)} \ln(1 + x_n(t)), \\ -y''_n(t) = \dfrac{1}{6n^3\sqrt[3]{t^2}(1+t)}\Big(1 + x_{3n}(t) + x'_{4n}(t) + \dfrac{1}{3n^2 y_{3n}(t)} + \dfrac{1}{4n^3 y'_{2n}(t)}\Big)^{\frac{1}{5}} + \\ \qquad\qquad \dfrac{1}{6e^{3t}(1+t)} \ln(1 + y'_{2n}(t)), \\ x_n(0) = \dfrac{1}{3} x_n(1), \quad x'_n(\infty) = \dfrac{1}{n}, \quad y_n(0) = \dfrac{3}{4} y_n(1), \quad y'_n(\infty) = \dfrac{1}{2n}, \quad n = 1, 2, \dots . \end{cases} \tag{56}$$

Proposition 1 Infinite system (56) has a minimal positive solution $(x_n(t), y_n(t))$ satisfying $x_n(t), x_n'(t), y_n(t), y_n'(t) \geq \frac{1}{2n}$ for $0 \leq t < +\infty$ $(n = 1, 2, 3, \ldots)$.

Proof Let $E = c_0 = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \|\cdot\|)$ is a real Banach space. Choose $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \ldots\}$. It is easy to verify that P is a normal cone in E with normal constant 1. Now we consider infinite system (56), which can be regarded as a BVP of form (1) in E with $\alpha_1 = \frac{1}{3}$, $\beta_1 = \frac{3}{4}$, $\xi_1 = 1$, $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, $y_\infty = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$. In this situation, $x = (x_1, \ldots, x_n, \ldots)$, $u = (u_1, \ldots, u_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$, $v = (v_1, \ldots, v_n, \ldots)$, $f = (f_1, \ldots, f_n, \ldots)$, in which

$$\begin{aligned} f_n(t,x,u,y,v) =& \frac{1}{3n^2\sqrt{t}(1+t)}\Big(2 + x_n + y_n + u_{2n} + v_{3n} + \frac{1}{2n^2 x_n} + \frac{1}{8n^3 u_{2n}}\Big)^{\frac{1}{3}} + \\ & \frac{1}{3e^{2t}(1+t)} \ln(1 + x_n), \end{aligned} \tag{57}$$

$$\begin{aligned} g_n(t,x,u,y,v) =& \frac{1}{6n^3\sqrt[3]{t^2}(1+t)}\Big(1 + x_{3n} + u_{4n} + \frac{1}{3n^2 y_{3n}} + \frac{1}{4n^3 v_{2n}}\Big)^{\frac{1}{5}} + \\ & \frac{1}{6e^{3t}(1+t)} \ln(1 + v_{2n}). \end{aligned} \tag{58}$$

Let $x_0^* = x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, $y_0^* = y_\infty = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$. Then $P_{0\lambda} = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_n \geq \frac{\lambda}{n},\ n = 1, 2, 3, \ldots\}$, $P_{1\lambda} = \{y = (y_1, y_2, \ldots, y_n, \ldots) : y_n \geq \frac{\lambda}{2n},\ n = 1, 2, 3, \ldots\}$, for $\lambda > 0$. By simple computation, we have $D_0 = \frac{1}{2}, D_1 = 3, \lambda_0^* = \frac{1}{2}, \lambda_1^* = 1$. It is clear that $f, g \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P_{1\lambda} \times P_{1\lambda}, P]$ for any $\lambda > 0$. Notice that $e^{3t} > \sqrt[3]{t^2}$, $e^{2t} > \sqrt{t}$ for $t > 0$, by (57) and (58), we get

$$\|f(t,x,u,y,v)\| \leq \frac{1}{3\sqrt{t}}\Big[\Big(\frac{7}{2} + \|x\| + \|u\| + \|v\| + \|y\|\Big)^{\frac{1}{3}} + \ln(1 + \|x\|)\Big], \tag{59}$$

and

$$\|g(t,x,u,y,v)\| \leq \frac{1}{6\sqrt[3]{t^2}}\Big[\Big(4 + \|x\| + \|u\|\Big)^{\frac{1}{5}} + \ln(1 + \|v\|)\Big], \tag{60}$$

which imply that (H_1) is satisfied for $a_0(t) = 0$, $b_0(t) = c_0(t) = \frac{1}{3\sqrt{t}}$, $a_1(t) = 0$, $b_1(t) = c_1(t) = \frac{1}{6\sqrt[3]{t^2}}$ and

$$h_0(u_0, u_1, u_2, u_3) = \Big(\frac{7}{2} + u_0 + u_1 + u_2 + u_3\Big)^{\frac{1}{3}} + \ln(1 + u_0),$$

$$h_1(u_0, u_1, u_2, u_3) = \Big(4 + u_0 + u_1\Big)^{\frac{1}{5}} + \ln(1 + u_3).$$

Let

$$f^1 = \{f_1^1, f_2^1, \ldots, f_n^1, \ldots\}, \quad f^2 = \{f_1^2, f_2^2, \ldots, f_n^2, \ldots\},$$

$$g^1 = \{g_1^1, g_2^1, \ldots, g_n^1, \ldots\}, \quad g^2 = \{g_1^2, g_2^2, \ldots, g_n^2, \ldots\},$$

where

$$f_n^1(t,x,u,y,v) = \frac{1}{3n^2\sqrt{t}(1+t)}\Big(2 + x_n + y_n + u_{2n} + v_{3n} + \frac{1}{2n^2 x_n} + \frac{1}{8n^3 u_{2n}}\Big)^{\frac{1}{3}}, \tag{61}$$

$$f_n^2(t,x,u,y,v)=\frac{1}{3e^{2t}(1+t)}\ln(1+x_n), \tag{62}$$

$$g_n^1(t,x,u,y,v)=\frac{1}{6n^3\sqrt[3]{t^2}(1+t)}\Big(1+x_{3n}+u_{4n}+\frac{1}{3n^2y_{3n}}+\frac{1}{4n^3v_{2n}}\Big)^{\frac{1}{5}}, \tag{63}$$

$$g_n^2(t,x,u,y,v)=\frac{1}{6e^{3t}(1+t)}\ln(1+v_{2n}). \tag{64}$$

Let $t\in J_+$, and $R>0$ be given and $\{z^{(m)}\}$ be any sequence in $f^1(t,P_{0R}^*,P_{0R}^*,P_{1R}^*,P_{1R}^*)$, where $z^{(m)}=(z_1^{(m)},\ldots,z_n^{(m)},\ldots)$. By (61), we have

$$0\le z_n^{(m)}\le\frac{1}{3n^2\sqrt{t}}\Big(\frac{7}{2}+4R\Big)^{\frac{1}{3}},\quad n,m=1,2,3,\ldots. \tag{65}$$

So, $\{z_n^{(m)}\}$ is bounded and by the diagonal method together with the method of constructing subsequence, we can choose a subsequence $\{m_i\}\subset\{m\}$ such that

$$\{z_n^{(m)}\}\to\overline{z}_n\ \text{ as } i\to\infty,\quad n=1,2,3,\ldots, \tag{66}$$

which implies by (65)

$$0\le\overline{z}_n\le\frac{1}{3n^2\sqrt{t}}\Big(\frac{7}{2}+4R\Big)^{\frac{1}{3}},\quad n=1,2,3,\ldots. \tag{67}$$

Hence $\overline{z}=(\overline{z}_1,\ldots,\overline{z}_n,\ldots)\in c_0$. It is easy to see from (65)–(67) that

$$\|z^{(m_i)}-\overline{z}\|=\sup_n|z_n^{(m_i)}-\overline{z}_n|\to 0 \text{ as } i\to\infty.$$

Thus, we have proved that $f^1(t,P_{0R}^*,P_{0R}^*,P_{1R}^*,P_{1R}^*)$ is relatively compact in c_0.

For any $t\in J_+$, $R>0$, x, y, $\overline{x}$, $\overline{y}\in D\subset P_{0R}^*$, we have by (62)

$$\begin{aligned}|f_n^2(t,x,u,y,v)-f_n^2(t,\overline{x},\overline{u},\overline{y},\overline{v})|&=\frac{1}{3e^{2t}(1+t)}|\ln(1+x_n)-\ln(1+\overline{x}_n)|\\&\le\frac{1}{3e^{2t}(1+t)}\frac{|x_n-\overline{x}_n|}{1+\xi_n},\end{aligned} \tag{68}$$

where ξ_n is between x_n and $\overline{x}_n$. By (68), we get

$$\|f^2(t,x,u,y,v)-f^2(t,\overline{x},\overline{u},\overline{y},\overline{v})\|\le\frac{1}{3e^{2t}(1+t)}\|x-\overline{x}\|,\ x,y,\overline{x},\overline{y}\in D. \tag{69}$$

In the same way, we can prove that $g^1(t,P_{0R}^*,P_{0R}^*,P_{1R}^*,P_{1R}^*)$ is relatively compact in c_0, and we can also get

$$\|g^2(t,x,u,y,v)-g^2(t,\overline{x},\overline{u},\overline{y},\overline{v})\|\le\frac{1}{6e^{3t}(1+t)}\|v-\overline{v}\|,\ x,y,\overline{x},\overline{y}\in D. \tag{70}$$

Thus, by (69) and (70), it is easy to see that (H$_2$) holds for $L_{00}(t)=\frac{1}{3e^{2t}(1+t)}$, $L_{10}(t)=\frac{1}{6e^{3t}(1+t)}$. Our conclusion follows from Theorem 1. □

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