*Journal of Mathematical Research & Exposition*
Jul., 2011, Vol. 31, No. 4, pp. 605–616
DOI:10.3770/j.issn:1000-341X.2011.04.004
Http://jmre.dlut.edu.cn

# A New Method of Constructing Bivariate Vector Valued Rational Interpolation Function

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**Abstract** At present, the methods of constructing vector valued rational interpolation function in rectangular mesh are mainly presented by means of the branched continued fractions. In order to get vector valued rational interpolation function with lower degree and better approximation effect, the paper divides rectangular mesh into pieces by choosing nonnegative integer parameters $d_1$ $(0 \le d_1 \le m)$ and $d_2$ $(0 \le d_2 \le n)$, builds bivariate polynomial vector interpolation for each piece, then combines with them properly. As compared with previous methods, the new method given by this paper is easy to compute and the degree for the interpolants is lower.

**Keywords** bivariate vector valued rational interpolation; nonnegative integer parameter; divide piece; primary function; interpolation formula.

**Document code** A
**MR(2010) Subject Classification** 41A20; 65D05
**Chinese Library Classification** O241.5

## 1. Introduction

Wynn [2] observed that the Samelson inverse may be used to solve the problem of rational interpolation of vectors. Graves-Morris [3], and Roberts and Graves-Morris [4], first solved vector-valued rational interpolation problem by using Thiele-type continued fraction and applied this method to model analysis of vibrating structures. Zhu and Gu defined bivariate Thiele-type vector valued rational interpolation by using Thiele-type branched continued fractions for two-variable functions in [5]. With the development of science and technology, the application range of vector valued rational interpolation and approximation is further extended, for example, vector valued rational interpolation and approximation has been applied to the structural dynamic analysis [6–8], graphics processing [9, 10] and so on. The paper mainly discusses a new method of constructing bivariate vector valued rational interpolation function, and the case of rational interpolation in higher dimension will be discussed in another paper.

Received October 17, 2009; Accepted January 12, 2011
Supported by Shanghai Natural Science Foundation (Grant No. 10ZR1410900), Key Disciplines of Shanghai Municipality (Grant No. S30104), the Anhui Provincial Natural Science Foundation (Grant No. 070416227) and Students' Innovation Foundation of Hefei University of Technology (Grant No. XS08079).
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Given a plane point set $\prod_{x,y}^{m,n} = \{(x_i, y_j)|i = 0, 1, \ldots, m; j = 0, 1, \ldots, n.x_i \neq x_j, y_i \neq y_j\}$, its corresponding bivariate finite vector set is $\mathbf{V}_{x,y}^{m,n} = \{\mathbf{V}^{i,j}|\mathbf{V}^{i,j} = \mathbf{V}(x_i, y_j) \in C^d, (x_i, y_j) \in \prod_{x,y}^{m,n}\}$.

$$\mathbf{R}(x, y) = \frac{\mathbf{N}(x, y)}{D(x, y)} = \frac{(N_1(x, y), N_2(x, y), \ldots, N_d(x, y))}{D(x, y)}, \tag{1}$$

where $N_k(x, y)$ $(k = 1, 2, \ldots, d)$ and $D(x, y)$ are bivariate polynomials.

The problem of bivariate vector valued rational interpolation is to look for a vector valued rational function $\mathbf{R}(x, y)$ which satisfies interpolation conditions

$$\mathbf{R}(x_i, y_j) = \frac{\mathbf{N}(x_i, y_j)}{D(x_i, y_j)} = \mathbf{V}(x_i, y_j), (x_i, y_j) \in \prod\nolimits_{x,y}^{m,n}. \tag{2}$$

Note

$$\mathbf{R}(x, y) = (R_1(x, y), R_2(x, y), \ldots, R_d(x, y)), \mathbf{V}(x, y) = (V_1(x, y), V_2(x, y), \ldots, V_d(x, y)),$$

then the formula (2) can be written in the following form

$$R_k(x_i, y_j) = V_k(x_i, y_j), k = 1, 2, \ldots, d; (x_i, y_j) \in \prod\nolimits_{x,y}^{m,n}.$$

The study of interpolation problem (1), (2) was presented on the basis of Thiele-type vector continued fraction [5]. But the algorithm demands that the vector in the denominator is not zero vector in each step of computation, which cannot be predicted. Therefore, a lot of improved methods come out [1], such as composition algorithm and so on. Since it needs to compute the generalized inverse of vector, the degree of constructed vector valued rational interpolation function is higher and the number of arithmetical operations is larger. To solve this problem, the paper presents a new method of constructing vector valued rational interpolation function. The algorithm does not require any condition for the given data, the rational interpolation function can be presented by the formula and the denominator of vector valued rational function is always greater than zero. The degree of the constructed rational interpolation function is lower and the number of arithmetical operations required is not larger. The main idea of the algorithm goes as follows: firstly, introduce two nonnegative integer parameters $d_1$ $(0 \leq d_1 \leq m)$ and $d_2$ $(0 \leq d_2 \leq n)$; Secondly, divide the rectangular mesh into pieces; thirdly, construct vector valued polynomial $\mathbf{P}_{i,j}(x, y)$ for each piece by using the bivariate polynomial Lagrange or Newton interpolation formula; fourthly, construct the corresponding rational fraction primary function $B_{i,j}(x, y)$; finally, the interpolation formula in rectangular mesh is presented as follows

$$\mathbf{R}(x, y) = \sum_{i=0}^{m-d_1} \sum_{j=0}^{n-d_2} B_{i,j}(x, y)\mathbf{P}_{i,j}(x, y). \tag{3}$$

## 2. Preliminaries

Suppose $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$, the formed rectangular mesh $\prod_{x,y}^{m,n}$ is given in the following array

$$\begin{array}{cccc} (x_0, y_0) & (x_1, y_0) & \cdots & (x_m, y_0) \\ (x_0, y_1) & (x_1, y_1) & \cdots & (x_m, y_1) \\ \vdots & \vdots & & \vdots \\ (x_0, y_n) & (x_1, y_n) & \cdots & (x_m, y_n). \end{array} \tag{4}$$

Let

$$w_i(x) = (x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_m), \quad i = 0, 1, \ldots, m, \tag{5}$$

$$\overline{w_j}(y) = (y - y_0) \cdots (y - y_{j-1})(y - y_{j+1}) \cdots (y - y_n), \quad j = 0, 1, \ldots, n. \tag{6}$$

Then the formula (5) is a polynomial of degree $m$ about variable $x$, while the formula (6) is a polynomial of degree $n$ about variable $y$. Moreover, their leading coefficients are all 1.

If $m$ and $n$ are all odd, let

$$Q(x) = -w_0(x) + w_1(x) - w_2(x) + \cdots - w_{m-1}(x) + w_m(x), \tag{7}$$

$$\overline{Q}(y) = -\overline{w_0}(y) + \overline{w_1}(y) - \overline{w_2}(y) + \cdots - \overline{w_{n-1}}(y) + \overline{w_n}(y). \tag{8}$$

Obviously, $Q(x)$ is a polynomial of degree $m-1$, while $\overline{Q}(y)$ is a polynomial of degree $n-1$.

If $m$ and $n$ are all even, let

$$Q(x) = w_0(x) - w_1(x) + w_2(x) - w_3(x) + \cdots - w_{m-1}(x) + w_m(x), \tag{9}$$

$$\overline{Q}(y) = \overline{w_0}(y) - \overline{w_1}(y) + \overline{w_2}(y) - \overline{w_3}(y) + \cdots - \overline{w_{n-1}}(y) + \overline{w_n}(y). \tag{10}$$

Since the number of $\overline{w}_i(x)$ as well as $\overline{w_j}(y)$ is odd in the formulas (9) and (10), respectively, $Q(x)$ is a polynomial of degree $m$ and $\overline{Q}(y)$ is a polynomial of degree $n$.

In order to obtain a unified form, $w_i(x)$ can be expressed in the following form

$$\begin{aligned} w_i(x) &= \prod_{l=0}^{i-1}(x - x_l) \prod_{k=i+1}^{m} (x - x_k) = (-1)^{m-i} \prod_{l=0}^{i-1}(x - x_l) \prod_{k=i+1}^{m} (x_k - x) \\ &= (-1)^{m-i} u_i(x), \quad i = 0, 1, \ldots, m, \end{aligned} \tag{11}$$

where

$$u_i(x) = \prod_{l=0}^{i-1}(x - x_l) \prod_{k=i+1}^{m} (x_k - x), \quad i = 0, 1, \ldots, m. \tag{12}$$

No matter $m$ is odd or even, the formulas (7) and (9) can both be expressed in the same form as follows

$$Q(x) = \sum_{i=0}^{m} u_i(x), \tag{13}$$

where $u_i(x)$ is given in the formula (12).

With the similar method, the formulas (8) and (10) can be expressed in the unified form

$$\overline{Q}(y) = \sum_{j=0}^{n} u_j(y),$$

where

$$u_j(y) = \prod_{l=0}^{j-1}(y - y_l) \prod_{k=j+1}^{n} (y_k - y), \quad j = 0, 1, \ldots, n. \tag{14}$$

Introduce nonnegative integer parameters $d_1$ $(0 \leq d_1 \leq m)$ and $d_2$ $(0 \leq d_2 \leq n)$, so that the mesh (4) is divided into several pieces. One of them is shown as follows

$$\begin{matrix} (x_i, y_j) & (x_{i+1}, y_j) & \cdots & (x_{i+d_1}, y_j) \\ (x_i, y_{j+1}) & (x_{i+1}, y_{j+1}) & \cdots & (x_{i+d_1}, y_{j+1}) \\ \vdots & \vdots & & \vdots \\ (x_i, y_{j+d_2}) & (x_{i+1}, y_{j+d_2}) & \cdots & (x_{i+d_1}, y_{j+d_2}), \end{matrix} \tag{15}$$

$i = 0, 1, \ldots, m - d_1$; $j = 0, 1, \ldots, n - d_2$. Then the formulas (12) and (14) can be changed into the following forms

$$u_i(x) = \prod_{l=0}^{i-1}(x - x_l) \prod_{k=i+d_1+1}^{m} (x_k - x), \quad i = 0, 1, \ldots, m - d_1, \tag{16}$$

$$u_j(y) = \prod_{l=0}^{j-1}(y - y_l) \prod_{k=j+d_2+1}^{n} (y_k - y), \quad j = 0, 1, \ldots, n - d_2. \tag{17}$$

Note

$$D(x, y) = \prod_{i=0}^{m-d_1} \prod_{j=0}^{n-d_2} u_i(x)u_j(y) = \prod_{i=0}^{m-d_1} \prod_{j=0}^{n-d_2} u_{i,j}(x, y), \tag{18}$$

where

$$u_{i,j}(x, y) = u_i(x)u_j(y). \tag{19}$$

In the formula (16), the constructed sum function $\sum_{i=0}^{m-d_1} u_i(x) > 0$ was proved in [11]. For $u_j(y)$, there is completely similar conclusion, namely $\sum_{j=0}^{n-d_2} u_j(y) > 0$. Therefore in the formula (18), the given bivariate polynomial $D(x, y) > 0$.

If $q(x, y) \in Q_k(2)$, where $Q_k(2)$ denotes the set of all the bivariate polynomials of degree $k$ about the variable $x, y$, respectively, then

$$q(x, y) = \sum_{l=0}^{k} g_l(y)x^l \text{ (or } \sum_{l=0}^{k} g_l(x)y^l),$$

where $g_l(y)$ is a polynomial of degree $k$ about variable $y$.

**Lemma 2.1** *Take $m = n = k$ in the mesh point set (4), then the bivariate polynomial $q(x, y)$ which is determined by $(k + 1)^2$ points in rectangular mesh (4) is unique.*

**Proof** Suppose there are two different polynomials $q_1(x, y)$ and $q_2(x, y)$, which are determined by $(k + 1)^2$ points in rectangular mesh (4). $y = y_j$ $(j = 0, \ldots, k)$ is an equation of a straight line which parallels with the abscissa axis. Considering

$$\eta(x, y) = q_1(x, y) - q_2(x, y),$$

one can obtain

$$\eta(x_i, y_j) = q_1(x_i, y_j) - q_2(x_i, y_j) = 0.$$

While $\eta(x, y_j) = \sum_{l=0}^{k} g_l(y_j)x^l$ is a polynomial of degree $k$ about variable $x$, there are $k+1$ zero points on the straight line $y = y_j$. Therefore $g_l(y_j) = 0$, $l = 0, 1, \ldots, k$; $j = 0, \ldots, k$. But $g_l(y)$ is a polynomial of degree $k$ about variable $y$ for arbitrary $l$, which has $k+1$ zero points, so $g_l(y) \equiv 0$. Thereby $\eta(x, y) \equiv 0$, namely $q_1(x, y) \equiv q_2(x, y)$.

## 3. Interpolation formula for bivariate vector valued rational function

The given vector rectangular mesh is arranged in the following array

$$\begin{array}{ccccc}
 & x = x_0 & x = x_1 & \cdots & x = x_m \\
y = y_0 & \mathbf{V}(x_0, y_0) & \mathbf{V}(x_1, y_0) & \cdots & \mathbf{V}(x_m, y_0) \\
y = y_1 & \mathbf{V}(x_0, y_1) & \mathbf{V}(x_1, y_1) & \cdots & \mathbf{V}(x_m, y_1) \\
\vdots & \vdots & \vdots & & \vdots \\
y = y_n & \mathbf{V}(x_0, y_n) & \mathbf{V}(x_1, y_n) & \cdots & \mathbf{V}(x_m, y_n).
\end{array} \tag{20}$$

Introduce two nonnegative integer parameters $d_1$ $(0 \le d_1 \le m)$ and $d_2$ $(0 \le d_2 \le n)$. Considering a piece of the mesh (20)

$$\begin{array}{ccccc}
 & x = x_i & x = x_{i+1} & \cdots & x = x_{i+d_1} \\
y = y_j & \mathbf{V}(x_i, y_j) & \mathbf{V}(x_{i+1}, y_j) & \cdots & \mathbf{V}(x_{i+d_1}, y_j) \\
y = y_{j+1} & \mathbf{V}(x_i, y_{j+1}) & \mathbf{V}(x_{i+1}, y_{j+1}) & \cdots & \mathbf{V}(x_{i+d_1}, y_{j+1}) \\
\vdots & \vdots & \vdots & & \vdots \\
y = y_{j+d_2} & \mathbf{V}(x_i, y_{j+d_2}) & \mathbf{V}(x_{i+1}, y_{j+d_2}) & \cdots & \mathbf{V}(x_{i+d_1}, y_{j+d_2}),
\end{array} \tag{21}$$

$i = 0, 1, \ldots, m - d_1$; $j = 0, 1, \ldots, n - d_2$. For the corresponding piece (21), one can obtain the following interpolation formula by making use of interpolation formula of Lagrange polynomial

$$\mathbf{P}_{i,j}(x, y) = \sum_{s=i}^{i+d_1} \sum_{k=j}^{j+d_2} l_s(x)\bar{l}_k(y)\mathbf{V}(x_s, y_k), \tag{22}$$

where $l_s(x)$, $\bar{l}_k(y)$ are primary functions of Lagrange type of the univariate polynomial interpolation.

Obviously, $\mathbf{P}_{i,j}(x, y)$ is a bivariate vector polynomial, whose degree does not exceed $d_1$ about variable $x$, and whose degree does not exceed $d_2$ about variable $y$.

For the mesh point set which is obtained by deleting the piece of mesh point set (21) from the mesh point set (20), one can define the following polynomials (see (15)–(18))

$$u_i(x) = \prod_{l=0}^{i-1}(x - x_l) \prod_{k=i+d_1+1}^{m} (x_k - x), \quad i = 0, 1, \ldots, m - d_1, \tag{23}$$

$$u_j(y) = \prod_{l=0}^{j-1}(y - y_l) \prod_{k=j+d_2+1}^{n} (y_k - y), \quad j = 0, 1, \ldots, n - d_2, \tag{24}$$

$$D(x,y) = \sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} u_{i,j}(x,y) = \sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} u_i(x)u_j(y). \tag{25}$$

Making use of $u_i(x)$, $u_j(y)$ and $D(x,y)$, one can construct a rational fraction function as follows

$$B_{i,j}(x,y) = \frac{u_i(x)u_j(y)}{D(x,y)}, \quad i=0,1,\ldots,m-d_1;\ j=0,1,\ldots,n-d_2. \tag{26}$$

One can prove that $B_{i,j}(x,y)$ possesses the following properties

1) $B_{i,j}(x_s,y_t)\begin{cases} \neq 0, & (x_s,y_t)\in(15),\\ =0, & \text{otherwise.}\end{cases}$

In fact, from the formula (23), when $s=i,i+1,\ldots,i+d_1$, $u_i(x_s)\neq 0$; or else, $u_i(x_s)=0$. In the same way, when $t=j,j+1,\ldots,j+d_2$, $u_j(y_t)\neq 0$; or else, $u_j(y_t)=0$. Then from the formula (25), one can obtain that property (1) is true.

One can obtain the following property by direct verification

2) $\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} B_{i,j}(x,y)\equiv 1$.

Multiply $\mathbf{P}_{i,j}(x,y)$ in the formula (22) by rational fraction function $B_{i,j}(x,y)$ in the formula (26), and then summarize them. In this way, one can obtain the following bivariate vector valued rational function

$$\mathbf{R}(x,y) = \sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} B_{i,j}(x,y)\mathbf{P}_{i,j}(x,y). \tag{27}$$

**Theorem 3.1** *For the given rectangular mesh set (20), the vector valued rational function which is determined by the formula (27) satisfies the following interpolation conditions*

$$\mathbf{R}(x_i,y_j) = \mathbf{V}(x_i,y_j), \quad i=0,1,\ldots,m;\ j=0,1,\ldots,n.$$

**Proof** Suppose $I=\{0,1,\ldots,m-d_1\}$, $I'=\{0,1,\ldots,n-d_2\}$. For an arbitrary point $(x_\alpha,y_\beta)$, $0\le\alpha\le m$; $0\le\beta\le n$ in rectangular mesh, let

$$J_\alpha=\{i\in I:\alpha-d_1\le i\le\alpha\}, \quad J_\beta=\{i\in I':\beta-d_2\le j\le\beta\}.$$

Then when $i\in J_\alpha$, $j\in J_\beta$, according to the formulas (16) and (17), one can obtain

$$u_i(x_\alpha)=0,\ i\neq\alpha;\quad u_i(x_\alpha)\neq 0,\ i=\alpha;$$

$$u_j(y_\beta)=0,\ j\neq\beta;\quad u_j(y_\beta)\neq 0,\ j=\beta.$$

Hence $D(x_\alpha,y_\beta)=\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} u_i(x_\alpha)u_j(y_\beta)\neq 0$. Since $J_\alpha$ and $J_\beta$ are nonempty sets, we have

$$\mathbf{R}(x_\alpha,y_\beta) = \frac{1}{D(x_\alpha,y_\beta)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} u_i(x_\alpha)u_j(y_\beta)\mathbf{P}_{i,j}(x_\alpha,y_\beta)$$

$$= \mathbf{V}(x_\alpha,y_\beta)\frac{\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} u_i(x_\alpha)u_j(y_\beta)}{D(x_\alpha,y_\beta)} = \mathbf{V}(x_\alpha,y_\beta).$$

The formula (27) is an interpolation formula of bivariate vector valued rational function, and the bivariate polynomial in the formula (25) is always greater than zero. The vector valued rational function constructed by the formula (27) not only keeps the feature of univariate vector rational interpolation, but also possesses the advantage that it is easy to compute and the degree is lower.

The specific algorithm goes like this:

1) According to the specific demand, choose $d_1$ and $d_2$, then compute $\mathbf{P}_{i,j}(x,y)$ by the formula (22);

2) Compute $u_i(x)$ and $u_j(y)$ by the formulas (23) and (24), then compute $D(x,y)$ by the formula (25);

3) Give the expression of $B_{i,j}(x,y)$ by the formula (26);

4) Compute $\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2} B_{i,j}(x,y)\mathbf{P}_{i,j}(x,y)=\mathbf{R}(x,y)$.

It is worthwhile to point out that one can construct vector valued rational functions of different types by choosing $d_1$ and $d_2$ in the formula (27), say, choosing $d_1 = m$ and $d_2 = 0$, namely, dividing pieces based on the array row makes the formula (22) change into the following form

$$\mathbf{P}_{0,j}(x,y)=\sum_{s=0}^{m} l_s(x)\mathbf{V}(x_s,y_j), \quad j=0,1,\ldots,n. \tag{28}$$

The formula (25) changes into the following form

$$D(x,y)=\sum_{j=0}^{n} u_j(y). \tag{29}$$

In the similar way, choosing $d_1 = 0$ and $d_2 = n$, namely, dividing pieces based on the array column makes the formula (22) change into the following form

$$\mathbf{P}_{i,0}(x,y)=\sum_{k=0}^{n} \bar{l}_k(y)\mathbf{V}(x_i,y_k), \quad i=0,1,\ldots,m. \tag{30}$$

The formula (25) changes into the following form

$$D(x,y)=\sum_{i=0}^{m} u_i(x). \tag{31}$$

In actual computation, we make the convention that

$$\prod_{i=0}^{-k}(t-t_i)\equiv 1, \quad \prod_{l=k}^{k-1}(t_k-t)\equiv 1,\ k\geq 1, k\in Z.$$

**Example 1** Given data

| | $x_0=-2$ | $x_1=-1$ | $x_2=0$ |
|---|---|---|---|
| $y_0=0$ | $(0,0)$ | $(12,6)$ | $(2,2)$ |
| $y_1=1$ | $(6,0)$ | $(6,0)$ | $(6,0)$ |
| $y_2=2$ | $(-2,2)$ | $(12,6)$ | $(24,24)$. |

With the algorithm in [12], one can obtain

$$\mathbf{R}(x,y)=(x+y-2)(x+y-1)[\frac{(3y-2y^2,y^2-y)}{(3-2y)^2+(y-1)^2}+$$

$$\frac{(10x+20)(6x+y+10,8x-2y+10)}{(6x+y+10)^2+(8x-2y+10)^2}]+$$
$$(x+y+2)(x+y+1)(x+y)(1-x,y-1).$$

Divide pieces based on the array row, here $m = n = 2$, $d_1 = 2, d_2 = 0$. From the formula (28), one can obtain

$$\mathbf{P}_{0,0}(x,y) = (-11x^2 - 21x + 2, -5x^2 - 9x + 2), \quad \mathbf{P}_{0,1}(x,y) = (6,0),$$
$$\mathbf{P}_{0,2}(x,y) = (-x^2 + 11x + 24, 7x^2 + 25x + 24).$$

From the formula (29), one can obtain

$$D(x,y) = \sum_{j=0}^{2} u_j(y) = y^2 - 3y + 2 + 2y - y^2 + y^2 - y = y^2 - 2y + 2.$$

From the formula (26), one can obtain

$$B_{0,0}(x,y) = \frac{1}{D}(y^2 - 3y + 2),\ B_{0,1}(x,y) = \frac{1}{D}(2y - y^2),\ B_{0,2}(x,y) = \frac{1}{D}(y^2 - y).$$

From the formula (27), one can obtain

$$\begin{aligned}\mathbf{R}(x,y) =& \sum_{j=0}^{2} B_{0,j}(x,y)\mathbf{P}_{0,j}(x,y)\\ =& \frac{1}{D}(-12x^2y^2 - 10xy^2 + 34x^2y + 20y^2 + 52xy - 22x^2 - 18y - 42x + 4\\ & 2x^2y^2 + 16xy^2 + 8x^2y - 10x^2 + 2xy + 26y^2 - 18x - 30y + 4).\end{aligned}$$

Through direct verification, one knows $\mathbf{R}(x_i, y_j) = \mathbf{V}(x_i, y_j)$ $(i, j = 0, 1, 2)$, and $\mathbf{R}(x, y)$ is independent of arbitrarily adjusting the array row.

Divide pieces based on the array column

$$\begin{aligned}\mathbf{R}(x,y) =& \frac{1}{x^2+2x+2}(-6x^2y^2 + 22x^2y + 2xy^2 + 14y^2 + 28xy - 10x^2 - 18x - 6y + 4,\\ & 8x^2y^2 + 28xy^2 - 4x^2y - 4x^2 - 22xy + 26y^2 - 6x - 30y + 4).\end{aligned}$$

Direct verification shows that $\mathbf{R}(x_i, y_j) = \mathbf{V}(x_i, y_j)$ $(i, j = 0, 1, 2)$, and $\mathbf{R}(x, y)$ is independent of arbitrarily adjusting the array column.

Choose $d_1 = 2$, $d_2 = 1$, and the mesh is divided into two pieces. In this way, the formula (22) changes into the following form

$$\mathbf{P}_{i,j}(x,y) = \sum_{s=i}^{i+2}\sum_{k=j}^{j+1} l_s(x)\bar{l}_k(y)\mathbf{V}(x_s, y_k),\ i = 0;\ j = 0, 1.$$

Then substituting data into the above formula, one can obtain

$$\mathbf{P}_{0,0}(x,y) = (11x^2y - 11x^2 + 21xy - 21x + 4y + 2, 5x^2y - 5x^2 + 9xy - 9x - 2y + 2),$$

$$\mathbf{P}_{0,1}(x,y) = (-x^2y + 11xy + x^2 - 11x + 18y - 12, 7x^2y + 25xy - 7x^2 - 25x + 24y - 24).$$

From the formula (25), one can obtain

$$D(x,y) = \sum_{j=0}^{1} u_j(y) = 2 - y + y = 2.$$

Vector valued rational interpolation degenerates into vector polynomial interpolation. In order to get rational interpolation function, we can zoom up (down) $u_0(y)$ or $u_1(y)$ properly. The simplest method is to zoom up (down) $u_1(y)$ to $\overline{u}_1(y) = 2u_1(y) = 2y$ (or $\overline{u}_1(y) = \frac{1}{2}u_1(y) = \frac{1}{2}y$). But it is necessary to make sure that $u_0(y) + \overline{u}_1(y) = 2 + y$ (or $u_0(y) + \overline{u}_1(y) = 2 - \frac{1}{2}y$) is positive in the interval which includes interpolation intervals. Take $D = \frac{4-y}{2}$, from the formula (26), one can obtain $B_{0,0}(x,y) = \frac{2-y}{D}, B_{0,1}(x,y) = \frac{\frac{1}{2}y}{D}$. One can obtain the following formula by substituting the above data in the formula (27)

$$\mathbf{R}(x,y) = \frac{1}{D}(\mathbf{P}_{0,0}(x,y)(2-y) + \mathbf{P}_{0,1}(x,y)\frac{1}{2}y).$$

Choose $d_1 = d_2 = 0$, namely not dividing pieces. From the formulas (22)-(24), one can obtain $D(x,y) = (y^2 - 2y + 2)(x^2 + 2x + 2)$. At the moment, $\mathbf{P}_{i,j}(x,y) = \mathbf{V}(x_i, y_j)$, $i = 0,1,2$; $j = 0,1,2$. From (25) and (27), one can obtain

$$\begin{aligned}\mathbf{R}(x,y) =& \frac{1}{D}(-6x^2y^2 + 32x^2y + 16xy^2 + 40y^2 + 32xy - 20x^2 - 36y - 36x + 8,\\ & 16x^2y^2 + 56xy^2 - 8x^2y - 44xy + 52y^2 - 8x^2 - 60y - 12x + 8).\end{aligned}$$

From Example 1, one can obtain that the bivariate vector valued rational interpolation functions are different when choosing different nonnegative integer parameters $d_1$ and $d_2$. The fact demonstrates that the general uniqueness of vector valued rational interpolation function is not true, and that the uniqueness is determined by the method adopted. The type of interpolation function is based on choosing parameters $d_1$ and $d_2$, but there is one common ground that the numerator polynomial of vector rational fraction function is a bivariate vector polynomial whose degree is $m$ about variable $x$ and also whose degree is $n$ about variable $y$.

## 4. Uniqueness and characterization theorem

From the above discussion, one can obtain that the type of vector valued rational interpolation function is related with the method adopted. The constructed bivariate vector valued rational interpolation function is unique when the same method is adopted. We will discuss about square field ($m = n$) in the following part.

**Theorem 4.1** *When $d_1$ and $d_2$ are determined, the bivariate vector valued rational interpolation function is unique in the formula (27).*

**Proof** Suppose there are two vector valued rational functions which satisfy interpolation conditions as follows

$$\mathbf{r}_1(x,y) = \frac{1}{D_1(x,y)} \sum_{i=0}^{n-d_1} \sum_{j=0}^{n-d_2} u_i(x)u_j(y)\mathbf{P}_{i,j}(x,y),$$

and

$$\mathbf{r}_2(x,y)=\frac{1}{D_2(x,y)}\sum_{i=0}^{n-d_1}\sum_{j=0}^{n-d_2}u_i(x)u_j(y)\overline{\mathbf{P}}_{i,j}(x,y).$$

As for given $(n+1)^2$ points, from Lemma 2.1, one can obtain $D_1(x,y)\equiv D_2(x,y)$, and $D(x_i,y_j)>0$ $(i,j=0,1,\ldots,n)$.

Let

$$\mathbf{R}(x,y)=\mathbf{r}_1(x,y)-\mathbf{r}_2(x,y)=\frac{1}{D}\sum_{i=0}^{n-d_1}\sum_{j=0}^{n-d_2}u_i(x)u_j(y)(\mathbf{P}_{i,j}(x,y)-\overline{\mathbf{P}}_{i,j}(x,y)),\tag{32}$$

namely

$$\sum_{i=0}^{n-d_1}\sum_{j=0}^{n-d_2}u_i(x_s)u_j(y_t)(\mathbf{P}_{i,j}(x_s,y_t)-\overline{\mathbf{P}}_{i,j}(x_s,y_t))=0.\tag{33}$$

It is noticed that the formula (33) is a bivariate vector polynomial, each component of which is a polynomial of degree $n$ about variable $x$ and $y$, respectively. According to the proof method in Lemma 2.1, one can prove that they are always zero. Therefore, $\mathbf{r}_1(x,y)\equiv\mathbf{r}_2(x,y)$.

It is easy to prove the following theorem.

**Theorem 4.2** *Suppose the bivariate vector valued rational interpolation function $\mathbf{R}(x,y)$ is defined in the formula (27).*

*1) If $m-d_1$, $n-d_2$ are all even, then $\mathbf{R}(x,y)$ is of type $[(m+n)/(m+n-d_1-d_2)]$.*

*2) For $m-d_1$, $n-d_2$, if one is odd and the other is even, then $\mathbf{R}(x,y)$ is of type $[(m+n)/(m+n-d_1-d_2-1)]$.*

*3) If $m-d_1$, $n-d_2$ are all odd, then $\mathbf{R}(x,y)$ is of type $[(m+n)/(m+n-d_1-d_2-2)]$.*

In Example 1, $m=n=2$. If one chooses $d_1=2$ and $d_2=0$, then $\mathbf{R}(x,y)$ is of type [4/2]; If one chooses $d_1=d_2=0$, then $\mathbf{R}(x,y)$ is of type [4/4].

## 5. Error estimation

**Theorem 5.1** *Suppose $\mathbf{R}(x,y)=\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}B_{i,j}(x,y)\mathbf{P}_{i,j}(x,y)$ is a bivariate vector function which satisfies interpolation conditions in the formula (2), and $\mathbf{V}(x,y)\in C^{d_1+d_2+2}[a,b]\times[c,d]$, then $\forall(x,y)\in[a,b]\times[c,d]$, we have*

$$\begin{aligned}\mathbf{V}(x,y)-\mathbf{R}(x,y)=&\frac{(-1)^{m-d_1}}{(d_1+1)!}\frac{\omega_m(x)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^iu_j(y)\frac{\partial^{d_1+1}}{\partial x^{d_1+1}}\mathbf{V}(\xi_i,y)+\\&\frac{(-1)^{n-d_2}}{(d_2+1)!}\frac{\omega_n(y)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^ju_i(x)\frac{\partial^{d_2+1}}{\partial y^{d_2+1}}\mathbf{V}(x,\eta_j)-\\&\frac{(-1)^{m+n-d_1-d_2}}{(d_1+1)!(d_2+1)!}\frac{\omega_m(x)\omega_n(y)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^{i+j}\frac{\partial^{d_1+d_2+2}}{\partial x^{d_1+1}\partial y^{d_2+1}}\mathbf{V}(\overline{\xi}_i,\overline{\eta}_j),\end{aligned}$$

*where $\xi_i,\overline{\xi}_i\in[a,b]$, $\eta_j,\overline{\eta}_j\in[c,d]$,*

$$\omega_{d_1+1}(x)=(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d_1}),\ \omega_{d_2+1}(y)=(y-y_j)(y-y_{j+1})\cdots(y-y_{j+d_2}),$$

$$\omega_m(x)=(x-x_0)(x-x_1)\cdots(x-x_m),\ \omega_n(y)=(y-y_0)(y-y_1)\cdots(y-y_n).$$

**Proof** We have

$$\begin{aligned}\mathbf{V}(x,y)-\mathbf{R}(x,y)=&\mathbf{V}(x,y)-\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}B_{i,j}(x,y)\mathbf{P}_{i,j}(x,y)\\=&\frac{\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}u_{i,j}(x,y)(\mathbf{V}(x,y)-\mathbf{P}_{i,j}(x,y))}{D(x,y)}.\end{aligned}$$

Using the bivariate vector Newton error formula

$$\begin{aligned}\mathbf{V}(x,y)-\mathbf{P}_{i,j}(x,y)=&\frac{\omega_{d_1+1}(x)}{(d_1+1)!}\frac{\partial^{d_1+1}}{\partial x^{d_1+1}}\mathbf{V}(\xi,y)+\frac{\omega_{d_2+1}(y)}{(d_2+1)!}\frac{\partial^{d_2+1}}{\partial y^{d_2+1}}\mathbf{V}(x,\eta)-\\&\frac{\omega_{d_1+1}(x)\omega_{d_2+1}(y)}{(d_1+1)!(d_2+1)!}\frac{\partial^{d_1+d_2+2}}{\partial x^{d_1+1}\partial y^{d_2+1}}\mathbf{V}(\overline{\xi},\overline{\eta}).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{V}(x,y)-\mathbf{R}(x,y)=&\frac{(-1)^{m-d_1}}{(d_1+1)!}\frac{\omega_m(x)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^i u_j(y)\frac{\partial^{d_1+1}}{\partial x^{d_1+1}}\mathbf{V}(\xi_i,y)+\\&\frac{(-1)^{n-d_2}}{(d_2+1)!}\frac{\omega_n(y)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^j u_i(x)\frac{\partial^{d_2+1}}{\partial y^{d_2+1}}\mathbf{V}(x,\eta_j)-\\&\frac{(-1)^{m+n-d_1-d_2}}{(d_1+1)!(d_2+1)!}\frac{\omega_m(x)\omega_n(y)}{D(x,y)}\sum_{i=0}^{m-d_1}\sum_{j=0}^{n-d_2}(-1)^{i+j}\frac{\partial^{d_1+d_2+2}}{\partial x^{d_1+1}\partial y^{d_2+1}}\mathbf{V}(\overline{\xi}_i,\overline{\eta}_j),\end{aligned}$$

where $\xi_i,\overline{\xi}_i\in[a,b],\eta_j,\overline{\eta}_j\in[c,d]$,

$$\omega_{d_1+1}(x)=(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d_1}),\omega_{d_2+1}(y)=(y-y_j)(y-y_{j+1})\cdots(y-y_{j+d_2}),$$

$$\omega_m(x)=(x-x_0)(x-x_1)\cdots(x-x_m),\omega_n(y)=(y-y_0)(y-y_1)\cdots(y-y_n).$$

For example, as for the bivariate vector function $V(x,y)=(\frac{x+y}{1+x^2},\cos(\frac{\pi}{2}(x+y)))$ whose interpolation nodes are listed as follows

| | $x_0=0$ | $x_1=1$ | $x_2=2$ |
|---|---|---|---|
| $y_0=0$ | $(0,1)$ | $(\frac{1}{2},0)$ | $(\frac{2}{5},-1)$ |
| $y_1=1$ | $(1,0)$ | $(1,-1)$ | $(\frac{3}{5},0)$ |
| $y_2=2$ | $(2,-1)$ | $(\frac{3}{2},0)$ | $(\frac{4}{5},1).$ |

Choosing $d_1=0$, $d_2=2$, from the formula (27), one can obtain bivariate vector valued rational interpolation function as follows

$$\mathbf{R}_1(x,y)=\frac{(\frac{7}{10}x^2y-\frac{11}{5}xy+2y+\frac{3}{5}x-\frac{1}{10}x^2,-x^2y^2+2x^2y+2xy^2-2xy-2x-2y+2)}{x^2-2x+2}.$$

With the algorithm of the branched continued fraction in [5], one can obtain a bivariate vector rational interpolation function $\mathbf{R}_2(x,y)$. Making use of these two methods, one can obtain error estimation values listed in the following Table 1.

| $(x,y)$ | $\mathbf{V}(x,y)-\mathbf{R}_1(x,y)$ | $\mathbf{V}(x,y)-\mathbf{R}_2(x,y)$ |
|---|---|---|
| (0.4,1.6) | (0.1100,-0.1694) | (1.9865,0.3370) |
| (0.8,0.8) | (0.0464,2.1377*e-004) | (0.1475,-0.4525) |
| (0.8,1.2) | (0.0257,-0.0369) | (0.0085,0.3775) |
| (0.8,1.6) | (0.0050,0.0125) | (2.5352,1.3702) |
| (1.2,1.2) | (0.0051,2.1377*e-004) | (0.0117,0.6352) |
| (1.2,1.6) | (0.0275,0.0510) | (2.5759,2.5066) |
| (1.6,1.6) | (0.0612,0.0808) | (2.4939,3.5852) |
| (1.6,1.8) | (0.0774,0.0513) | (2.3751,-0.6472 ) |

Table 1 Error Estimation Values

From Table 1, one can obtain that the approximation effect by adopting the method presented in this paper is much better than the approximation effect with the method in [5]. In addition, there are three advantages about the method presented in this paper: 1) It is easy to compute; 2) The number of arithmetical operations required is less and 3) The degree is lower.

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