Journal of Mathematical Research & Exposition
Jul., 2011, Vol. 31, No. 4, pp. 617–626
DOI:10.3770/j.issn:1000-341X.2011.04.005
Http://jmre.dlut.edu.cn

Removable Edges in a 5-Connected Graph

Li Qiong XU[1,*], Xiao Feng GUO[2]
1. *School of Sciences, Jimei University, Fujian* 361023*, P. R. China;*
2. *School of Mathematics Sciences, Xiamen University, Fujian* 361005*, P. R. China*

Abstract An edge e of a k-connected graph G is said to be a removable edge if $G \ominus e$ is still k-connected, where $G \ominus e$ denotes the graph obtained from G by deleting e to get $G-e$, and for any end vertex of e with degree $k-1$ in $G-e$, say x, delete x, and then add edges between any pair of non-adjacent vertices in $N_{G-e}(x)$. The existence of removable edges of k-connected graphs and some properties of 3-connected and 4-connected graphs have been investigated [1, 11, 14, 15]. In the present paper, we investigate some properties of 5-connected graphs and study the distribution of removable edges on a cycle and a spanning tree in a 5-connected graph. Based on the properties, we proved that for a 5-connected graph G of order at least 10, if the edge-vertex-atom of G contains at least three vertices, then G has at least $(3|G|+2)/2$ removable edges.

Keywords 5-connected graph; removable edge; edge-vertex-atom.

Document code A
MR(2010) Subject Classification 05C15
Chinese Library Classification O157.5

1. Introduction

Graph theoretic terminology used here generally follows that of Bondy [2]. We consider only finite and simple graphs.

Connectivity of graphs is a fundamental topic in graph theory research. For properties and constructions of several classes of k-edge-connected graphs and k-connected graphs, many investigations have been made. The concepts of contractible edges and removable edges of k-connected graphs are very important in studying the constructions of k-connected graphs and in proving some properties of k-connected graphs by induction.

For removable edges of k-connected graphs, Holton et al. [6] first defined removable edges in a 3-connected graph. Later, Yin [17] defined removable edges in a 4-connected graph. The concept of removable edges in a 3-connected graph and a 4-connected graph can be generalized to k-connected graphs [16].

Definition 1 ([16]) *Let G be a k-connected graph, and let e be an edge of G. Let $G \ominus e$ denote*

Received July 8, 2009; Accepted January 18, 2010
Supported by the National Natural Science Foundation of China (Grant No. 10831001) and the Science-Technology Foundation for Young Scientists of Fujian Province (Grant No. 2007F3070).
* Corresponding author
E-mail address: xuliqiong@jmu.edu.cn (L. Q. XU)

the graph obtained from G by the following operation: (1) delete e from G to get $G-e$; (2) for any end vertex of e with degree $k-1$ in $G-e$, say x, delete x, and then add edges between any pair of non-adjacent vertices in $N_{G-e}(x)$. If $G\ominus e$ is k-connected, then e is said to be a removable edge of G, otherwise e is said to be non-removable. The set of all non-removable edges of G and the set of all removable edges of G are denoted by $E_N(G)$ and $E_R(G)$ respectively.

Barnette and Grunbaum [1] proved that a 3-connected graph of order at least five has a removable edge. Based on the fact and the above graph operation, a constructive characterization of minimally 3-connected graphs was given by Dawes [3], which differs from the characterization provided by Tutte [13].

In [17], Yin also proved that a 4-connected graph without removable edge is either K_5 or K_6 by removing a 1-factor. Based on this result, he provided a constructive characterization of 4-connected graphs, which is simpler than Slater's method [10]. And then, We proved that a 5-connected graph G has no removable edge if and only if $G\cong K_6$. Using this result, we gave the constructive characterization of 5-connected graphs. Recently, Su et al. [12] proved that a k-connected graph without removable edge is either $K_{(k+1)}$ (when k is even) or the graph obtained from $K_{(k+2)}$ by removing a 1-factor. Based on this result, the constructive characterization of k-connected graphs is given.

For the removable edges and non-removable edges of a k-connected graph G, the following result was given in [16].

Theorem 1 ([16]) *Let G be a k-connected graph of order at least $k+3$ ($k\geq 3$) and $e=xy\in E(G)$. Then e is non-removable if and only if there exists $S\subseteq V(G)$ with $|S|=k-1$ such that $G-e-S$ has exactly two components A, B with $|A|\geq 2$ and $|B|\geq 2$, moreover $x\in A$, $y\in B$.*

Without specific statement, in the following G always denotes a 5-connected graph. The vertex set and edge set of G are denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G are denoted, respectively, by $|G|$ and $|E(G)|$. The neighborhood of $x\in G$ is denoted by $\Gamma_G(x)$ and the degree of x is denoted by $d_G(x)$. For a nonempty subset N of $V(G)$, the induced subgraph by N in G is denoted by $[N]$. For a subset S of $V(G)$, $G-S$ denotes the graph obtained by deleting all the vertices in S from G together with all the incident edges. If $G-S$ is disconnected, we say that S is a vertex-cut of G. $\delta(G)$ denotes the minimum degree of $V(G)$. The girth of G is the length of a shortest cycle in G and is denoted by $g(G)$. Let A, $B\subset V(G)$ such that $A\neq\emptyset$, $B\neq\emptyset$, $A\cap B=\emptyset$. Define $[A,B]=\{xy\in E(G)|x\in A, y\in B\}$. For $e\in E(G)$ and $S\subset V(G)$ such that $|S|=4$, if $G-e-S$ has exactly two connected components, say A and B, such that $|A|\geq 2$ and $B\geq 2$, then we say that (e,S) is a separating pair and $(e,S;A,B)$ is a separating group, in which A and B are called the edge-vertex-cut fragments. An edge-vertex-cut fragments of G with a minimum number of vertices is called an edge-vertex-atom of G.

Let $E_0\subset E_N(G)$ such that $E_0\neq\emptyset$ and let $(xy,S;A,B)$ be a separating group of G such that $x\in A$ and $y\in B$. If $xy\in E_0$, then A and B are called E_0-edge-vertex-cut fragments. An E_0-edge-vertex-cut fragment is called E_0-edge-vertex-cut end-fragment of G if it does not contain any other E_0-edge-vertex-cut fragment of G as a proper subset. It is easy to see that any E_0-edge-vertex-cut fragment of G contains such an end-fragment.

Removable edges in 3-connected graphs and 4-connected graphs have been studied extensively [1, 11, 14, 15, 17]. In the present paper, we investigate some properties of 5-connected graphs and study the distribution of removable edges on a cycle and a spanning tree in a 5-connected graph. On the basis of the properties, we proved that for a 5-connected graph G of order at least 10, if the edge-vertex-atom of G contains at least three vertices, then G has at least $(3|G|+2)/2$ removable edges.

2. The properties of removable edges in a 5-connected graph

Lemma 2 *Let G be a 5-connected graph of order at least 10, an edge-vertex-atom of which contains at least three vertices. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$. Then every edge in $[\{x\}, S]$ is removable.*

Proof By contradiction. Assume that there is an edge in $[\{x\}, S]$, say xu, is non-removable. So there is a corresponding separating group $(xu, T; C, D)$ such that $x \in C$, $u \in D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T), \quad X_2 = (D \cap S) \cup (S \cap T) \cup (A \cap T),$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T), \quad X_4 = (C \cap S) \cup (S \cap T) \cup (B \cap T).$$

Obviously, $x \in A \cap C$. Since $X_1 \cup \{y, u\}$ is a vertex-cut of G and G is 5-connected, we have that $|X_1| \geq 3$. Next we will distinguish the following cases to proceed the proof.

Case 1 $y \in B \cap C$. Then X_4 is a vertex-cut of $G - xy$. Since G is 5-connected, we have that $|X_4| \geq 4$. Since $|X_2| + |X_4| = |S| + |T| = 8$, we have that $|X_2| \leq 4$. Thus $A \cap D = \emptyset$ (otherwise, X_2 would be a vertex-cut of G, which contradicts that G is 5-connected).

Assume that $B \cap D = \emptyset$. Then $|D| = |S \cap D| \geq 3$, and so $|S \cap C| + |S \cap T| = |S| - |S \cap D| \leq 1$, $|S \cap T| + |A \cap T| = |X_2| - |S \cap D| \leq 1$. Thus $|X_1| \leq 2$, which contradicts that $|X_1| \geq 3$.

Otherwise, $B \cap D \neq \emptyset$. Since X_3 is a vertex-cut of G, $|X_3| \geq 5$. Since $|X_1| + |X_3| = 8$ and $|X_1| \geq 3$, we have that $|X_1| = 3$ and $|X_3| = 5$. If $|A \cap C| \geq 2$, then $X_1 \cup \{x\}$ is a vertex-cut of G with 4 vertices, a contradiction. Hence $|A \cap C| = 1$. Then $|A \cap T| = |A| - |A \cap C| \geq 2$. So we have that $|S \cap C| + |S \cap T| = |X_1| - |A \cap T| \leq 1$ and $|S \cap T| + |B \cap T| = |T| - |A \cap T| \leq 2$, and so $|X_4| = |S \cap C| + |S \cap T| + |B \cap T| \leq 3$, which contradicts that $|X_4| \geq 4$.

Case 2 $y \in B \cap T$.

We claim that $A \cap T \neq \emptyset$ and $S \cap C \neq \emptyset$. Otherwise, one of $A \cap T$ and $S \cap C$, say $A \cap T$, is empty. Since $A \cap C \neq \emptyset$ and A is a connected subgraph of G, we have that $A \cap D = \emptyset$, and so $|A| = |A \cap C| \geq 3$. Since $|X_1| = |S \cap C| + |S \cap T| \geq 3$ and $u \in S \cap D$, noting that $|S| = |S \cap C| + |S \cap T| + |S \cap D| = 4$, we have that $|X_1| = |S \cap C| + |S \cap T| = 3$ and $|S \cap D| = 1$, and thus $X_1 \cup \{x\}$ would be a vertex-cut of G. However, $|X_1 \cup \{x\}| = 4$, which contradicts that G is 5-connected. Therefore, $A \cap T \neq \emptyset$. Obviously, $|A \cap T| \leq 3$.

Now we distinguish the following cases.

Case 2.1 $|A \cap T| = 1$. Then $|S \cap C| + |S \cap T| = |X_1| - |A \cap T| \geq 2$. And since $|S| = |S \cap C| + |S \cap T| + |S \cap D| = 4$, we have that $|S \cap D| \leq 2$.

Case 2.1.1 $|S \cap D| = 1$. Since $|S \cap T| + |S \cap D| = |S| - |S \cap C| \leq 3$ and $|S \cap T| + |B \cap T| = |T| - |A \cap T| = 3$, then $|X_2| \leq 4$, $|X_3| = 4$. So $A \cap D = \emptyset = B \cap D$, $|D| = |A \cap D| + |S \cap D| + |B \cap D| = 1$, which contradicts that $|D| \geq 2$.

Case 2.1.2 $|S \cap D| = 2$. Then we have that $|S \cap C| + |S \cap T| = 2$ and $|X_1| = |S \cap C| + |S \cap T| + |A \cap T| = 3$. An argument similar to that used in case 1 can lead to that $|A \cap C| = 1$. And since $|S \cap T| + |S \cap D| = |S| - |S \cap C| \leq 3$, we have that $|X_2| \leq 4$. By noticing that G is 5-connected, we have that $A \cap D = \emptyset$. Then, $|A| = |A \cap T| + |A \cap C| = 2$, which contradicts that $|A| \geq 3$.

Case 2.2 $|A \cap T| = 2$. Then $|S \cap T| \leq 1$. Then, we will discuss the following cases.

Case 2.2.1 $|S \cap T| = 1$. Then we have that $|B \cap T| = 1$ and $|S \cap C| + |S \cap D| = 3$. By noticing that $S \cap C \neq \emptyset$ and $S \cap D \neq \emptyset$, we may assume that $|S \cap C| = 1$ and $|S \cap D| = 2$, then $|X_3| = 4$ and $|X_4| = 3$, so $B \cap D = \emptyset$ and $B \cap C = \emptyset$. Thus, $|B| = |B \cap C| + |B \cap D| + |B \cap T| = 1$, which contradicts that $|B| \geq 2$.

Case 2.2.2 $|S \cap T| = 0$. We have that $|B \cap T| = 2$.

Assume that $|S \cap C| = 1$. Then, we have that $|X_1| = 3$ and $|X_4| = 3$. An argument analogous to that used in case 1 can lead to that $|A \cap C| = 1$ and $B \cap C = \emptyset$. Thus, $|C| = |A \cap C| + |S \cap C| + |B \cap C| = 2$, a contradiction.

Assume that $|S \cap C| \geq 2$. Then, we have that $|S \cap D| \leq 2$ and $|X_2| = |X_3| \leq 4$. An argument analogous to that used in case 2.2.1 can lead to that $A \cap D = \emptyset$ and $B \cap D = \emptyset$. Thus, $|D| = |A \cap D| + |S \cap D| + |B \cap D| \leq 2$, a contradiction.

Case 2.3 $|A \cap T| = 3$. Then $|S \cap T| = 0$, $|B \cap T| = 1$. Since $S \cap C \neq \emptyset$, $S \cap D \neq \emptyset$, we have $|X_3| \leq 4$, $|X_4| \leq 4$. Thus $B \cap C = \emptyset$, $B \cap D = \emptyset$. So $|B| = |B \cap T| = 1$, a contradiction.

The proof is now completed. □

The next two results are consequences of Lemma 2.

Corollary 3 *Let G be a 5-connected graph of order at least 10 with $\delta(G) \geq 6$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$. Then every edge in $[\{x\}, S]$ is removable.*

Proof If $\delta(G) \geq 6$, we claim that the edge-vertex-atom of G contains at least three vertices. Otherwise, the edge-vertex-atom of G contains two vertices, say A, we take its separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$. Assume that $A = \{x, z\}$. Since G is 5-connected and $|S| = 4$, we have that $d_G(z) = 5$, which contradicts that $\delta(G) \geq 6$. From Lemma 2, the Corollary holds. □

By a similar argument, the following result can be obtained easily.

Corollary 4 *Let G be a 5-connected graph of order at least 10 with $g(G) \geq 4$. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$. Then every edge in $[\{x\}, S]$ is removable.*

Lemma 5 *Let G be a 5-connected graph of order at least 10, an edge-vertex-atom of which contains at least three vertices. Let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$. Then $E(G[S]) \subseteq E_R(G)$.*

Proof By contradiction. Assume that there is an edge in $E(G[S])$, say uv, is non-removable. So there is a corresponding separating group $(uv, T; C, D)$ such that $u \in C$, $v \in D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T), \quad X_2 = (D \cap S) \cup (S \cap T) \cup (A \cap T),$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T), \quad X_4 = (C \cap S) \cup (S \cap T) \cup (B \cap T).$$

Obviously, $u \in S \cap C$ and $v \in D \cap S$. We discuss the following cases.

Case 1 $x \in A \cap C$ and $y \in B \cap C$.

Then we have that $|X_1| \geq 4$ and $|X_4| \geq 4$. Since $|X_1| + |X_3| = |S| + |T| = |X_2| + |X_4| = 8$, we have $|X_2| \leq 4$ and $|X_3| \leq 4$. Thus $A \cap D = \emptyset = B \cap D$. Since $|D| = |A \cap D| + |S \cap D| + |B \cap D| = |S \cap D| \geq 3$, we have $|S \cap C| + |S \cap T| = |S| - |S \cap D| \leq 1$ and $|A \cap T| + |S \cap T| = |X_2| - |S \cap D| \leq 1$. Thus, $|X_1| \leq 2$, which contradicts $|X_1| \geq 4$.

Case 2 $x \in A \cap C$ and $y \in B \cap T$.

Then we have that $|X_1| \geq 4$. Since $|X_1| + |X_3| = 8$, we have that $|X_3| \leq 4$, thus $B \cap D = \emptyset$. If $B \cap C \neq \emptyset$, an argument analogous to that used in case 1 can lead to a contradiction. So $B \cap C = \emptyset$. Then $|B| = |B \cap T| \geq 3$. Noting that $|S \cap T| + |A \cap T| = |T| - |B \cap T| \leq 1$, we have that $|S \cap C| = |X_1| - |S \cap T| - |A \cap T| \geq 3$. Then $|S \cap D| = |S| - |S \cap T| - |S \cap C| \leq 1$. Hence $|X_2| = |A \cap T| + |S \cap T| + |S \cap D| \leq 2$, then $A \cap D = \emptyset$, thus $|D| = |A \cap D| + |S \cap D| + |B \cap D| = |S \cap D| \leq 1$, which contradicts that $|D| \geq 2$.

Case 3 $x \in A \cap T$ and $y \in B \cap T$.

Assume that $A \cap C \neq \emptyset$. Then $|X_1| \geq 4$, thus $B \cap D = \emptyset$. If $B \cap C \neq \emptyset$, an argument analogous to that used in case 1 can lead to a contradiction. If $B \cap C = \emptyset$, a similar argument used in case 2 can lead to a contradiction. Hence $A \cap C = \emptyset$. Similarly, $B \cap C = \emptyset$. Hence, $|C| = |A \cap C| + |S \cap C| + |B \cap C| = |S \cap C| \geq 3$. Noticing that $v \in S \cap D$ and $|S| = 4$, we have that $|C| = |S \cap C| = 3$, $|S \cap T| = 0$ and $|S \cap D| = 1$. Since $x \in A \cap T$, $y \in B \cap T$, it follows that $|X_2| \leq 4$, $|X_3| \leq 4$, and then $A \cap D = \emptyset = B \cap D$. Hence, $|D| = |A \cap D| + |S \cap D| + |B \cap D| = 1$, which contradicts that $|D| \geq 2$.

The proof of other cases can reduce to the above case. The proof is now completed. □

From Lemma 5 we can deduce the following result by a similar argument used in Corollary 3.

Corollary 6 *Let G be a 5-connected graph of order at least 10, and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$. If $\delta(G) \geq 6$ or $g(G) \geq 4$, then $E(G[S]) \subseteq E_R(G)$.*

Let us note an immediate consequence of Corollary 3, Corollary 4 and Corollary 6, concerning the distribute of removable edges in a triangle of G.

Corollary 7 *Let G be a 5-connected graph of order at least 10. If $\delta(G) \geq 6$ or $g(G) \geq 4$, then* *every triangle of* G *contains at least one removable edge.*

3. Removable edges in a cycle of a 5-connected graph

Theorem 8 *Let* G *be a 5-connected graph of order at least 10 and* C *a cycle of* G*. If the edge-vertex-atom of* G *contains at least three vertices, then there are at least two removable edges of* G *in* C*.*

Proof By contradiction. Assume that there is at most one removable edge of G in C. Let $F = E(C) \cap E_R(G)$. Then $|F| \leq 1$. Denote $E(C) - F$ by E_0 and let $uw \in E_0$. We take a separating group $(uw, S'; A', B')$ such that $u \in A'$, $w \in B'$. From $|F| \leq 1$, we know that $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [B', S']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, A' must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say A. Then, we have that $(E(A) \cup [A, S]) \cap F = \emptyset$, and take a separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$ with $xy \in E_0$.

Let $xz \in E_0 \cap (E(A) \cup [A, S])$. Then obviously $z \notin S$. Otherwise, from Lemma 2, xz is a removable edge of G, a contradiction. We take a separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1$, $z \in B_1$. Then, we have that $x \in A \cap A_1$, $z \in A \cap B_1$. Let

$$X_1 = (A_1 \cap S) \cup (S \cap S_1) \cup (A \cap S_1), \quad X_2 = (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S),$$

$$X_3 = (B_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1), \quad X_4 = (B \cap S_1) \cup (S \cap S_1) \cup (A_1 \cap S).$$

From Lemma 2, we have that $y \notin B \cap S_1$, and so $y \in A_1 \cap B$. Since $A \cap B_1 \neq \emptyset$, we have that X_2 is a vertex-cut of $G - xz$, so $|X_2| \geq 4$. By an analogous argument, we can deduce that $|X_4| \geq 4$. Since $|X_2| + |X_4| = |S| + |S_1| = 8$, we can get that $|X_2| = |X_4| = 4$, then $|A_1 \cap S| = |A \cap S_1|$, $|B \cap S_1| = |B_1 \cap S|$. We claim that $A \cap B_1 = \{z\}$. Otherwise, $|A \cap B_1| \geq 2$. Then, $(xz, X_2; A \cap B_1, A_1 \cup B)$ is a separating group of G and $xz \in E_0$. It is easy to see that $A \cap B_1$ is an E_0-edge-vertex-cut fragment contained in A, which contradicts that A is an E_0-edge-vertex-cut end-fragment of G. Therefore, $A \cap B_1 = \{z\}$. Since $x \in A \cap A_1$, $z \in A \cap B_1$, we have $X_1 \cup \{y, z\}$ is a vertex-cut of G. Hence $|X_1| \geq 3$. We consider the following cases.

Case 1 $|X_1| \geq 4$. Since $|X_1| + |X_3| = |S| + |S_1| = 8$, $|X_3| \leq 4$. Then $B \cap B_1 = \emptyset$, and so $|B_1| = |A \cap B_1| + |S \cap B_1| + |B \cap B_1| = 1 + |S \cap B_1|$. Since $|B_1| \geq 3$, $|S \cap B_1| \geq 2$.

If $|S \cap B_1| \geq 3$, then $|S \cap A_1| + |S \cap S_1| = |S| - |S \cap B_1| \leq 1$, $|S_1 \cap A| + |S \cap S_1| = |X_2| - |S \cap B_1| \leq 1$. Hence $|X_1| \leq 2$, a contradiction.

If $|S \cap B_1| = 2$, then $|S \cap A_1| + |S \cap S_1| = |S| - |S \cap B_1| = 2$, $|S_1 \cap A| + |S \cap S_1| = |X_2| - |S \cap B_1| = 2$. Noting that $|X_1| \geq 4$, we have that $|S \cap A_1| = 2$, $|S \cap S_1| = 0$, $|A \cap S_1| = 2$ and $|B \cap S_1| = 2$. Let $A \cap S_1 = \{a, b\}$, $S \cap B_1 = \{c, d\}$ and $B \cap S_1 = \{e, f\}$. If $cd \notin E(G)$, it is easy to see that $ca, cf, da, de, df \in E(G)$. If $cd \in E(G)$, we claim that $ad, ac \in E(G)$. If not, there are two cases.

(1) $|N_G(a) \cap \{c, d\}| = 1$. Without loss of generality, we may assume that $ac \in E(G)$, $ad \notin E(G)$. Let $S' = (S_1 \setminus \{a\}) \cup \{z\}$, $A' = B_1 \setminus \{z\}$, $B' = G - ac - A' - S'$. Then $(ac, S'; A', B')$ is a separating group of G and $|A'| = 2$, a contradiction.

(2) $|N_G(a) \cap \{c, d\}| = 0$. Then, $(S_1 \setminus \{a\}) \cup \{z\}$ is a vertex-cut of G with cardinality 4, a contradiction.

Hence, $ac, ad \in E(G)$. Similarly $bc, bd \in E(G)$. By symmetry, we can show that $ec, ed, fc, fd \in E(G)$. From Lemma 3, $za, zb \in E_R(G)$. Since $E(C) \cap \{za, zb, zc, zd\} \neq \emptyset$ and $(E(A) \cup [A, S]) \cap F = \emptyset$, there holds $\{zc, zd\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we may assume that $zc \in E_N(G)$, and take a corresponding separating group $(zc, T; C, D)$ such that $z \in C$, $c \in D$. Let

$$Y_1 = (S_1 \cap C) \cup (S_1 \cap T) \cup (A_1 \cap T), \quad Y_2 = (A_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap D),$$

$$Y_3 = (S_1 \cap D) \cup (S_1 \cap T) \cup (B_1 \cap T), \quad Y_4 = (B_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap C).$$

Obviously, $x \in A_1 \cap C$, $c \in B_1 \cap D$. Since each of a, b is adjacent to both c and z, we have $a, b \in S_1 \cap T$. By an analogous argument used in Lemma 2 we can show that $|Y_1| = |Y_3| = 4$, $|Y_4| \geq 3$, $|Y_2| \leq 5$. Since $|B_1| = 3$, $zd \in E(G)$, we have $|B_1 \cap D| = 1$, $|B_1 \cap T| \leq 1$. Then, we consider the following cases.

Case 1.1 $|Y_4| \geq 4$. Then $|Y_2| \leq 4$, so $A_1 \cap D = \emptyset$. Since $|B_1 \cap D| = 1$, $|D| \geq 3$, it follows $|D \cap S_1| \geq 2$. Noting that $|S_1| = 4$ and $|S_1 \cap T| \geq 2$, so $|D \cap S_1| = 2 = |S_1 \cap T|$, $|S_1 \cap C| = 0$. Then $|Y_4| \leq 3$, a contradiction.

Case 1.2 $|Y_4| = 3$, then $|Y_2| = 5$. We discuss the following cases.

Case 1.2.1 $|B_1 \cap T| = 0$. Then $|B_1 \cap C| = 2$. Thus, $Y_4 \cup \{z\}$ is a vertex-cut of G with cardinality 4, a contradiction.

Case 1.2.2 $|B_1 \cap T| = 1$. Then, $|A_1 \cap T| + |S_1 \cap T| = 3$, $|S_1 \cap D| = |Y_2| - |A_1 \cap T| - |S_1 \cap T| = 2$, and then $|S_1 \cap T| \leq |S_1| - |S_1 \cap D| \leq 2$. Noting that $a, b \in S_1 \cap T$, thus $|S_1 \cap T| = 2$. Then, we have that $|A_1 \cap T| = 1$, $|S_1 \cap D| = 2$, $|S_1 \cap C| = 0$, and so $|Y_1| = 3$, a contradiction.

Case 2 $|X_1| = 3$. Then we claim that $|A_1 \cap A| = 1$. Otherwise, $X_1 \cup \{x\}$ is a vertex-cut of G with cardinality 4, a contradiction. Since $|A| \geq 3$ and $|A \cap B_1| = 1$, we have $|A \cap S_1| \geq 1$. Then $|A_1 \cap S| + |S_1 \cap S| = |X_1| - |A \cap S_1| \leq 2$, thus $|B_1 \cap S| = |S| - |A_1 \cap S| - |S_1 \cap S| \geq 2$. If $|B_1 \cap S| \geq 3$, then $|S \cap A_1| + |S \cap S_1| = |S| - |S \cap B_1| \leq 1$, $|A \cap S_1| = |X_2| - |S \cap S_1| - |S \cap B_1| \leq 1$, and so $|X_1| \leq 2$, a contradiction. If $|B_1 \cap S| = 2$, then $|B \cap S_1| = 2$. Noticing that $|X_3| = 5$, we have $|S \cap S_1| = 1$, $|S \cap A_1| = 1$, $|A \cap S_1| = 1$. Assume that $A \cap S_1 = \{a\}$, $S \cap A_1 = \{b\}$. If $ab \notin E(G)$, then, $(S \setminus \{b\}) \cup \{x\}$ is a vertex-cut of G with cardinality 4, a contradiction. If $ab \in E(G)$, then let $A' = A \setminus \{x\}$, $S' = (S \setminus \{b\}) \cup \{x\}$, $B' = G - ab - S' - A'$. We have that $(ab, S'; A', B')$ is a separating group of G with $|A'| = 2$, a contradiction.

The proof now is completed. □

Corollary 9 *Let G be a 5-connected graph of order at least 10 and C a cycle of G. If $\delta(G) \geq 6$ or $g(G) \geq 4$, then there are at least two removable edges of G in C.*

4. Removable edges in a spanning tree of a 5-connected graph

Theorem 10 *Let G be a 5-connected graph and T a spanning tree of G. If $\delta(G) \geq 6$, then there are at least two removable edges of G in T.*

Proof Clearly, $|G| \geq 7$. If $|G| = 7$, then $G = K_7$. Since every edge of K_7 is removable, the conclusion holds. Now we may assume that $|G| \geq 8$. By contradiction. Assume that there is at most one removable edge of G in T. Let $F = E(T) \cap E_R(G)$. Then $|F| \leq 1$. Denote $E(T) - F$ by E_0, we take a separating group $(uw, S'; A', B')$ such that $u \in A'$, $w \in B'$ and $uw \in E_0$. From $|F| \leq 1$, we know that $(E(A') \cup [A', S']) \cap F = \emptyset$ or $(E(B') \cup [B', S']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup [A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, A' must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say A. Then, we have that $(E(A) \cup [A, S]) \cap F = \emptyset$, and we take a separating group $(xy, S; A, B)$ such that $x \in A$, $y \in B$ with $xy \in E_0$.

Let $uz \in E_0 \cap (E(A) \cup [A, S])$. We take a separating group $(uz, T; C, D)$ such that $u \in C$, $z \in D$. Let

$$X_1 = (C \cap S) \cup (S \cap T) \cup (A \cap T), \quad X_2 = (A \cap T) \cup (S \cap T) \cup (D \cap S),$$

$$X_3 = (D \cap S) \cup (S \cap T) \cup (B \cap T), \quad X_4 = (B \cap T) \cup (S \cap T) \cup (C \cap S).$$

If $u = x$, it follows from Lemma 2 that $z \in A \cap D$, $y \in B \cap C$. By an analogous argument used in Theorem 8, we have that $|X_2| = |X_4| = 4$. We claim that $A \cap D = \{z\}$. Otherwise, $|A \cap D| \geq 2$. Let $A_1 = A \cap D$, $S_1 = X_2$ and $B_1 = G - A_1 - S_1 - xz$. Then we get an edge-vertex-cut fragment A_1 which is a proper subset of A. This is a contradiction. Thus, $A \cap D = \{z\}$, and then $d_G(z) = 5$, a contradiction.

If $u \neq x$, we consider the following cases.

Case 1 $uz \in E(A)$. Then $u \in A \cap C$, $z \in A \cap D$. Since $A \cap D \neq \emptyset$, X_2 is a vertex-cut of $G - uz$, then $|X_2| \geq 4$. If $|X_2| = 4$, by an argument similar to that used above, $A \cap D = \{z\}$, and then $d_G(z) = 5$, a contradiction. So $|X_2| \geq 5$.

Case 1.1 $x \in A \cap C$, $y \in B \cap C$. By a similar argument, we can get that $|X_4| \geq 4$. Noticing that $|X_2| + |X_4| = |S| + |T| = 8$, then $|X_2| \leq 4$, a contradiction.

Case 1.2 $x \in A \cap C$, $y \in B \cap T$. Since $|X_2| \geq 5$, we have $|S \cap D| > |B \cap T|$, $|X_4| \leq 3$. Thus $B \cap C = \emptyset$. Since $X_1 \cup \{y, z\}$ is a vertex-cut of G, there holds $|X_1| \geq 3$. Noticing that $|X_1| + |X_3| = |S| + |T| = 8$, we get $|X_3| \leq 5$.

If $|X_1| \geq 4$, then $|S \cap C| \geq |B \cap T|$, $|X_3| \leq 4$, and then $B \cap D = \emptyset$. Hence $|B| = |B \cap T| \geq 3$. Thus $|S| \geq |S \cap D| + |S \cap C| \geq 2|B \cap T| \geq 6$, which contradicts $|S| = 4$.

If $|X_1| = 3$, we claim that $|A \cap C| = 2$. Otherwise, $|A \cap C| \geq 3$, let $S_1 = X_1 \cup \{u\}$, $A_1 = A \cap C - \{u\}$, $B_1 = B \cup D$. Then we get an edge-vertex-cut fragment A_1 which is a proper subset of A, a contradiction. Hence $|A \cap C| = 2$, and then $d_G(x) = d_G(u) = 5$, a contradiction.

Case 1.3 $x \in A \cap T$, $y \in B \cap T$. By Lemma 5, $xy \in E_R(G)$, a contradiction.

Case 1.4 $x \in A \cap T$, $y \in B \cap C$. Using an argument analogous to the one in case 1.1 can lead to a contradiction.

Other cases such as $x \in A \cap T$, $y \in B \cap D$; $x \in A \cap D$, $y \in B \cap T$; $x \in A \cap D$, $y \in B \cap D$ can reduce to the above cases by symmetry.

Case 2 $uz \in [A, S]$. Then $u \in A \cap C$, $z \in S \cap D$.

Case 2.1 $x \in A \cap C$, $y \in B \cap C$. A contradiction yields by an analogous argument used in case 1.2.

Case 2.2 $x \in A \cap C$, $y \in B \cap T$. Since X_1 is a vertex-cut of $G - xy - uz$, $|X_1| \geq 3$. Suppose $|X_1| = 3$. If $|A \cap C| = 2$, then $d_G(x) = d_G(u) = 5$, a contradiction. Thus, $|A \cap C| \geq 3$. Let $S_1 = X_1 \cup \{u\}$, $A_1 = A \cap C - \{u\}$, $B_1 = B \cup D$. Then we get an edge-vertex-cut fragment A_1 which is a proper subset of A, a contradiction. Therefore $|X_1| \geq 4$. And then $|A \cap T| \geq |S \cap D|$, $|X_3| \leq 4$. Hence $B \cap D = \emptyset$.

Since $|X_2| + |X_4| = 8$, we have $|X_2| \leq 4$ or $|X_4| \leq 4$. Without loss of generality, we may assume that $|X_4| \leq 4$, then $B \cap C = \emptyset$. Thus, $|B| = |B \cap C| + |B \cap T| + |B \cap D| = |B \cap T| \geq 3$. Since $|T| = 4$, we have $|A \cap T| = 1$, $|S \cap T| = 0$, $|B \cap T| = 3$, $|S \cap D| = 1$, $|S \cap C| = 3$. Then $|X_4| = 6$, a contradiction.

Case 2.3 $x \in A \cap T$, $y \in B \cap C$. Then $|X_4| \geq 4$, and then $|S \cap C| \geq |A \cap T| \geq 1$. Noticing that $|X_2| + |X_4| = 8$, we have that $|X_2| \leq 4$, and then $A \cap D = \emptyset$. By a similar argument, $B \cap D = \emptyset$. Then $|D| = |S \cap D| \geq 3$. Since $|S| = 4$, we have $|S \cap D| = 3$, $|S \cap C| = 1$, $|S \cap T| = 0$. Therefore, $|A \cap T| = 1$, $|B \cap T| = 3$. Thus, $|X_1| = 2$, and $X_1 \cup \{z\}$ is a vertex-cut of G, a contradiction.

Case 2.4 $x \in A \cap T$, $y \in B \cap D$. By a similar argument, we have $|X_1| = |X_3| = 4$. If $|A \cap C| \geq 2$, then $(uz, X_1; A \cap C, B \cup D)$ is a separating group of G, and $A \cap C$ is a proper subset of A. This is a contradiction. If $|A \cap C| = 1$, then $d_G(u) = 5$, a contradiction.

Case 2.5 $x \in A \cap D$, $y \in B \cap T$. By a similar argument, we have that $|X_1| \geq 4$ and $|X_2| \geq 4$, then $|X_3| \leq 4$ and $|X_4| \leq 4$, hence $B \cap D = \emptyset$ and $B \cap C = \emptyset$. Therefore $|B| = |B \cap T| \geq 3$. Since $|X_3| \leq 4$, $|S \cap C| \geq |B \cap T| \geq 3$. Hence $|S \cap C| = 3 = |B \cap T|$, $|S \cap T| = 0$, $|A \cap T| = |S \cap D| = 1$. Thus $|X_2| = 2$, and $X_2 \cup \{y\}$ is a vertex-cut of G with cardinality 3, a contradiction.

Case 2.6 $x \in A \cap D$, $y \in B \cap D$. By an analogous argument used in case 2.4, we can get a contradiction.

The proof now is completed. □

By a similar argument we also have the following results.

Theorem 11 *Let G be a 5-connected graph and T a spanning tree of G. If $\delta(G) \geq 6$, then there are at least two removable edges of G in $G - E(T)$.*

Now we have the main theorems of this paper.

Theorem 12 *Let G be a 5-connected graph of order at least 10 and $\delta(G) \geq 6$. Then $E_R(G) \geq$*

$2|G| + 2$.

Proof Let $G' = G[E_N(G)]$. From Theorem 8 and Theorem 10, G' is a forest, then $|E_N(G)| \leq |G| - 2$. Thus, $E_R(G) = |E(G) - E_N(G)| \geq 3|G| - (|G| - 2) = 2|G| + 2$. □

By a similar argument we also have the following results.

Theorem 13 *Let G be a 5-connected graph of order at least 10. If the edge-vertex-atom of G contains at least three vertices, then $E_R(G) \geq (3|G| + 2)/2$.*

Theorem 14 *Let G be a 5-connected graph of order at least 10 and $g(G) \geq 4$. Then $E_R(G) \geq (3|G| + 4)/2$.*

References

[1] BARBETTE D W, GRÜNBAUM B. *On Steinitz's Theorem Concerning Convex 3-Polytopes and on Some Properties of Planar Graphs* [M]. Springer, Berlin, 1966.

[2] BONDY J A, Murty U S R. *Graph Theory with Applications* [M]. American Elsevier Publishing Co., Inc., New York, 1976.

[3] DAWES R W. *Minimally 3-connected graphs* [J]. J. Combin. Theory Ser. B, 1986, **40**(2): 159–168.

[4] DIRAC G A. *Minimally 2-connected graphs* [J]. J. Reine Angew. Math., 1967, **228**: 204–216.

[5] HABIB M, PÉROCHE B. *A construction method for minimally k-edge-connected graphs* [J]. Ann. Discrete Math., 1980, **9**: 199–204.

[6] HOLTON D A, JACKSON B, SAITO A. et al. *Removable edges in 3-connected graphs* [J]. J. Graph Theory, 1990, **14**(4): 465–473.

[7] MADER W. *A reduction method for edge-connectivity in graphs* [J]. Ann. Discrete Math., 1978, **3**: 145–164.

[8] MADER W. *Generalizations of critical connectivity of graphs* [J]. Discrete Math., 1988, **72**(1-3): 267–283.

[9] MARTINOV N. *Uncontractible 4-connected graphs* [J]. J. Graph Theory, 1982, **6**(3): 343–344.

[10] SLATER P J. *A classification of 4-connected graphs* [J]. J. Combinatorial Theory Ser. B, 1974, **17**: 281–298.

[11] SU Jianji. *The number of removable edges in 3-connected graphs* [J]. J. Combin. Theory Ser. B, 1999, **75**(1): 74–87.

[12] SU Jianji, GUO Xiaofeng, XU Liqiong. *Removable edges in a k-connected graph and a construction method for k-connected graphs* [J]. Discrete Math., 2009, **309**(10): 3161–3165.

[13] TUTTE W T. *A theory of 3-connected graphs* [J]. Nederl. Akad. Wetensch. Proc. Ser. A, 1961, **64**: 441–455.

[14] WU Jichang, LI Xueliang, SU Jianji. *The number of removable edges in a 4-connected graph* [J]. J. Combin. Theory Ser. B, 2004, **92**(1): 13–40.

[15] WU Jichang, LI Xueliang, WANG Lusheng. *Removable edges in a cycle of a 4-connected graph* [J]. Discrete Math., 2004, **287**(1-3): 103–111.

[16] XU Liqiong, GUO Xiaofeng. *Removable edges in a 5-connected graph and a construction method of 5-connected graphs* [J]. Discrete Math., 2008, **308**(9): 1726–1731.

[17] YIN Jianhua. *Removable edges and constructions of 4-connected graphs* [J]. J. Sys. Sci. Math. Scis., 1999, **19**(4): 434–438.

[18] ZHANG Fuji, GUO Xiaofeng, CHEN Rongsi. *A Construction Method for Critically k-Edge-Connected Graphs* [M]. New York Acad. Sci., New York, 1989.

[19] ZHANG Fuji, GUO Xiaofeng. *Reducible chains in several types of 2-connected graphs* [J]. Discrete Math., 1992, **105**(1-3): 285–291.