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Existence of Solutions to a p-Laplacian Equation with Integral Initial Condition

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Abstract In this paper, a class of one-dimension p-Laplacian equation with nonlocal initial value is studied. The existence of solutions for such a problem is obtained by using the topological degree method.

Keywords integral initial condition; existence; topological degree.

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1. Introduction

The problems of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics can be reduced to nonlocal problems with integral boundary conditions. For boundary-value problems with integral boundary conditions and comments on their importance, we refer the readers to [1–4] and the references therein. For more information about the general theory of integral equations and their relation with boundary-value problems, readers may refer to [5–7].

In recent years, the existence and multiplicity of positive solutions for nonlocal problems have attracted great attention to many mathematicians. Readers may refer to [8–16] and references therein. On the other hand, initial-value problems with integral conditions constitute a very interesting and important class of problems. However, to the best of our knowledge, the integral initial value problems to one-dimension p-Laplacian equation have not been studied.

The purpose of this paper is to investigate the existence of solutions to the following one-dimension p-Laplacian equation:

$$(\phi(u'))' = -f(t,u), \quad t \in (0,1), \tag{1.1}$$

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with integral initial value

$$u(0) = \int_0^1 g(s)u(s)\mathrm{d}s, \tag{1.2}$$

$$u'(0) = A, \tag{1.3}$$

where $\phi(s) = \phi_p(s) = |s|^{p-2}s, \phi^{-1}(s) = \phi_q(s) = |s|^{q-2}s, p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $g(s) \in L^1([0,1])$ and A is a real constant. The main arguments are based upon Leray-Schauder topological degree.

The paper is organized as follows. We shall introduce some necessary lemmas in the rest of this section. In Section 2, we provide some necessary preliminaries and in Section 3, the main results will be stated and proved.

For application in what follows, we state some properties of completely continuous operators.

Lemma 1.1 *Suppose that X is a Banach space and A is a completely continuous operator from X to X .Then for any $\lambda \neq 0$, only one of the following statements holds:*

(i) For any $y \in X$, there exists a unique $x \in X$, such that $(A - \lambda I)x = y$;

(ii) There exists an $x \in X$, $x \neq 0$, such that $(A - \lambda I)x = 0$.

Lemma 1.2 *Assume that $H : [0,1] \times \overline{\Omega} \to E$ is completely continuous. Let $h_t(x) = x - H(t,x)$. If for all $t \in [0,1]$, $p \notin h_t(\partial\Omega)$, then $\deg(h_t, \Omega, p)$ is a constant, $\forall 0 \leq t \leq 1$.*

Lemma 1.3 *If $\deg(f, \Omega, p) \neq 0$, then the equation $f(x) = p$ must admit at least one solution in Ω.*

2. Preliminaries

Let I denote the real interval $[0,1]$, and $C(I)$ denote the Banach space of all continuous $u : I \to \mathbb{R}$, equipped with the norm $\|u\| = \max\{|u(t)|; t \in I\}$, for any $u \in C(I)$.

Consider the following problem:

$$(\phi(x'))' = -y(t), \quad t \in (0,1), \tag{2.1}$$

$$x(0) = \int_0^1 g(s)x(s)\mathrm{d}s, \tag{2.2}$$

$$x'(0) = A, \tag{2.3}$$

where $y(t) \in C(I)$, $\int_0^1 g(s)\mathrm{d}s \neq 1$.

Integrating the Eq.(2.1) from 0 to t to obtain

$$\phi(x'(t)) - \phi(x'(0)) = -\int_0^t (y(s))\mathrm{d}s,$$

using the initial condition (2.3), we have

$$x'(t) = \phi^{-1}(\phi(A) - \int_0^t y(s)\mathrm{d}s).$$

Integrating the above equality from 0 to t again, we obtain

$$x(t) - \int_0^1 g(s)x(s)\mathrm{d}s = \int_0^t \phi^{-1}(\phi(A) - \int_0^\tau y(s)\mathrm{d}s)\mathrm{d}\tau. \tag{2.4}$$

Let $F(t) := \int_0^t \phi^{-1}(\phi(A) - \int_0^\tau y(s)\mathrm{d}s)\mathrm{d}\tau$. Define an operator $K : C(I) \to C(I)$ by

$$(Kx)(t) = \int_0^1 g(s)x(s)\mathrm{d}s,$$

then (2.4) can be rewritten as

$$(I - K)x(t) = F(t). \tag{2.5}$$

Thus $x(t)$ is a solution to (2.1)–(2.3) if and only if it is a solution to (2.5).

Lemma 2.1. *$I - K$ is a Fredholm operator.*

Proof To prove that $I - K$ is a Fredholm operator, we need only to show that K is completely continuous.

It is easy to see from the definition of K that K is a bounded linear operator from $C(I)$ to $C(I)$. Obviously, $\dim R(K) = 1$. So K is a completely continuous operator.

This completes the proof. □

Lemma 2.2 *Problems (2.1)–(2.3) admits a unique solution.*

Proof Since Problems (2.1)–(2.3) is equivalent to problem (2.5), we need only to show that problem (2.5) has a unique solution.

Using Lemma 2.1 and Alternative Theorem, it is sufficient for us to prove that

$$(I - K)x(t) = 0 \tag{2.6}$$

has a trivial solution $x \equiv 0$ only.

On the contrary, suppose (2.6) has a nontrivial solution μ, then μ is a constant, and we have

$$I\mu = K\mu = \mu.$$

The definition of K and the above equality yield

$$[1 - \int_0^1 g(s)\mathrm{d}s]\mu = 0,$$

which is a contradiction with the assumptions $\int_0^1 g(s)\mathrm{d}s \neq 1$ and $\mu \not\equiv 0$.

Thus we complete the proof. □

3. Main results

Consider the following problem

$$(\phi(u'))' = -f(t, u), \quad t \in (0, 1), \tag{3.1}$$

$$u(0) = \int_0^1 g(s)u(s)\mathrm{d}s, \tag{3.2}$$

$$u'(0) = A, \tag{3.3}$$

In the following, we will assume that the following conditions hold.

(H_1) $\int_0^1 |g(s)|\mathrm{d}s = M < 1$;

(H_2) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H_3) $|f(t,x)| \le c_1\phi(|x|) + c_2,\ c_1, c_2 > 0$ and $c_1 < \phi(\frac{1-M}{2^{q-1}})$.

From Lemma 2.2 we know that $u(t)$ is a solution to Problems (3.1)–(3.3) if and only if it is a solution to the following integral equation

$$(I-K)u(t) = \int_0^t \phi^{-1}(\phi(A) - \int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau. \tag{3.4}$$

Define an operator $T:\ C(I) \to C(I)$ by

$$(Tu)(t) = \int_0^t \phi^{-1}(\phi(A) - \int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau,$$

then (3.4) can be rewritten as

$$(I-K)u(t) = (Tu)(t).$$

In order to prove the existence of solutions to (3.4), we need the following lemmas.

Lemma 3.1 *T is completely continuous.*

Proof For any ball $B_1 = \{u \in C(I); \|u\| \le R_1\}$, set $M_1 = \max_{\substack{0\le s\le 1\\ u\in B_1}} |f(s,u(s))|$, then we have for any $u \in B_1$,

$$\begin{aligned}|(Tu)(t)| \le& \int_0^t |\phi^{-1}(\phi(A) - \int_0^\tau f(s,u(s))\mathrm{d}s)|\mathrm{d}\tau \le \int_0^1 \phi^{-1}(|\phi(A) - \int_0^\tau f(s,u(s))\mathrm{d}s|)\mathrm{d}\tau\\ \le& \int_0^1 \phi^{-1}(|\phi(A)| + |\int_0^\tau f(s,u(s))\mathrm{d}s|)\mathrm{d}\tau \le \int_0^1 \phi^{-1}(|\phi(A)| + \int_0^1 |f(s,u(s))|\mathrm{d}s)\mathrm{d}\tau\\ \le& \phi^{-1}(|\phi(A)| + M_1).\end{aligned}$$

This shows that $T(B_1)$ is uniformly bounded.

Since $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, it is uniformly continuous on $[0,1] \times [-R_1, R_1]$. Thus for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t_1,\ t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}|(Tu)(t_1) - (Tu)(t_2)| =& |\int_{t_1}^{t_2} \phi^{-1}(\phi(A) - \int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau|\\ \le& \int_{t_1}^{t_2} \phi^{-1}(|\phi(A)| + \int_0^\tau |f(s,u(s))|\mathrm{d}s)\mathrm{d}\tau\\ \le& M_2|t_1 - t_2|,\end{aligned}$$

where $M_2 = \phi^{-1}(|\phi(A)| + M_1)$. This implies that $T(B_1)$ is equicontinuous on $[0,1]$.

There fore, $T : C(I) \to C(I)$ is completely continuous. The Proof of Lemma 3.1 is completed. □

Theorem *Assume conditions (H_1)–(H_3) hold, then (3.1)–(3.2) admits at least one solution.*

Proof Lemmas 2.1 and 3.1 imply that the operator $K + T$ is completely continuous. It suffices for us to prove the following equation

$$(I-(K+T))u=0 \tag{3.5}$$

has at least one solution.

Define $H:[0,1]\times C[0,1]\to C[0,1]$ as

$$H(\sigma,u)=\sigma(K+T)u,$$

and it is clear that H is completely continuous.

Set $h_\sigma(u)=u-H(\sigma,u)$, then we have

$$h_0(u)=u,\quad h_1(u)=[I-(K+T)]u.$$

To apply the Leray-Schauder degree to h_σ, we need only to show that there exists a ball $B_R(\theta)$ in $C[0,1]$ whose radius R will be fixed later, such that $\theta\notin h_\sigma(\partial B_R(\theta))$.

Choosing $R>\frac{2^{q-1}\phi^{-1}(|\phi(A)|+c_2)}{1-M-2^{q-1}\phi^{-1}(c_1)}$, then for any fixed $u\in\partial B_R(\theta)$, there exists a $t_0\in[0,1]$ such that $|u(t_0)|=R$. By directly calculating, we have

$$\begin{aligned}|(h_\sigma u)(t_0)|&=|u(t_0)-\sigma[\int_0^1 g(s)u(s)\mathrm{d}s+\int_0^{t_0}\phi^{-1}(\phi(A)-\int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau]|\\&\geq|u(t_0)|-|\sigma[\int_0^1 g(s)u(s)\mathrm{d}s+\int_0^{t_0}\phi^{-1}(\phi(A)-\int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau]|\\&\geq R-|\int_0^1 g(s)u(s)\mathrm{d}s|-|\int_0^{t_0}\phi^{-1}(\phi(A)-\int_0^\tau f(s,u(s))\mathrm{d}s)\mathrm{d}\tau|\\&\geq(1-M)R-\int_0^1\phi^{-1}(|\phi(A)|+\int_0^1|f(s,u(s))|\mathrm{d}s)\mathrm{d}\tau.\end{aligned} \tag{3.6}$$

From (H_3), we have

$$\begin{aligned}|(h_\sigma u)(t_0)|&\geq(1-M)R-\int_0^1\phi^{-1}\Big(|\phi(A)|+\int_0^1\big(c_1\phi(|u|)+c_2\big)\mathrm{d}s\Big)\mathrm{d}\tau\\&\geq(1-M)R-\int_0^1\phi^{-1}\Big((|\phi(A)|+c_2)+c_1\phi(\|u\|)\Big)\mathrm{d}\tau\\&>0.\end{aligned}$$

This implies $h_\sigma u\neq\theta$, and hence we obtain $\theta\notin h_\sigma(\partial B_R(\theta))$.

Lemma 1.2 shows that

$$\deg(h_1,B_R(\theta),\theta)=\deg(h_0,B_R(\theta),\theta)=1\neq0.$$

Using Lemma 1.3 we know that (3.5) admits a solution $u\in B_R(\theta)$, which implies that (3.1)–(3.3) also admits a solution in $B_R(\theta)$. □

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