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Action of $\mathcal{U}_q(g)$ on Its Positive Part $\mathcal{U}_q^+(g)$

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Abstract In this paper, two kinds of skew derivations of a type of Nichols algebras are introduced, and then the relationship between them is investigated. In particular they satisfy the quantum Serre relations. Therefore, the algebra generated by these derivations and corresponding automorphisms is a homomorphic image of the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$, which proves the Nichols algebra becomes a $\mathcal{U}_q(g)$-module algebra. But the Nichols algebra considered here is exactly $\mathcal{U}_q^+(g)$, namely, the positive part of the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$, it turns out that $\mathcal{U}_q^+(g)$ is a $\mathcal{U}_q(g)$-module algebra.

Keywords Nichols algebra; Yetter-Drinfeld module; skew derivation; quantum group.

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1. Introduction

In [1], skew derivations of so called twisted Hopf algebras were introduced, and the corresponding skew differential operator algebras were studied, where the concept of a (K, c, I, χ)-twisted Hopf algebra with generators was improved to include some important examples such as the free algebras, the polynomial algebras, Lusztig's algebra in [2], Ringel composition algebra $\mathscr{C}(\Lambda)$ and Ringel-Hall algebra $\mathscr{H}(\Lambda)$ in [3] and [4], Rosso's quantum shuffle algebra $T(V)$ in [5]. In particular, the author focused on the algebra $\mathscr{W}(\mathscr{C}(\Lambda), I^{im})$, which is generated by the left multiplication operators $(\theta_i)_l$ for $i \in I^{re}$, and the left skew derivations ${}_i\delta$ for $i \in I^{im}$, where $\theta_i,\ i \in I$, is a minimal system of generators of $\mathscr{H}(\Lambda)$. It turns out that these skew derivations satisfy the quantum Serre relations, and hence the algebra generated by these derivations is a homomorphic image of $\mathscr{U}^+$ associated to Λ.

In recent years, Nichols algebras are becoming very interesting objects to be studied. When the braiding is just a trivial, or more generally a symmetric braiding, then the Nichols algebra is nothing but a symmetric algebra, but when the braiding is not a symmetry, a Nichols algebra could have a much richer structure. In general, the first part of classification problem of pointed

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Hopf algebras by the lifting method is to determine the structure of the corresponding Nichols algebras [6]. From this point of view, Nichols algebras are the key to the structure of pointed Hopf algebras [7, 8]. One of the important techniques to study the structure of Nichols algebras is skew derivations, their extension could track back to [9].

In this paper, we pay attention to a Nichols algebra $\mathfrak{B}(V)$, which is a particular example of a twisted Hopf algebra considered in [1]. For $1 \le i \le n$, let σ_i be an automorphism of the Nichols algebra, and the (id, σ_i)-derivation D_i discussed in [6, 10] is also the particular case of the derivation δ_i appearing in [1].

In Section 2, we define $(\sigma_i^{-1}, \mathrm{id})$-derivation X_i, and for $1 \le i \le n$ we prove that D_i, X_i, σ_i satisfy the following relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i,\ \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1, \tag{1.1}$$

$$\sigma_i D_j \sigma_i^{-1} = q_i^{-a_{ij}} D_j, \quad \sigma_i X_j \sigma_i^{-1} = q_i^{a_{ij}} X_j, \tag{1.2}$$

$$X_i D_j - D_j X_i = \delta_{ij}\frac{\sigma_i - \sigma_i^{-1}}{q_i - q_i^{-1}}. \tag{1.3}$$

If we denote

$$k_i := \sigma_i^{-1},\ e_i := -\sigma_i^{-1} D_i,\ f_i := -X_i\sigma_i,\ 1 \le i \le n,$$

then the relations (1.1), (1.2), (1.3) can be written as

$$k_i k_j = k_j k_i,\ k_i k_i^{-1} = k_i^{-1} k_i = 1, \tag{1.4}$$

$$k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j,\ k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \tag{1.5}$$

$$e_i f_j - f_j e_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}. \tag{1.6}$$

Denote by U the subalgebra of $\mathrm{End}_{\Bbbk}\mathfrak{B}(V)$ generated by these generators k_i, k_i^{-1}, e_i, f_i, $1 \le i \le n$ with relations (1.4)–(1.6), we prove that the Nichols algebra $\mathfrak{B}(V)$ becomes a left U-module algebra.

In Section 3, by a straightforward computation, we also prove that the derivations e_i, f_i satisfy the quantum Serre relations, that is

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} e_i^{1-a_{ij}-s} e_j e_i^s = 0,\ i \ne j,$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} f_i^{1-a_{ij}-s} f_j f_i^s = 0,\ i \ne j.$$

Therefore the skew differential operator algebra U with the quantum Serre relations is a homomorphic image of the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$, which endows the Nichols algebra a left $\mathcal{U}_q(g)$-module structure. Furthermore, the Nichols algebra we considered here is exactly $\mathcal{U}_q^+(g)$, that is, the positive part of the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$. Therefore, it follows that $\mathcal{U}_q^+(g)$ is a $\mathcal{U}_q(g)$-module algebra.

Throughout this paper, the ground field $\Bbbk$ is $\mathbb{C}$, the field of complex numbers. We refer to [11–13] for the notation and basic properties of Hopf algebras and quantum groups.

2. Skew derivations of a type of Nichols algebras

Let $n \in \mathbb{Z}$, $d \in \mathbb{N}$, $q \in \mathbb{C}$ and not algebraic over $\mathbb{Q}$. As usual, we define

$$[n]_d = \frac{q^{dn} - q^{-dn}}{q^d - q^{-d}},\ [n]_d! = [n]_d[n-1]_d \cdots [1]_d,$$

and the Gauss binomial coefficients

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[n]_d[n-1]_d \cdots [n-j+1]_d}{[j]_d!},\ 1 \le j \le n,$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix}_d = 1,\ \begin{bmatrix} n \\ j \end{bmatrix}_d = 0$ if $j > n$ (see [14]). In particular, we have the following two useful identities [13],

$$\sum_{s=0}^{r} (-1)^s q^{\pm \mathrm{d}s(r-1)} \begin{bmatrix} r \\ s \end{bmatrix}_d = 0,\ r \ge 1, \tag{2.1}$$

$$\begin{bmatrix} r \\ j \end{bmatrix}_d \begin{bmatrix} r-j \\ m \end{bmatrix}_d \begin{bmatrix} j \\ k \end{bmatrix}_d = \begin{bmatrix} r-m-k \\ j-k \end{bmatrix}_d \begin{bmatrix} r-m \\ k \end{bmatrix}_d \begin{bmatrix} r \\ m \end{bmatrix}_d. \tag{2.2}$$

Denote by $A = (a_{ij})$ a Cartan matrix of a simple finite-dimensional Lie algebra, namely, (a_{ij}) is an $n \times n$ indecomposable matrix with integer entries such that $a_{ii} = 2$ and $a_{ij} \le 0$, for $i \ne j$, and $(d_1, d_2, \ldots, d_n)$ is a vector with relatively prime entries d_i such that the matrix $(d_i a_{ij})$ is symmetric and positive definite. Denote $q_i := q^{d_i}$ and $q_{ij} := q_i^{a_{ij}} = q^{d_i a_{ij}}$.

Let Γ be a group. We will write ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ for the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$, and say that $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a Yetter-Drinfeld module over Γ. If $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, then the $\Bbbk\Gamma$-module V is just a Γ-graded vector space $V = \oplus_{g\in\Gamma} V_g$, where $V_g = \{v \in V | \delta(v) = g \otimes v\}$. We define a linear isomorphism $c : V \otimes V \to V \otimes V$ by $c(x \otimes y) = g.y \otimes x$, for all $x \in V_g,\ g \in \Gamma,\ y \in V$. Then (V, c) is a braided vector space, that is, c is a solution of the braided equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

In the following, let Γ be an abelian group. Consider the braided vector space (V, c), where V is a Yetter-Drinfeld module over $\Bbbk\Gamma$ with a basis $x_1, x_2, \ldots, x_n$ and the braiding c is given by

$$c(x_i \otimes x_j) = g_i.x_j \otimes x_i := q^{d_i a_{ij}} x_j \otimes x_i. \tag{2.3}$$

Then the Nichols algebra $\mathfrak{B}(V)$ associated to the braided vector space (V, c) is

$$\mathfrak{B}(V) = \Bbbk\langle x_1, x_2, \ldots, x_n | (ad_c x_i)^{1-a_{ij}} x_j = 0, \quad i \ne j\rangle,$$

see e.g., [6], where $(ad_c x_i)x_j$ is the braided adjoint representation of x_i, namely,

$$(ad_c x_i)x_j = \mu(\mathrm{id} - c)(x_i \otimes x_j),$$

where μ is the multiplication and c is the braiding.

Let σ_i be an automorphism of $\mathfrak{B}(V)$ given by the action of g_i. That is, $\sigma_i(x_j) = g_i.x_j = q^{d_i a_{ij}} x_j$. For $a \in \mathfrak{B}(V)$, by induction on r, we get that

$$(ad_c x_i)^r a = \sum_{s=0}^{r} (-1)^s q_i^{s(r-1)} \begin{bmatrix} r \\ s \end{bmatrix}_{d_i} x_i^{r-s} \sigma_i^s(a) x_i^s.$$

Therefore, $(ad_c x_i)^{1-a_{ij}} x_j = 0$ implies that

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} x_i^{1-a_{ij}-s} x_j x_i^s = 0. \tag{2.4}$$

It follows that $\mathfrak{B}(V)$ is $\mathcal{U}_q^+(g)$, the positive part of the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$ (see [6]).

Let X_i be a linear map from $\mathfrak{B}(V)$ to itself defined by

$$X_i(a) = \frac{\sigma_i^{-1}(a) x_i - x_i a}{q_i - q_i^{-1}},$$

for all $a \in \mathfrak{B}(V)$. Recall that if τ, σ are two automorphisms of an algebra R and the (τ, σ)-derivation of R is a linear map from R to itself such that

$$D(ab) = \tau(a) D(b) + D(a) \sigma(b),$$

for all $a, b \in R$.

Proposition 2.1 *For all $1 \le i \le n$, the map X_i is a $(\sigma_i^{-1}, \mathrm{id})$-derivation of $\mathfrak{B}(V)$.*

Proof Note that $X_i(1) = 0$, and for any $a, b \in \mathfrak{B}(V)$,

$$\begin{aligned} X_i(ab) &= \frac{\sigma_i^{-1}(ab) x_i - x_i ab}{q_i - q_i^{-1}} = \frac{(\sigma_i^{-1}(a) x_i - x_i a) b}{q_i - q_i^{-1}} + \frac{\sigma_i^{-1}(a)(\sigma_i^{-1}(b) x_i - x_i b)}{q_i - q_i^{-1}} \\ &= \sigma_i^{-1}(a) X_i(b) + X_i(a) b, \end{aligned}$$

which completes the proof. □

Proposition 2.2 *For all $1 \le i \le n$, there exists a uniquely determined (id, σ_i)-derivation $D_i : \mathfrak{B}(V) \to \mathfrak{B}(V)$ with $D_i(x_j) = \delta_{ij}$ (Kronecker δ) for all $1 \le j \le n$.*

In fact, the Proposition 2.2 above has been stated, for example in [6, 10], and the derivations of more general algebras have been considered in [1] with different approaches.

Proposition 2.3 *For all $1 \le i, j \le n$, we have*

$$\sigma_i D_j \sigma_i^{-1} = q_i^{-a_{ij}} D_j, \quad \sigma_i X_j \sigma_i^{-1} = q_i^{a_{ij}} X_j. \tag{2.5}$$

Proof To prove $\sigma_i D_j \sigma_i^{-1} = q_i^{-a_{ij}} D_j$, note that $\mathfrak{B}(V)$ is generated as an algebra by $x_1, x_2, \ldots, x_n$, therefore, it is enough to check that it holds for all monomials $x_{j_1} x_{j_2} \cdots x_{j_m}$ in $\mathfrak{B}(V)$. By induction on the length m, if $m = 1$, it is easy to check that $D_j \sigma_i(x_{j_1}) = q_{ij} \sigma_i D_j(x_{j_1})$, since the two sides are both equivalent to q_{ij} if $j_1 = j$, but 0 otherwise. Assume it holds for all monomials with the length at most m. For the case $m+1$, denote $x_{j_1} x_{j_2} \cdots x_{j_{m+1}} = a x_{j_{m+1}}$ with the element a

the length of m. On the one hand,

$$\begin{aligned}D_j\sigma_i(ax_{j_{m+1}}) &= D_j(q_{ij_{m+1}}\sigma_i(a)x_{j_{m+1}})\\ &= q_{ij_{m+1}}\delta_{jj_{m+1}}\sigma_i(a) + q_{ij_{m+1}}q_{jj_{m+1}}D_j\sigma_i(a)x_{j_{m+1}}.\end{aligned}$$

On the other hand, by assumption on m,

$$\begin{aligned}q_{ij}\sigma_iD_j(ax_{j_{m+1}}) &= q_{ij}\sigma_i(\delta_{jj_{m+1}}a + D_j(a)q_{jj_{m+1}}x_{j_{m+1}})\\ &= q_{ij}\delta_{jj_{m+1}}\sigma_i(a) + q_{ij_{m+1}}q_{jj_{m+1}}q_{ij}\sigma_iD_j(a)x_{j_{m+1}}\\ &= q_{ij}\delta_{jj_{m+1}}\sigma_i(a) + q_{ij_{m+1}}q_{jj_{m+1}}D_j\sigma_i(a)x_{j_{m+1}}.\end{aligned}$$

Therefore, $D_j\sigma_i(ax_{j_{m+1}}) = q_{ij}\sigma_iD_j(ax_{j_{m+1}})$. Thus by induction on m, we conclude that $D_j\sigma_i = q_{ij}\sigma_iD_j$ holds for all $1 \le i,\, j \le n$.

Next we prove the equality $\sigma_iX_j\sigma_i^{-1} = q_i^{a_{ij}}X_j$, for any $a \in B(V)$. In fact,

$$\begin{aligned}\sigma_iX_j\sigma_i^{-1}(a) &= \frac{\sigma_i((\sigma_j^{-1}\sigma_i^{-1})(a)x_j - x_j\sigma_i^{-1}(a))}{q_j - q_j^{-1}} = \frac{\sigma_j^{-1}(a)\sigma_i(x_j) - \sigma_i(x_j)a}{q_j - q_j^{-1}}\\ &= \frac{q_{ij}(\sigma_j^{-1}(a)x_j - x_ja)}{q_j - q_j^{-1}} = q_{ij}X_j(a).\end{aligned}$$

This completes the proof. □

Proposition 2.4 *For all $1 \le i,\, j \le n$, we have*

$$X_iD_j - D_jX_i = \delta_{ij}\frac{\sigma_i - \sigma_i^{-1}}{q_i - q_i^{-1}}. \tag{2.6}$$

Proof For any $a \in \mathfrak{B}(V)$ and by (2.5)

$$\begin{aligned}(X_iD_j - D_jX_i)(a) &= X_i(D_j(a)) - D_j(X_i(a))\\ &= \frac{1}{q_i - q_i^{-1}}(\sigma_i^{-1}(D_j(a))x_i - x_iD_j(a) - D_j(\sigma_i^{-1}(a)x_i - x_ia))\\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}}(\sigma_i(a) - \sigma_i^{-1}(a)) = \delta_{ij}\frac{\sigma_i - \sigma_i^{-1}}{q_i - q_i^{-1}}(a).\end{aligned}$$

The proof is completed. □

Denote

$$k_i := \sigma_i^{-1},\ e_i := -\sigma_i^{-1}D_i,\ f_i := -X_i\sigma_i,\ 1 \le i \le n.$$

It is easy to check that e_i is a (k_i, id)-derivation and f_i is an (id, k_i^{-1})-derivation. It follows from (2.5) and (2.6) that

$$k_ik_j = k_jk_i,\ k_ik_i^{-1} = k_i^{-1}k_i = 1, \tag{2.7}$$

$$k_ie_jk_i^{-1} = q_i^{a_{ij}}e_j,\ k_if_jk_i^{-1} = q_i^{-a_{ij}}f_j, \tag{2.8}$$

$$e_if_j - f_je_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}. \tag{2.9}$$

Denote by U the subalgebra of $\mathrm{End}_{\Bbbk}\mathfrak{B}(V)$ generated by all elements k_i, k_i^{-1}, e_i, f_i, $1 \le i \le n$. It is clear that the Nichols algebra $\mathfrak{B}(V)$ is a left U-module.

3. Module algebras

In this section, we denote by $\widetilde{U}$ the algebra generated by elements K_i, K_i^{-1}, E_i, F_i, $1 \leq i \leq n$ with the relations

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1, \tag{3.1}$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,\ K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j, \tag{3.2}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}. \tag{3.3}$$

It is known that $\widetilde{U}$ is a Hopf algebra, with comultiplication $\triangle$, antipode S and counit ε given by

$$\triangle(K_i) = K_i \otimes K_i,\ \triangle(E_i) = E_i \otimes 1 + K_i \otimes E_i,\ \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$S(K_i) = K_i^{-1},\ S(E_i) = -K_i^{-1}E_i,\ S(F_i) = -F_iK_i,$$

$$\varepsilon(K_i) = 1,\ \varepsilon(E_i) = 0,\ \varepsilon(F_i) = 0.$$

Let

$$u_{ij}^+ := \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s}E_jE_i^s, \quad i \neq j,$$

$$u_{ij}^- := \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF_i^s, \quad i \neq j.$$

It is well-known that

$$\triangle(u_{ij}^+) = u_{ij}^+ \otimes 1 + K_i^{1-a_{ij}} \otimes u_{ij}^+, \tag{3.4}$$

$$\triangle(u_{ij}^-) = u_{ij}^- \otimes K_i^{a_{ij}-1}K_j^{-1} + 1 \otimes u_{ij}^-, \tag{3.5}$$

see e.g., [13]. Therefore the ideal I generated by u_{ij}^+, u_{ij}^- is a Hopf ideal, and the Hopf quotient algebra $\widetilde{U}/I$ is exactly the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U}_q(g)$.

Considering the relations (2.7), (2.8) and (2.9) above, we have an epimorphism from $\widetilde{U}$ to U given by: $K_i \mapsto k_i$, $E_i \mapsto e_i$, $F_i \mapsto f_i$. Therefore $\mathfrak{B}(V)$ is also a left $\widetilde{U}$-module, with the module structure induced by $K_i.a = k_i(a)$, $E_i.a = e_i(a)$, $F_i.a = f_i(a)$. In particular, we have the following result

Proposition 3.1 *The Nichols algebra $\mathcal{B}(V)$ is a left $\widetilde{U}$-module algebra.*

Proof The conclusion can be verified directly by using the definition of module algebra, since e_i is a (k_i, id)-derivation and f_i is an (id, k_i^{-1})-derivation. □

We hope that the Nichols algebra $\mathcal{B}(V)$ is also a left $\mathcal{U}_q(g)$-module algebra. To prove this, it suffices to prove that $u_{ij}^+.a = 0$ and $u_{ij}^-.a = 0$, for all $a \in \mathcal{B}(V)$.

As the classical case, we denote

$$v_{ij}^+ := \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} e_i^{1-a_{ij}-s}e_je_i^s, \quad i \neq j,$$

$$v_{ij}^- := \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} f_i^{1-a_{ij}-s} f_j f_i^s, \quad i \neq j.$$

Lemma 3.2 *We have* $v_{ij}^+(x_h) = 0,\ v_{ij}^-(x_h) = 0$, *for* $1 \le h \le n$.

Proof It is obvious that $v_{ij}^+(x_h) = 0$, for $1 \le h \le n$.

Now we prove that $v_{ij}^-(x_h) = 0$, for $1 \le h \le n$. Firstly, by induction on r, we conclude that for any $a \in \mathcal{B}(V)$,

$$f_i^r(a) = \frac{1}{(q_i - q_i^{-1})^r} \sum_{s=0}^{r} (-1)^{r-s} q_i^{s(r-1)} \begin{bmatrix} r \\ s \end{bmatrix}_{d_i} x_i^s \sigma_i^s(a) x_i^{r-s}.$$

In particular,

$$f_i^{1-a_{ij}}(x_j) = \frac{1}{(q_i - q_i^{-1})^{1-a_{ij}}} (ad_c x_i)^{1-a_{ij}} x_j = 0, \quad i \neq j.$$

Therefore,

$$\begin{aligned} f_i^{1-a_{ij}-s} f_j f_i^s(x_h) =& \alpha_{ij} \sum_{t=0}^{1-a_{ij}-s} \sum_{k=0}^{s} (-1)^{1-a_{ij}-t-k} q_i^{(t+k)(s+a_{ih})-k} \begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} \cdot \\ & (q_i^{sa_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^{s-k} x_j x_i^{1-a_{ij}-s-t}), \end{aligned}$$

where $\alpha_{ij} = \frac{1}{(q_i - q_i^{-1})^{1-a_{ij}} (q_j - q_j^{-1})}$.

Note that

$$\begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} = 0$$

if $t > 1 - a_{ij} - s$ or $k > s$, so we can express $f_i^{1-a_{ij}-s} f_j f_i^s(x_h)$ as the form

$$\begin{aligned} f_i^{1-a_{ij}-s} f_j f_i^s(x_h) =& \alpha_{ij} \sum_{t+k=0}^{1-a_{ij}} (-1)^{1-a_{ij}-t-k} q_i^{(t+k)(s+a_{ih})-k} \begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} \cdot \\ & (q_i^{sa_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^{s-k} x_j x_i^{1-a_{ij}-s-t}). \end{aligned}$$

We have

$$\begin{aligned} v_{ij}^-(x_h) =& \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} f_i^{1-a_{ij}-s} f_j f_i^s(x_h) \\ =& \alpha_{ij} \sum_{t+k=0}^{1-a_{ij}} \sum_{s=0}^{1-a_{ij}} (-1)^{s+1-a_{ij}-t-k} q_i^{(t+k)(s+a_{ih})-k} \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} \cdot \\ & (q_i^{sa_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^{s-k} x_j x_i^{1-a_{ij}-s-t}). \end{aligned}$$

It suffices to sum over all s with $k \le s \le 1 - a_{ij} - t$, otherwise,

$$\begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} = 0.$$

So we get

$$v_{ij}^-(x_h)$$

$$
\begin{aligned}
&= \alpha_{ij} \sum_{t+k=0}^{1-a_{ij}} \sum_{s=k}^{1-a_{ij}-t} (-1)^{s+1-a_{ij}-t-k} q_i^{(t+k)(s+a_{ih})-k} \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-s \\ t \end{bmatrix}_{d_i} \cdot \\
&\quad \begin{bmatrix} s \\ k \end{bmatrix}_{d_i} (q_i^{sa_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^{s-k} x_j x_i^{1-a_{ij}-s-t}) \\
&= \alpha_{ij} \sum_{t+k=0}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-t-k} (-1)^{1-a_{ij}-t+u} q_i^{(t+k)(u+k+a_{ih})-k} \begin{bmatrix} 1-a_{ij} \\ u+k \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-u-k \\ t \end{bmatrix}_{d_i} \cdot \\
&\quad \begin{bmatrix} u+k \\ k \end{bmatrix}_{d_i} (q_i^{(u+k)a_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^u x_j x_i^{1-a_{ij}-u-k-t}) \\
&= \alpha_{ij} \sum_{t+k=0}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-t-k} (-1)^{1-a_{ij}-t+u} q_i^{(t+k)(u+k+a_{ih})-k} \begin{bmatrix} 1-a_{ij}-t-k \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ k \end{bmatrix}_{d_i} \cdot \\
&\quad \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} (q_i^{(u+k)a_{ij}} q_{jh} x_i^t x_j x_i^k x_h x_i^{1-a_{ij}-t-k} - x_i^{t+k} x_h x_i^u x_j x_i^{1-a_{ij}-u-k-t}) \quad \text{(by (2.2))} \\
&= \alpha_{ij} \sum_{m=0}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t} \begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \cdot \\
&\quad \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} (q_i^{(u+m-t)a_{ij}} q_{jh} x_i^t x_j x_i^{m-t} x_h x_i^{1-a_{ij}-m} - x_i^m x_h x_i^u x_j x_i^{1-a_{ij}-u-m}).
\end{aligned}
$$

Denote

$$
\begin{aligned}
\Phi := &q_{jh} \sum_{m=0}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t+(u+m-t)a_{ij}} \cdot \\
&\begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} x_i^t x_j x_i^{m-t} x_h x_i^{1-a_{ij}-m},
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega := &\sum_{m=0}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t} \cdot \\
&\begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} x_i^m x_h x_i^u x_j x_i^{1-a_{ij}-u-m}.
\end{aligned}
$$

In this case, we see that $v_{ij}^-(x_h)$ has the form $v_{ij}^-(x_h) = \alpha_{ij}(\Phi - \Omega)$. In the following, we prove that $\Phi = 0$ and $\Omega = 0$ both hold.

Now consider Φ. Take $m = 1 - a_{ij}$ in the expression of Φ, and hence $u = 0$. We have

$$
\begin{aligned}
&q_{jh} q_i^{(1-a_{ij})a_{ih}} \sum_{t=0}^{1-a_{ij}} (-1)^{1-a_{ij}-t} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} x_i^t x_j x_i^{1-a_{ij}-t} x_h \\
&\quad = q_{jh} q_i^{(1-a_{ij})a_{ih}} ((ad_c x_i)^{1-a_{ij}} x_j) x_h = 0.
\end{aligned}
$$

Therefore, Φ is reduced to the form

$$
\Phi = q_{jh} \sum_{m=0}^{-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t+(u+m-t)a_{ij}} \cdot
$$

$$\begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} x_i^t x_j x_i^{m-t} x_h x_i^{1-a_{ij}-m}$$

$$=q_{jh} \sum_{m=0}^{-a_{ij}} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t} q_i^{m(m-t+a_{ih})+(m-t)(a_{ij}-1)} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} \cdot$$

$$x_i^t x_j x_i^{m-t} x_h x_i^{1-a_{ij}-m} \Big(\sum_{u=0}^{1-a_{ij}-m} (-1)^u q_i^{u(a_{ij}+m)} \begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \Big) = 0. \quad \text{(by (2.1))}$$

To deal with Ω, we take $m = 0$ in the expression of Ω and get $t = 0$. Therefore

$$(-1)^{1-a_{ij}} x_h \sum_{u=0}^{1-a_{ij}} (-1)^u \begin{bmatrix} 1-a_{ij} \\ u \end{bmatrix}_{d_i} x_i^u x_j x_i^{1-a_{ij}-u} = x_h((ad_c x_i)^{1-a_{ij}} x_j) = 0,$$

and then Ω is just of the form

$$\Omega = \sum_{m=1}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t} \cdot$$

$$\begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} x_i^m x_h x_i^u x_j x_i^{1-a_{ij}-u-m}.$$

It is easy to check that

$$\begin{bmatrix} 1-a_{ij}-t \\ m-t \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{d_i} = \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{d_i} \begin{bmatrix} m \\ t \end{bmatrix}_{d_i}.$$

Together with this equivalence, we have

$$\Omega = \sum_{m=1}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} \sum_{t=0}^{m} (-1)^{1-a_{ij}-t+u} q_i^{m(u+m-t+a_{ih})-m+t} \cdot$$

$$\begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{d_i} \begin{bmatrix} m \\ t \end{bmatrix}_{d_i} x_i^m x_h x_i^u x_j x_i^{1-a_{ij}-u-m}$$

$$= \sum_{m=1}^{1-a_{ij}} \sum_{u=0}^{1-a_{ij}-m} (-1)^{1-a_{ij}+u} q_i^{m(u+m+a_{ih})-m} \begin{bmatrix} 1-a_{ij}-m \\ u \end{bmatrix}_{d_i} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{d_i} \cdot$$

$$x_i^m x_h x_i^u x_j x_i^{1-a_{ij}-u-m} \Big(\sum_{t=0}^{m} (-1)^t q_i^{t(1-m)} \begin{bmatrix} m \\ t \end{bmatrix}_{d_i} \Big) = 0. \quad \text{(by (2.1))}$$

Consequently, we complete the proof. □

Theorem 3.3 *We have $u_{ij}^+.a = 0$, $u_{ij}^-.a = 0$, for all $a \in \mathcal{B}(V)$, therefore $\mathcal{B}(V)$ is a left $\mathcal{U}_q(g)$-module algebra.*

Proof By Lemma 3.2, $u_{ij}^+.x_h = v_{ij}^+(x_h) = 0$, and $u_{ij}^-.x_h = v_{ij}^-(x_h) = 0$, for $1 \le h \le n$. Note that the Nichols algebra $\mathcal{B}(V)$ is a left $\widetilde{U}$-module algebra, therefore for $a, b \in \mathcal{B}(V)$, by (3.4) and (3.5)

$$u_{ij}^+.(ab) = (u_{ij}^+.a)b + (K_i^{1-a_{ij}}.a)(u_{ij}^+.b),$$

$$u_{ij}^-.(ab) = (u_{ij}^-.a)(K_i^{a_{ij}-1} K_j^{-1}.b) + a(u_{ij}^-.b).$$

Therefore by induction on the length of monomials in $\mathcal{B}(V)$, it can be concluded that all the monomials are zero under the action of u_{ij}^+, resp. u_{ij}^-, and hence $u_{ij}^+.a = 0$, $u_{ij}^-.a = 0$ for all $a \in \mathcal{B}(V)$. It follows that $\mathcal{B}(V)$ is also a left $\mathcal{U}_q(g)$-module algebra. □

Note that the Nichols algebra $\mathcal{B}(V)$ we consider here is exactly $\mathcal{U}_q^+(g)$, it follows from Theorem 3.3 that $\mathcal{U}_q^+(g)$ is a left $\mathcal{U}_q(g)$-module algebra.

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