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Finite Groups Whose Nontrivial Normal Subgroups Have Order Two

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Abstract In this paper, we investigate the structure of the groups whose nontrivial normal subgroups have order two. Some properties of this kind of groups are obtained.

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1. Introduction

In [1], the authors investigated the structure of finite groups whose non-trivial normal subgroups have the same order. In particular, they presented the following result.

Theorem 1.1 *(A) Let G be a finite soluble group which has a unique non-trivial normal subgroup. Then*

(i) G is a cyclic p-group of order p^2 for some prime p;

(ii) $G = P{:}Q \cong Z_p^n{:}Z_q$, $p \neq q$, Z_q acts irreducibly on Z_p^n.

(B) Let G be a finite insoluble group which has a unique non-trivial normal subgroup K. Then G/K is simple, and one of the following holds:

(i) K is soluble, G is perfect and G/K is a non-abelian simple group. Furthermore,

(a) $K = Z(G) \cong Z_p$, G is a covering group of G/K;

(b) $K \cong Z_p^n$ with $n > 1$, G/K acts irreducibly on K.

(ii) K is insoluble and one of the following holds:

(a) K is simple and G is an almost simple group;

(b) $K = T_1 \times \cdots \times T_n \cong T^n$ with $T_i \cong T$ simple, $n > 1$, G/K acts transitively on $\{T_1, T_2, \ldots, T_n\}$. Furthermore, $G/K \cong Z_p$ with $p = n$, or G/K is a non-abelian simple subgroup of $\mathrm{Out}(T) \wr S_n$.

Remark (1) Z_m denotes a cyclic group of order m. The symbol $A{:}B$ means a splitting extension

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of a group A by a group B, $A{:}B{:}C = A{:}(B{:}C)$. A covering group H of a simple group G is perfect and a central extension of G (see [3, p.43, Sect.1.5]). A group G is called almost simple if there is a non-abelian simple subgroup N such that $N \trianglelefteq G \lesssim \mathrm{Aut}(N)$.

(2) For part (A)(ii) in Theorem 1.1, since $Q \cong Z_q$ acts irreducibly on $P \cong Z_p^n$, by [2, Theorems 2.3.2 and 2.3.3], $Q \lesssim Z_{(p^n-1)}$, and q does not divide $p^d - 1$ for any $d < n$.

In this note, we continue the work of [1], and investigate the structure of the finite groups whose nontrivial normal subgroups have order two. Some properties of this kind of groups are obtained.

In the sequel, G always denotes a finite group whose nontrivial normal subgroups have order two, and we use $nn(G) = 2$ to denote such a group G with this property. The letters p, q, r always denote the primes, and G_p denotes a Sylow p-subgroup of G.

2. Main results and proofs

Recall here that a group G is said to be decomposable if it can be expressed as a direct product of its two non-trivial normal subgroups; otherwise, G is called indecomposable. Let $1 \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_l = G$ be a chief series of G. Then l is called the length of a chief series of G, and we use $l(G)$ to denote this integer.

Theorem 2.1 *Let G be a finite group with $nn(G) = 2$. Then $2 \le l(G) \le 3$. In particular, $l(G) = 2$ if and only if $G = N \times K$, where N and K are two simple subgroups of different orders.*

Proof Since $nn(G) = 2$, $l(G) \le 3$. If $l(G) = 1$, then G is simple, contrary to the hypothesis that $nn(G) = 2$. Thus $2 \le l(G) \le 3$.

Suppose $l(G) = 2$. Let $1 \trianglelefteq N \trianglelefteq G$ be a chief series of G. Since $nn(G) = 2$, there is another minimal normal subgroup K of G. Then $N \times K$ is a normal subgroup of G. If $|N| = |K|$, since $nn(G) = 2$, $N \times K$ is a proper normal subgroup of G. However, this is contrary to hypothesis that $l(G) = 2$. Thus $|N| \ne |K|$. Again, by the hypothesis that $nn(G) = 2$, $G = N \times K$, where N and K are both simple. □

Theorem 2.2 *Let G be a finite group with $nn(G) = 2$. Then one of the following holds:*

(A) G is decomposable, and G satisfies one of the following:

(i) $G = A \times B \times C$, $|A| = |B| = |C|$, A, B and C are non-abelian simple, or $A \cong B \cong C \cong Z_p$;

(ii) $G = A \times B$, A is simple and B has a unique non-trivial normal subgroup B_1 with $|A| = |B_1|$, the detailed structure of B is given in Theorem 1.1;

(iii) $G = A \times B$, A and B are both simple with $|A| \ne |B|$.

(B) G is indecomposable, and G satisfies one of the following:

(i) G has a unique minimal normal subgroup;

(ii) G has at least two minimal normal subgroups.

Remark The detailed structural information of the groups in part(B) of Theorem 2.2 is given in Theorems 2.3–2.6.

Proof Let G be a decomposable group. Write $G = A \times B$. Since $nn(G) = 2$, one of A and B is a minimal normal subgroup of G, say A. If $B = B_1 \times B_2$ for suitable non-trivial subgroups B_1 and B_2, since $l(G) \leq 3$, $G = A \times B_1 \times B_2$. Since $nn(G) = 2$, $|A| = |B_1| = |B_2|$. Furthermore, A, B_1 and B_2 are non-abelian simple subgroups, or $A \cong B_1 \cong B_2 \cong Z_p$ for some prime p. Then part(A)(i) holds.

Suppose that B is indecomposable. If B is not simple, then B has a unique normal subgroup B_1, and B_1 is obviously normal in G. Since $nn(G) = 2$, $|A| = |B_1|$, this is part(A)(ii). Finally, if B is simple, then $|A| \neq |B|$, as in part(A)(iii).

Part(B) of the theorem is obvious. □

Theorem 2.3 *Let G be indecomposable with $nn(G) = 2$. Suppose that A and B are two distinct minimal normal subgroups of G. Then $|A| = |B|$, $G/(A \times B)$ is simple, $A \times B$ is the unique maximal normal subgroup of G, and one of the following holds:*

(i) A and B are both non-abelian simple, $\{A, B, A \times B\}$ includes all the non-trivial normal subgroups of G, $C_G(A) = B$ and $C_G(B) = A$;

(ii) $A \times B = Z(G) \cong Z_p^2$ and G is a covering group of the non-abelian simple group $G/(A \times B)$, or $G \cong Z_p^2{:}Z_q$ with $q|(p-1)$;

(iii) $A \cong B \cong Z_p^n$ with $n > 1$, $G \cong (Z_p^n \times Z_p^n){:}Z_q$ with $q \neq p$, or $G/(A \times B)$ is isomorphic to a non-abelian simple subgroup of $\mathrm{GL}(n, p)$.

Remark Since G is indecomposable, by Theorem 2.1, $l(G) = 3$.

Proof Let A and B be two distinct minimal normal subgroups of G. Since G is indecomposable, $A \times B$ is a proper normal subgroup of G. Then $G/(A \times B)$ is simple since $l(G) = 3$. Since $nn(G) = 2$, A and B have the same order, and each minimal normal subgroup of G is contained in $A \times B$. Let K be a non-trivial normal subgroup G which does not contain $A \times B$. If $K \not\leq A \times B$, since $G/(A \times B)$ is simple, $G = K(A \times B)$. Let K_1 be a minimal normal subgroup of G contained in K. Then $|K_1| = |A| = |B|$ since $nn(G) = 2$. Clearly, $K_1 \leq A \times B$. Without loss of generality, we may suppose $A \times B = K_1 \times B$. Then $G = K(A \times B) = K(K_1 \times B) = K \times B$, which means G is decomposable, a contradiction. This implies that any normal subgroup of G is contained in $A \times B$, and so $A \times B$ is the unique maximal normal subgroup of G.

Suppose that A is non-abelian. Then B is also non-abelian since $|A| = |B|$. Then, by [4, Chap.I, Theorem 9.12], $\{A, B, A \times B\}$ includes all the non-trivial normal subgroups of G. Since $C_G(A) \trianglelefteq G$ and A is non-abelian, $C_G(A) = B$. Similarly, $C_G(B) = A$. This is part(i) of the theorem.

Suppose that A is an abelian subgroup. Then $A \cong B \cong Z_p^n$ for some prime p.

Suppose that $A \cong B \cong Z_p$. Since $G/(A \times B)$ is simple, $G/(A \times B)$ is non-abelian simple or of prime order. If $G/(A \times B)$ is non-abelian simple, since $G/C_G(A) \lesssim Aut(A) \cong Z_{(p-1)}$, $A \leq Z(G)$, and so $A \times B = Z(G)$. In this case, if $Z(G) \not\leq \Phi(G)$, without loss of generality, we suppose $A \not\leq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = A{:}M = A \times M$, G is decomposable, a contradiction. Thus, $\Phi(G) = Z(G)$, and $G/Z(G)$ is non-abelian simple. It follows that G is perfect and G is a covering group of $G/Z(G)$, this is part(ii). Let $G/(A\times B)$ be of prime order. Then $G/(A\times B)\cong Z_q$ for some prime q. If $p=q$, then G is a p-group of order p^3. Furthermore, if G is abelian, G is cyclic since G is indecomposable. However, this is also a contradiction since any cyclic group has a unique minimal normal subgroup. On the other hand, if G is non-abelian, then G is isomorphic to one of $\{Q_8, D_8, Z_{p^2}{:}Z_p, Z_p^2{:}Z_p\}$, and the latter two groups in the set are extra-special groups of order p^3 with p odd. This is impossible since each of the above four groups has a unique minimal normal subgroup which is the center. These contradictions imply that $p\neq q$. Then $G\cong Z_p^2{:}Z_q$. By hypothesis, Z_q acts reducibly on Z_p^2. Since G is indecomposable, $q|(p-1)$. Then we have part(ii).

Suppose that $A\cong B\cong Z_p^n$, $n>1$. Then $G/(A\times B)\cong G/C_G(A)\lesssim Aut(A)\cong \mathrm{GL}(n,p)$. Since $G/(A\times B)$ acts irreducibly on A, if $G/(A\times B)$ is cyclic, then $|G/(A\times B)|=q$ with $q\neq p$ ($p=q$ will lead to a contradiction that the length of chief series of G is more than 5, contrary to Theorem 2.1). Then $G\cong (Z_p^n\times Z_p^n){:}Z_q$, by [2, Theorems 2.3.2 and 2.3.3], $Z_q\lesssim Z_{(p^n-1)}$, and q does not divide p^d-1 for any $d<n$. On the other hand, if $G/(A\times B)$ is not abelian, then $G/(A\times B)$ is isomorphic to an irreducible non-abelian simple subgroup of $\mathrm{GL}(n,p)$, and part(iii) holds. The proof is completed. □

According to the classification of the groups of order p^3 (see [5, p.64 and p.65]), the following result regarding nilpotent groups is obvious.

Theorem 2.4 *Let G be a finite nilpotent group with $nn(G)=2$, and let G be indecomposable. Then G is a p-group of order p^3, where p is a prime. Then one of the following holds:*

(i) $G\cong Z_{p^3}$;

(ii) $p=2$, $G\cong D_8$ *or* Q_8;

(iii) $p>2$, $G\cong Z_p^2{:}Z_p$ *or* $Z_{p^2}{:}Z_p$, *two extra-special groups of order* p^3.

In the following, we will deal with the groups which are indecomposable and have a unique minimal normal subgroup.

Theorem 2.5 *Let G be a finite soluble group with $nn(G)=2$, not nilpotent. Suppose that G is indecomposable and has a unique minimal normal subgroup. Then one of the following holds:*

(A) $\Phi(G)=1$, *G satisfies one of the following:*

(i) $G\cong Z_p^n{:}Z_{q^2}$, Z_{q^2} *acts irreducibly on* Z_p^n;

(ii) $G\cong Z_p^n{:}Z_q^m{:}Z_r$, Z_p^n *is minimal normal in* $Z_p^n{:}Z_q^m{:}Z_r$, Z_r *acts irreducibly on* Z_q^m, *p,q and r are primes with $p\neq q$ and $q\neq r$.*

(B) $\Phi(G)\cong Z_p^n\neq 1$, $G=G_p{:}G_q$ *with* $p\neq q$, $G_q\cong Z_q$, $\Phi(G)=\Phi(G_p)$ *is minimal normal in* G, $1\lhd\Phi(G)\lhd G_p\lhd G$ *is the unique chief series of G. Furthermore, one of the following holds:*

(i) $\Phi(G)=Z(G)\cong Z_p$, $G/\Phi(G)\cong Z_p^m{:}Z_q$, Z_q *acts irreducibly on* Z_p^m;

(ii) $\Phi(G)\le Z(G_p)$, $Z(G)=1$, $G/\Phi(G)\cong Z_p^m{:}Z_q$, Z_q *acts irreducibly on* Z_p^m;

(iii) $C_G(\Phi(G))=\Phi(G)$, $G\cong Z_{p^2}{:}Z_q$ *and* $G/\Phi(G)$ *acts primitively on* $\Phi(G)$, *or* $G/\Phi(G)\cong Z_p^m{:}Z_q$ *and* $G/\Phi(G)$ *acts imprimitively on* $\Phi(G)$.

Proof Let N be the unique minimal normal subgroup of G. Then $N\cong Z_p^n$ for some prime p.

Since G is soluble, G/K is cyclic of prime order for any maximal normal subgroup K.

(A) Suppose that $\Phi(G) = 1$. Then there is a maximal subgroup M of G such that $G = N{:}M$. Let T be a minimal normal subgroup of M. Then $T \cong Z_q^m$ for some prime q. Since $l(G) = 3$, $M/T \cong Z_r$ for some prime r. If $p = q$, then $r \neq p$ since G is not nilpotent, $G = G_p{:}G_r$. Since $\Phi(G) = 1$ and $G_p \trianglelefteq G$, $\Phi(G_p) = 1$ and $G_p = N \times T$ is an elementary abelian subgroup. It follows that G has at least two minimal normal subgroups N and T, contrary to our hypothesis. Thus, $p \neq q$. If M is abelian, since $nn(G) = 2$, it follows that $q = r$, and $M \cong Z_{q^2}$. Then $G \cong Z_p^n{:}Z_{q^2}$. By [2, Theorems 2.3.2 and 2.3.3], $q^2|(p^n - 1)$, but q^2 does not divide $p^d - 1$ for any $d < n$, therefore part(A)(i) holds. If M is not abelian, $M \cong Z_q^m{:}Z_r$ with $q \neq r$, and $G \cong Z_p^n{:}Z_q^m{:}Z_r$, part(A)(ii) holds.

(B) Suppose that $\Phi(G) \neq 1$. Since $\Phi(G)$ contains no Sylow subgroup of G and $nn(G) = 2$, $|G|$ is divisible by at most two primes. Since G is not nilpotent, $|G|$, thus $|G/\Phi(G)|$, has exactly two prime divisors.

Suppose that $\Phi(G)$ is not minimal normal in G. Since $G/\Phi(G)$ has exactly two different prime divisors, it follows that the length of a chief series of G is at least four. This is contrary to Theorem 2.1. Thus, $\Phi(G) \cong Z_p^n$ which is minimal normal in G, where p is a prime. We know $l(G) = 3$. Let $1 \trianglelefteq \Phi(G) \trianglelefteq K \trianglelefteq G$ be a chief series of G. Since G is soluble, $K/\Phi(G)$ is abelian, and so K is nilpotent. It follows that $K := G_p$ is a p-group since G has a unique minimal normal subgroup. Then $G = G_p{:}G_q$ for some prime $q \neq p$, $1 \lhd \Phi(G) \lhd G_p \lhd G$ is the unique chief series of G. Furthermore, $G_q \cong Z_q$ since $l(G) = 3$. If $\Phi(G_p) = 1$, since $\Phi(G)$ is normalized by G_q, by Maschke's Theorem ([5,VIII,Theorem 2.2]), G has at least two minimal normal subgroups which are both contained in G_p, a contradiction. Thus $\Phi(G_p) = \Phi(G)$.

Suppose firstly that $C_G(\Phi(G)) = G$. Then $\Phi(G) \leq Z(G)$ and $\Phi(G) \cong Z_p$ since $\Phi(G)$ is minimal normal in G. If $\Phi(G) < Z(G)$, $Z(G) \cong Z_{p^2}$ since G has a unique minimal normal subgroup and $nn(G) = 2$. It follows that $G/Z(G)$ is simple, and thus $G/Z(G)$ is cyclic and G is abelian, contrary to our hypothesis. Thus $\Phi(G) = Z(G) \cong Z_p$. $G/\Phi(G) = G/Z(G) \cong Z_p^m{:}Z_q$, Z_q acts irreducibly on Z_p^m, m is some suitable positive integer. This is part(B)(i).

Suppose secondly that $\Phi(G) < C_G(\Phi(G)) < G$. It is easy to see that $Z(G) = 1$, and $1 \lhd \Phi(G) \lhd C_G(\Phi(G)) \lhd G$ is a chief series of G since $l(G) = 3$. Thus $C_G(\Phi(G)) = G_p$, that is, $\Phi(G) \leq Z(G_p)$. Also, we have $G/\Phi(G) \cong Z_p^m{:}Z_q$ for some positive integer m. Since $l(G) = 3$, Z_q acts irreducibly on Z_p^m. This is part(B)(ii).

Suppose finally that $C_G(\Phi(G)) = \Phi(G)$. Let $1 \trianglelefteq \Phi(G) \trianglelefteq K \trianglelefteq G$ be a chief series of G. Then $K = G_p$. If $G/\Phi(G)$ acts primitively on $\Phi(G)$, by [2, Theorem 2.5.10], $K/\Phi(G) \cong Z_p$. Thus, $G/\Phi(G) \cong Z_p{:}Z_q$ which is not nilpotent, $q|(p-1)$. Since $\Phi(G) = \Phi(G_p)$ and $nn(G) = 2$, G_p is cyclic, and thus $G_p \cong Z_{p^2}$, that is, $G \cong Z_{p^2}{:}Z_q$. If $G/\Phi(G)$ acts imprimitively on $\Phi(G)$, it is easy to see $G/\Phi(G) \cong Z_p^m{:}Z_q$. This proves part(B)(iii). □

Theorem 2.6 *Let G be a finite insoluble group with $nn(G) = 2$. Suppose that G has a unique minimal normal subgroup N. Then one of the following holds:*

(i) G/N has a unique normal subgroup, G/N is as the group in Theorem 1.1;

(ii) G/N is a direct product of two simple subgroups of the same order.

Proof Since $nn(G)=2$, if G/N has a unique minimal normal subgroup, G/N is as the group in Theorem 1.1. Otherwise, if G/N has at least two minimal normal subgroups, say, K_1/N and K_2/N, again by the hypothesis that $nn(G)=2$, $G/N=K_1/N\times K_2/N$ with property that $|K_1/N|=|K_2/N|$. Clearly, in this case, K_1/N and K_2/N are both simple. The proof is completed. $\square$

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