Journal of Mathematical Research & Exposition
Jul., 2011, Vol. 31, No. 4, pp. 713–720
DOI:10.3770/j.issn:1000-341X.2011.04.017
Http://jmre.dlut.edu.cn

An Iterative Scheme for the System of Generalized Variational Inequalities in Banach Spaces

Ying LIU
College of Mathematics and Computer, Hebei University, Hebei 071002*, P. R. China*

Abstract In this paper, we propose an iterative method of approximating solutions for a class of the system of generalized variational inequalities and give a convergence result for the iterative method in uniformly convex and uniformly smooth Banach spaces.

Keywords generalized variational inequality; generalized f-projection operator; compact operator; lower semi-continuity; positively homogeneous.

Document code A
MR(2010) Subject Classification 47H09; 47H05; 47J25; 47J05
Chinese Library Classification O177.91

1. Introduction

Let B be a Banach space, B^* be the dual space of B. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of B^* and B. Let K be a nonempty closed convex subset of B and $T : K \to B^*$ be an operator. The following problem:

$$\text{Find } x^* \in K, \quad \text{such that } \langle Tx^*, y - x^* \rangle \geq 0, \ \text{for all } y \in K \tag{1.1}$$

is called classical variational inequality problem.

The variational inequality (1.1) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design [1–9]. Variational inequalities have been extended and generalized in many directions using novel and innovative techniques. Most recently, applying the f-projection operator, Wu and Huang [10] proposed an iterative method of approximating solutions for the generalized variational inequality problem: find $x^* \in K$ such that

$$\langle Ax^*, y - x^* \rangle + f(y) - f(x^*) \geq 0, \quad \forall y \in K, \tag{1.2}$$

where $A : K \to B^*$ is an operator. A convergence result for this iterative method was also given in compact subsets of Banach spaces. They proved the following theorem:

Theorem K1 ([10, Theorem 4.1]) *Let K be a nonempty compact convex subset of a uniformly*

Received July 28, 2009; Accepted April 26, 2010
Supported by the Natural Science Youth Foundation of Hebei Province (Grant Nos. A2011201053; A2010000191) and the Natural Science Youth Foundation of Hebei Education Commission (Grant No. 2010110).
E-mail address: ly_cyh2007@yahoo.com.cn

convex and uniformly smooth Banach space B with dual space B^* and $0 \in K$. Let $A : K \to B^*$ be a continuous mapping and $f : K \to R$ be convex, lower semi-continuous and positively homogeneous. Suppose that

(1) $f(x) \geq 0$ for all $x \in K$ and $f(0) = 0$;

(2) For any $x \in K$,

$$\langle J(x - \pi_K^f(Jx - \rho Ax)), J^*(Jx - J(x - \pi_K^f(Jx - \rho Ax)))\rangle \geq 0.$$

Let $x_0 \in K$ and the sequence $\{x_n\}$ be generated by the following iteration scheme:

$$x_{n+1} = \pi_K^f(Jx_n - \alpha_n J(x_n - \pi_K^f(Jx_n - \rho Ax_n))), \quad n = 0, 1, 2, \ldots,$$

where $\{\alpha_n\}$ satisfies the conditions:

(a) $0 \leq \alpha_n \leq 1$, for all $n = 0, 1, 2, \ldots$; (b) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$.

Then generalized variational inequality (1.2) has a solution $x^* \in K$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^*$, as $i \to \infty$.

In addition, Fan [9] defined a Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \pi_K(Jx_n - \beta Tx_n), \quad n = 1, 2, 3, \ldots, \tag{1.3}$$

where $\{\alpha_n\}$ satisfies: $0 < a \leq \alpha_n \leq b < 1$ for all $n \in N$, for some positive numbers $a, b \in (0, 1)$ satisfying $a < b$. He established some existence results and convergence of the above iterative scheme (1.3) for variational inequalities (1.1) in noncompact subsets of Banach spaces.

Motivated by these facts, our purpose in this paper is to establish some existence results of solutions and the convergence of an iterative scheme in noncompact subsets of Banach spaces for the following system of the generalized variational inequalities:

$$\text{Find } x^* \in K, \text{ such that } \begin{cases} \langle Tx^*, y - x^*\rangle + f(y) - f(x^*) \geq 0, \\ \langle Ax^*, y - x^*\rangle + f(y) - f(x^*) \geq 0, \end{cases} \forall y \in K. \tag{1.4}$$

The results presented in this paper generalize the corresponding results of [8–10].

2. Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively.

Let X, Y be Banach spaces, $T : D(T) \subset X \to Y$. The operator T is said to be compact if it is continuous and maps the bounded subsets of $D(T)$ onto the relatively compact subsets of Y.

We denote by $J : B \to 2^{B^*}$ the normalized duality mapping from B to 2^{B^*}, defined by

$$J(x) := \{v \in B^* : \langle v, x\rangle = \|v\|^2 = \|x\|^2\}, \quad \forall x \in B.$$

The duality mapping J has the following properties:

(i) If B is smooth, then J is single-valued;

(ii) If B is strictly convex, then J is one-to-one;

(iii) If B is reflexive, then J is surjective;

(iv) If B is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of B.

Let B be a reflexive, strictly convex, smooth Banach space and J the duality mapping from B into B^*. Then J^* is also single-valued, one-to-one, surjective, and it is the duality mapping from B^* into B, i.e., $J^*J = I_B$; $JJ^* = I_{B^*}$.

When $\{x_n\}$ is a sequence in B, we denote strong convergence of $\{x_n\}$ to $x \in B$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $U = \{x \in B : \|x\| = 1\}$. A Banach space B is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in U$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in U and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. A Banach space B is said to be smooth provided $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$.

Alber [2, 4] introduced the functional $V : B^* \times B \to R$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in B^*$ and $x \in B$.

Definition 2.1 ([9]) *If B is a uniformly convex and uniformly smooth Banach space, the generalized projection $\pi_K : B^* \to K$ is a mapping that assigns an arbitrary point $\phi \in B^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem*

$$V(\phi, \pi_K(\phi)) = \inf_{y\in K} V(\phi, y).$$

The operator π_K is J fixed in each point $x \in K$, i.e., $\pi_K(Jx) = x$ (see [8]).

Recently, by employing the functional $V(\phi, x)$, many authors established some existence results and iterative methods for the classical variational inequality problems (1.1) in reflexive, strictly convex and smooth Banach spaces, see [8, 9] and the references therein. However, we can not solve the generalized variational inequality (1.2) by using the functional $V(\phi, x)$. The primary reason is that there is a nonlinear item $f(y) - f(x^*)$ in (1.2). Therefore, for solving the generalized variational inequality (1.2), Wu and Huang [10] introduced the functional $G : B^* \times K \to R\bigcup\{+\infty\}$ defined by

$$G(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2 + 2\rho f(x),$$

where $\phi \in B^*$, $x \in B$, $\rho > 0$ is a constant and $f : K \subset B \to R\bigcup\{+\infty\}$ is proper, convex and lower semi-continuous. It is easy to see that $G(\phi, x) \geq (\|\phi\| - \|x\|)^2 + 2\rho f(x)$ and $G(\phi, x) = V(\phi, x) + 2\rho f(x)$.

From the definitions of G and f, it is easy to have the following properties:

(i) $G(\phi, x)$ is convex and continuous with respect to ϕ when x is fixed;

(ii) $G(\phi, x)$ is convex and lower-semi-continuous with respect to x when ϕ is fixed;

(iii) $(\|\phi\| - \|x\|)^2 + 2\rho f(x) \leq G(\phi, x) \leq (\|\phi\| + \|x\|)^2 + 2\rho f(x)$.

Remark 2.1 If $f \equiv 0$, then $G(\phi, x) = V(\phi, x)$, $\forall \phi \in B^*, x \in K$.

Definition 2.2 ([10]) *Let B be a Banach space with dual space B^* and K be a nonempty, closed and convex subset of B. We say that $\pi_K^f : B^* \to 2^K$ is a generalized f-projection operator* *if*

$$\pi_K^f \phi = \{u \in K : G(\phi, u) = \inf_{y \in K} G(\phi, y)\} \forall \phi \in B^*.$$

Definition 2.3 *We say that a Banach space B has the property (h) if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ implies $x_n \to x$.*

Remark 2.2 It is well known that any uniformly convex space has the property (h).

Theorem 2.1 ([10]) *If B is a reflexive Banach space with dual space B^* and K is a nonempty, closed and convex subset of B, then the following conclusions hold:*

(f1) For any given $\phi \in B^*, \pi_K^f \phi$ is a nonempty, closed and convex subset of K;

(f2) π_K^f is monotone, i.e., for any $\phi_1, \phi_2 \in B^*$, $x_1 \in \pi_K^f \phi_1$ and $x_2 \in \pi_K^f \phi_2$,

$$\langle x_1 - x_2, \phi_1 - \phi_2 \rangle \geq 0;$$

(f3) If B is smooth, then for any given $\phi \in B^*$, $x \in \pi_K^f \phi$ if and only if

$$\langle \phi - Jx, x - y \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in K;$$

(f4) If $f : K \to R \bigcup \{+\infty\}$ is positively homogeneous, i.e., $f(tx) = tf(x)$ for all $t > 0$ and $x \in K$ with $tx \in K$, then for any $\phi \in B^*$ and $x_1, x_2 \in \pi_K^f \phi$ with $x_1 \neq 0$ and $x_2 \neq 0$, we have $x_1 \neq \mu x_2$ for all $\mu \in (0, +\infty)$ with $\mu \neq 1$;

(f5) If $f : K \to R \bigcup \{+\infty\}$ is positively homogeneous and B is strictly convex, then the operator $\pi_K^f : B^* \to K$ is single-valued.

Remark 2.3 If $f \equiv 0$, then $\pi_K^f \phi = \pi_K \phi$, $\forall \phi \in B^*$.

By Theorem 2.1 (f3), it is easy to obtain the following result.

Theorem 2.2 ([10]) *Let A be an arbitrary operator acting from the reflexive smooth Banach space B to B^*, $\rho > 0$. Then the point $x^* \in K \subset B$ is a solution of the variational inequality*

$$\langle Ax^*, y - x^* \rangle + f(y) - f(x^*) \geq 0, \quad \forall y \in K,$$

if and only if x^* is a solution of the following inclusion

$$x \in \pi_K^f (Jx - \rho Ax).$$

Theorem 2.3 ([10]) *Let B be a reflexive and strictly convex Banach space with dual space B^* and K be a nonempty closed convex subset of B. Suppose that $f : K \to R \bigcup \{+\infty\}$ is proper, convex, lower semi-continuous, positively homogeneous and bounded from below. Then*

(i) $\pi_K^f : B^* \to K$ is norm-weak continuous;

(ii) Moreover, if B has the property (h), then $\pi_K^f : B^* \to K$ is continuous.

Theorem 2.4 ([10]) *If $f(x) \geq 0$ for all $x \in K$, then*

$$G(Jx, y) \leq G(\phi, y) + 2\rho f(y), \quad \forall \phi \in B^*, \ y \in K, \ x \in \pi_K^f \phi.$$

Theorem 2.5 ([11]) *Let B be a uniformly convex Banach space and let $r > 0$. Then there* *exists a continuous strictly increasing convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in B : \|z\| \le r\}$.

Theorem 2.6 ([5]) *Let B be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \le \|\phi\|^2 + 2\langle \Phi, J^*(\phi + \Phi)\rangle, \quad \forall \phi, \Phi \in B^*.$$

3. Main results

For any $x_0 \in K$, we define the iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)\pi_K^f(Jx_n - \rho A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\pi_K^f(Jy_n - \rho T y_n), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy:

$$0 < \alpha_n < 1, \text{ and } \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0; \quad 0 < \beta_n < 1 \text{ and } \liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0.$$

Theorem 3.1 *Let B be a uniformly convex and uniformly smooth Banach space. Let K be a nonempty, closed convex subset of B and $0 \in K$. Assume that T, A are two operators of K into B^* and $f : K \to R$ is proper, convex, lower semi-continuous, positively homogeneous and bounded from below. Suppose that*

(1) $f(x) \ge 0$ for all $x \in K$ and $f(0) = 0$;

(2) For any $x \in K$, $\langle Tx, J^*(Jx - \rho Tx)\rangle \ge 0$, and $\langle Ax, J^*(Jx - \rho Ax)\rangle \ge 0$;

(3) $J - \rho T : K \to B^*$ is compact, and A is continuous.

Then the system of generalized variational inequality (1.4) has a solution $x^* \in K$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ defined by (3.1) such that $x_{n_i} \to x^*$, as $i \to \infty$.

Proof From the definition of G, Theorems 2.4 and 2.6, we have

$$\begin{aligned} \|\pi_K^f(Jy_n - \rho T y_n)\|^2 &= G(J\pi_K^f(Jy_n - \rho T y_n), 0) \\ &\le G(Jy_n - \rho T y_n, 0) = \|Jy_n - \rho T y_n\|^2 \\ &\le \|Jy_n\|^2 - 2\rho\langle Ty_n, J^*(Jy_n - \rho T y_n)\rangle \le \|y_n\|^2. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\|\pi_K^f(Jx_n - \rho A x_n)\|^2 \le \|x_n\|^2. \tag{3.3}$$

From the convexity of $\|\cdot\|^2$ and (3.3), we have

$$\|y_n\|^2 \le \beta_n\|x_n\|^2 + (1-\beta_n)\|\pi_K^f(Jx_n - \rho A x_n)\|^2 \le \|x_n\|^2. \tag{3.4}$$

Therefore,

$$\begin{aligned} \|x_{n+1}\|^2 &\le \alpha_n\|x_n\|^2 + (1-\alpha_n)\|\pi_K^f(Jy_n - \rho T y_n)\|^2 \\ &\le \alpha_n\|x_n\|^2 + (1-\alpha_n)\|y_n\|^2 \le \|x_n\|^2, \end{aligned} \tag{3.5}$$

for every $n \in N \bigcup \{0\}$. Therefore, $\{\|x_n\|\}$ is nonincreasing and hence there exists $\lim_{n\to\infty} \|x_n\|$. So, $\{x_n\}$, $\{y_n\}$, $\{\pi_K^f(Jx_n - \rho Ax_n)\}$, $\{\pi_K^f(Jy_n - \rho Ty_n)\}$ are bounded. From Theorem 2.5 and (3.2),(3.3), we have

$$\begin{aligned}\|x_{n+1}\|^2 \leq &\alpha_n\|x_n\|^2 + (1-\alpha_n)\|\pi_K^f(Jy_n - \rho Ty_n)\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|) \\ \leq &\alpha_n\|x_n\|^2 + (1-\alpha_n)\|y_n\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|) \\ \leq &\alpha_n\|x_n\|^2 + (1-\alpha_n)\{\beta_n\|x_n\|^2 + (1-\beta_n)\|\pi_K^f(Jx_n - \rho Ax_n)\|^2 - \\ &\beta_n(1-\beta_n)g(\|x_n - \pi_K^f(Jx_n - \rho Ax_n)\|)\}, -\alpha_n(1-\alpha_n)g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|) \\ \leq &\|x_n\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - \pi_K^f(Jx_n - \rho Ax_n)\|) - \\ &\alpha_n(1-\alpha_n)g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|). \end{aligned} \tag{3.6}$$

That is

$$(1-\alpha_n)\beta_n(1-\beta_n)g(\|x_n - \pi_K^f(Jx_n - \rho Ax_n)\|) \leq \|x_n\|^2 - \|x_{n+1}\|^2$$

and

$$\alpha_n(1-\alpha_n)g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|) \leq \|x_n\|^2 - \|x_{n+1}\|^2.$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0$, and $\lim_{n\to\infty} \|x_n\|$ exists, we have

$$\lim_{n\to\infty} g(\|x_n - \pi_K^f(Jx_n - \rho Ax_n)\|) = 0, \text{ and } \lim_{n\to\infty} g(\|x_n - \pi_K^f(Jy_n - \rho Ty_n)\|) = 0.$$

Applying the properties of g, we have

$$\lim_{n\to\infty} \|x_n - \pi_K^f(Jx_n - \rho Ax_n)\| = 0, \text{ and } \lim_{n\to\infty} \|x_n - \pi_K^f(Jy_n - \rho Ty_n)\| = 0. \tag{3.7}$$

Since

$$\|y_n - x_n\| = (1-\beta_n)\|\pi_K^f(Jx_n - \rho Ax_n) - x_n\|,$$

from (3.7), we have

$$\|y_n - x_n\| \to 0. \tag{3.8}$$

Since $\{y_n\}$ is bounded and $J - \rho T$ is compact on K, the sequence $\{Jy_n - \rho Ty_n\}$ must have a subsequence $\{Jy_{n_i} - \rho Ty_{n_i}\}$, which converges to a point $\phi \in B^*$. By the continuity of the projection operator π_K^f, we have

$$\lim_{i\to\infty} \pi_K^f(Jy_{n_i} - \rho Ty_{n_i}) = \pi_K^f\phi. \tag{3.9}$$

Let $x^* = \pi_K^f\phi$. Since

$$\|y_{n_i} - x^*\| \leq \|y_{n_i} - x_{n_i}\| + \|x_{n_i} - \pi_K^f(Jy_{n_i} - \rho Ty_{n_i})\| + \|\pi_K^f(Jy_{n_i} - \rho Ty_{n_i}) - x^*\|, \tag{3.10}$$

combining (3.7), (3.8) and (3.9) gives

$$\lim_{i\to\infty} y_{n_i} = x^*. \tag{3.11}$$

By the continuity properties of the operators π_K^f and $J - \rho T$, combining (3.9) and (3.11), we have

$$\pi_K^f(Jx^* - \rho Tx^*) = x^*. \tag{3.12}$$

From (3.11) and (3.8), we have

$$\lim_{i\to\infty} x_{n_i} = x^*. \tag{3.13}$$

By the continuity properties of the operators π_K^f, J and A, we have

$$\lim_{i\to\infty} \pi_K^f(Jx_{n_i} - \rho Ax_{n_i}) = \pi_K^f(Jx^* - \rho Ax^*). \tag{3.14}$$

Since

$$\|\pi_K^f(Jx_{n_i} - \rho Ax_{n_i}) - x^*\| \leq \|\pi_K^f(Jx_{n_i} - \rho Ax_{n_i}) - x_{n_i}\| + \|x_{n_i} - x^*\|,$$

it follows from (3.7) and (3.13)

$$\lim_{i\to\infty} \pi_K^f(Jx_{n_i} - \rho Ax_{n_i}) = x^*. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\pi_K^f(Jx^* - \rho Ax^*) = x^*. \tag{3.16}$$

Theorem 2.2, equalities (3.12) and (3.16) imply that x^* is a solution of the system of the generalized variational inequalities (1.4) and $x_{n_i} \to x^*$. $\square$

Remark 3.1 Note that $\{\alpha_n\} = \{\frac{1}{2} - \frac{1}{n}\}$ and $\{\beta_n\} = \{\frac{1}{2} - \frac{1}{n}\}$ is an example of the sequences α_n and β_n.

Theorem 3.2 *Let B be a uniformly convex and uniformly smooth Banach space. Let K be a nonempty, closed convex subset of B and $0 \in K$. Assume that T, A are two operators of K into B^* and $f : K \to R$ is proper, convex, lower semi-continuous, positively homogeneous and bounded from below. Suppose that*

(1) $f(x) \geq 0$ for all $x \in K$ and $f(0) = 0$;

(2) For any $x \in K$, $\langle Tx, J^*(Jx - \rho Tx)\rangle \geq 0$, and $\langle Ax, J^*(Jx - \rho Ax)\rangle \geq 0$;

(3) $J - \rho T : K \to B^*$ is compact, and A is continuous.

Moreover, if the solution of the the system of generalized variational inequality (1.4) is unique, denoted by x^*, then the sequence $\{x_n\}$ defined by (3.1) converges to x^*.

Proof Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$. Since $\{x_{n_k}\}$ is bounded, similarly to the proof of Theorem 3.1, we can obtain a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that $\lim_{j\to\infty} x_{n_j} = x^*$ and hence $\lim_{n\to\infty} x_n = x^*$. $\square$

If $f \equiv 0$ and $A \equiv 0$, then we obtain the following conclusion:

Theorem 3.3 ([9, Theorem 3.1]) *Let B be a uniformly convex and uniformly smooth Banach space and let K be a closed convex subset of B. Suppose that there exists a positive number β, such that*

$$\langle Tx, J^*(Jx - \beta Tx)\rangle \geq 0, \text{ for all } x \in K,$$

and $J - \beta T : K \to B^*$ is compact. Then the variational inequality (1.1) has a solution $x^* \in K$ and the sequence $\{x_n\}$ defined by (1.3) has a convergent subsequence $\{x_{n_i}\}$ that converges strongly to $x^* \in K$.

Proof Taking $f \equiv 0$ in Theorem 3.1, then we have $\pi_K^f = \pi_K$. Taking $A \equiv 0$, by $x_n \in K$ and the property of the operator π_K, we have $\pi_K J x_n = x_n$, So, $y_n = x_n$. Moreover, if $0 < a \le \alpha_n \le b < 1$ for all $n \in N$, for some positive numbers $a, b \in (0,1)$ satisfying $a < b$, then we have $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Thus, by Theorem 3.1, we can obtain the desired conclusion. □

Theorem 3.4 *Let H be a Hilbert space. Let K be a nonempty, closed convex subset of H and $0 \in K$. Assume that T, A are two operators of K into H and $f : K \to R$ is proper, convex, lower semi-continuous, positively homogeneous and bounded from below. Suppose that*

(1) $f(x) \ge 0$ for all $x \in K$ and $f(0) = 0$;

(2) For any $x \in K$, $\langle Tx, x\rangle \ge \rho\|Tx\|^2$, and $\langle Ax, x\rangle \ge \rho\|Ax\|^2$;

(3) $I - \rho T : K \to H$ is compact, and A is continuous.

Let the sequence $\{x_n\}$ be generated by the following iteration scheme:

$$\begin{cases} x_0 \in K \text{ chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)\pi_K^f(x_n - \rho A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)\pi_K^f(y_n - \rho T y_n), \end{cases}$$

where $\{\alpha_n\}$,$\{\beta_n\}$ satisfy:

$$0 < \alpha_n < 1, \text{ and } \liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0; \quad 0 < \beta_n < 1 \text{ and } \liminf_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n) > 0.$$

Then the system of generalized variational inequality (1.4) has a solution $x^* \in K$ and there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^*$, as $i \to \infty$.

Proof Since $H^* = H$ and $J^* = J = I$ for a Hilbert space H, the conclusion follows from Theorem 3.1. This completes the proof. □

References

[1] LI Jinlu. *The generalized projection operator on reflexive Banach spaces and its applications* [J]. J. Math. Anal. Appl., 2005, **306**(1): 55–71.

[2] ALBER Y, GUERRE-DELABRIERE S. *On the projection methods for fixed point problems* [J]. Analysis (Munich), 2001, **21**(1): 17–39.

[3] AL'BER Y I, NOTIK A I. *On some estimates for projection operators in Banach spaces* [J]. Comm. Appl. Nonlinear Anal., 1995, **2**(1): 47–55.

[4] AL'BER Y I, REICH S. *An iterative method for solving a class of nonlinear operator equations in Banach spaces* [J]. Panamer. Math. J., 1994, **4**(2): 39–54.

[5] ZHANG Shisheng. *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces* [J]. J. Math. Anal. Appl., 1997, **216**(1): 94–111.

[6] CHIDUME C E, LI Jinlu. *Projection methods for approximating fixed points of Lipschitz suppressive operators* [J]. Panamer. Math. J., 2005, **15**(1): 29–39.

[7] CHIDUME C E. *Iterative solutions of nonlinear equations in smooth Banach spaces* [J]. Nonlinear Anal., 1996, **26**(11): 1823–1834.

[8] LI Jinlu. *On the existence of solutions of variational inequalities in Banach spaces* [J]. J. Math. Anal. Appl., 2004, **295**(1): 115–126.

[9] FAN Jianghua. *A Mann type iterative scheme for variational inequalities in noncompact subsets of Banach spaces* [J]. J. Math. Anal. Appl., 2008, **337**(2): 1041–1047.

[10] WU Keqing, HUANG Nanjing. *Properties of the generalized f-projection operator and its applications in Banach spaces* [J]. Comput. Math. Appl., 2007, **54**(3): 399–406.

[11] XU Hongkun. *Inequalities in Banach spaces with applications* [J]. Nonlinear Anal., 1991, **16**(12): 1127–1138.