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Completely Non-Normal Toeplitz Operators

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Abstract In this paper, we show that the hyponormal Toeplitz operator T_φ with trigonometric polynomial symbol φ is either normal or completely non-normal. Moreover, if T_φ is non-normal, then $T_{\bar\varphi}$ has a dense set of cyclic vectors. Some general conditions are also considered.

Keywords Toeplitz operator; completely non-normal; hyponormal; cyclic.

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1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$, and $L^2(\mathbb{T})$ be the Banach space consisting of the square integrable functions with respect to the normalized arc length measure on $\mathbb{T}$, which is denoted by $d\mu = d\theta/2\pi$. We write H^2 for the classical Hardy space, $L^\infty(\mathbb{T})$ for the space consisting of the essentially bounded measurable functions on $\mathbb{T}$, and H^∞ for the space of the bounded analytic functions on the unit disk.

Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator on H^2 with symbol φ is the operator T_φ defined by $T_\varphi(g) := P(\varphi g)$, $\forall g \in H^2$, where P is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. A Toeplitz operator T_φ is said to be analytic if its symbol φ is in H^∞. Many basic facts about Toeplitz operators can be found in [7], [15] for example.

Let $\mathcal{B}(H)$ be the C*-algebra of all the bounded linear operators acting on the complex Hilbert space H. We say an operator $T \in \mathcal{B}(H)$ is normal, if $T^*T = TT^*$, and T is said to be hyponormal if its self-commutator $T^*T - TT^*$ is positive. It is a basic and natural problem to describe these algebra properties of T_φ by the symbol φ.

In the early 1960s, Brown and Halmos [2] completely characterized the normal Toeplitz operators by their symbols.

Theorem (BH) *Given $\varphi \in L^\infty(\mathbb{T})$, then the Toeplitz operator T_φ is normal if and only if $\varphi = \alpha + \beta\psi$, where α and β are complex numbers and ψ is a real-valued function in $L^\infty(\mathbb{T})$.*

An operator T in $\mathcal{B}(H)$ is called completely non-normal (or c.n.n.), if T has no non-trivial reducing subspace $\mathcal{M}$ such that the restriction of T to $\mathcal{M}$ is normal. Generally, that T is non-

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normal does not imply that T is c.n.n. However, Wu [14] proved that if φ is analytic, then the properties of non-normal and completely non-normal are equivalent. Naturally, we are interested in the following question:

Is every Toeplitz operator either normal or c.n.n.?

In fact, our study mainly concerns the restriction of the commutor $T_{\bar{\varphi}}T_{\varphi}-T_{\varphi}T_{\bar{\varphi}}$ on some reducing subspace of T_{φ}. Although the research on the reducing space of analytic Toeplitz operator T_{φ} can be found in many papers (see [3], [12] for example), the study for the Toeplitz operator with general symbol seems to be scarce from the literature. So there is much work to do to answer the question completely.

In this paper, we give an affirmative answer to the question under some hypothesis.

Theorem 1.1 *Let non-constant functions $\varphi,\ k \in H^{\infty}(\mathbb{T})$ with $\|k\|_{\infty} \leqslant 1$ such that $\varphi = k\bar{\varphi}$. Then T_{φ} is completely non-normal if and only if T_{φ} is non-normal.*

Theorem 1.2 *Suppose that $\varphi = \sum_{j=-n}^{m} \alpha_j z^j$ with $\alpha_{-n}\alpha_m \neq 0$, then T_{φ} is completely non-normal if φ satisfies one of the following conditions:*

(i) $m \neq n$;

(ii) $m = n$ *and* $|\alpha_{-n}| \neq |\alpha_m|$.

In Section 3, we consider the problem for hyponormal Toeplitz operators. We show that the properties of non-normal and completely non-normal are equivalent for the hyponormal Toeplitz operators with trigonometric polynomial symbols. As an application, we prove that if this kind of Toeplitz operator is non-normal, then it has a dense set of cyclic vectors.

2. The proof of theorems

We begin with some lemmas. Throughout this paper, denote by $\sigma_p(T)$ the point spectrum of the operator T.

Lemma 2.1 ([5]) *If φ is a function in $L^{\infty}(\mathbb{T})$ not almost everywhere zero, then either* $\operatorname{Ker} T_{\varphi} = \{0\}$ *or* $\operatorname{Ker} T_{\varphi}^{*} = \{0\}$.

The proof of above lemma can also be found in chapter 7 of [7].

Lemma 2.2 *Let $\varphi \in L^{\infty}(\mathbb{T})$ be a non-constant function, and M be a non-zero reducing subspace of T_{φ} such that $P_M T_{\varphi}|_M$ is normal. Then $\sigma_p(P_M T_{\varphi}|_M) = \emptyset$. In particular, $T_{\varphi}(M)$ is a dense subset of M.*

Proof Let $T_0 = P_M T_{\varphi}|_M$, and assume $\sigma_p(T_0) \neq \emptyset$. Then there exists $\lambda \in \sigma_p(P_M T_{\varphi}|_M)$ such that $\operatorname{Ker}(\lambda - T_0) = \operatorname{Ker}(\lambda - T_0)^{*} \neq \{0\}$. So $\operatorname{Ker}(\lambda - T_{\varphi}) \supseteq \operatorname{Ker}(\lambda - T_0) \neq \{0\}$ and $\operatorname{Ker}(\lambda - T_{\varphi})^{*} \supseteq \operatorname{Ker}(\lambda - T_0)^{*} \neq \{0\}$. On the other hand, since $\lambda - \varphi$ is not almost everywhere zero, Lemma 2.1 shows that either $\operatorname{Ker} T_{\lambda-\varphi} = \{0\}$ or $\operatorname{Ker} T_{\lambda-\varphi}^{*} = \{0\}$, which is a contradiction. Hence the proof is completed. □

Lemma 2.3 *Let $E = \{g \in H^2;\ \bar{k}^n g \in H^2,\ \forall n \geq 1\}$, where $k \in H^2$ is non-constant. Then $E = \{0\}$.*

Proof Suppose $E \neq \{0\}$. Since $\bar{k}^n T_z g = \bar{k}^n(zg) = z\bar{k}^n g \in H^2$ $(\forall g \in E)$ and $1 \notin E$, then E is a non-trivial invariant subspace of T_z. By Beurling theorem [1], there is an inner function $\psi \in H^\infty$, such that $E = \psi H^2$. Note that $\psi \cdot 1 \in \psi H^2$, then $\bar{k}\psi \in E$ by the definition of E. So there is a function $\rho \in H^2$ such that $\bar{k}\psi = \psi\rho \in H^2$, that is, $\bar{k} = \rho \in H^2$, which contradicts the fact that k is non-constant. Thus $E = \{0\}$. $\Box$

Given a function $\psi \in L^\infty(\mathbb{T})$, let S_ψ be the operator defined by

$$S_\psi h = (I-P)(\psi(I-P)(h)), \quad \forall h \in L^2(\mathbb{T}).$$

Now, we are ready to prove our results.

Proof of Theorem 1.1 We only need to prove the sufficiency. Assume M is a non-trivial reducing subspace of T_φ such that $P_M T_\varphi|_M$ is normal. Then we have

$$M \subseteq \bigcap_{r=1}^{\infty}\bigcap_{s=1}^{\infty} \mathrm{Ker}(T_\varphi^{*r}T_\varphi^s - T_\varphi^s T_\varphi^{*r}).$$

Write $\varphi = f + \bar{g}$ where $f, g \in H^2$ and $g(0) = 0$. Without loss of generality, assume f is not a constant, or else we consider $T_{\bar{\varphi}}$ instead. In the following, we prove the theorem in two cases.

(i) If $M \subseteq \mathrm{Ker}H_{\bar{f}}$, then $\bar{f}h \in H^2$ and $T_{\bar{\varphi}}h = P((\bar{f}+g)h) = \bar{\varphi}h \in M$, $\forall h \in M$. Replacing h by $\bar{\varphi}h$, we get

$$\bar{\varphi}^2 h = (\bar{f}^2 + 2\bar{f}g + g^2)h \in H^2.$$

Therefore, $\bar{f}^2 h \in H^2$. By induction we can get $h \in \{g \in H^2;\ \bar{f}^n g \in H^2,\ \forall n \geq 1\}$. Hence, $M = \{0\}$ follows from Lemma 2.3.

(ii) Assume that there exists a function $h_0 \in M$ such that $H_{\bar{f}}h_0 \neq 0$. Since $\varphi = k\bar{\varphi}$, $(I-P)(\bar{g} - k\bar{f}) = 0$, that is, there exists a function $\alpha \in H^2$ such that $\bar{g} = k\bar{f} + \alpha$. In view of the equality $H_{k\bar{f}} = S_k H_{\bar{f}}$, it is easy to check that

$$T_{\overline{\varphi}}T_\varphi - T_\varphi T_{\overline{\varphi}} = H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}} = H_{\bar{f}}^* H_{\bar{f}} - H_{k\bar{f}}^* H_{k\bar{f}} = H_{\bar{f}}^*(I - S_{\bar{k}}S_k)H_{\bar{f}}. \tag{1}$$

Since $I - S_{\bar{k}}S_k \geqslant 0$, we get

$$\|(I - S_{\bar{k}}S_k)^{1/2}H_{\bar{f}}h\|^2 = \langle H_{\bar{f}}^*(I - S_{\bar{k}}S_k)H_{\bar{f}}h,\ h\rangle = 0,\ \forall h \in M.$$

So $(I - S_{\bar{k}}S_k)H_{\bar{f}}h = 0$, or equivalently,

$$H_{\bar{f}}h = \bar{k}H_{k\bar{f}}h. \tag{2}$$

It follows that,

$$H_{k\bar{f}}h = S_k H_{\bar{f}}h = S_k(\bar{k}H_{k\bar{f}}h) = S_{|k|^2}H_{k\bar{f}}h.$$

Put $\bar{w}_0 = H_{k\bar{f}}h_0 = kH_{\bar{f}}h_0$, then $0 \neq w_0 \in zH^2$ and $S_{|k|^2}\bar{w}_0 = \bar{w}_0$. Notice that

$$\begin{aligned}\bar{z}w_0 = T_{\bar{z}}w_0 &= T_{\bar{z}}(\overline{(I-P)(\bar{w}_0)}) = T_{\bar{z}}(\overline{(I-P)(|k|^2\bar{w}_0)}) \\ &= T_{\bar{z}}T_{|k|^2}(w_0) = T_{|k|^2\bar{z}}w_0 = T_{|k|^2}(\bar{z}w_0).\end{aligned}$$

The second and fifth equalities follow from that $T_{\bar{z}}P(\bar{x}) = T_{\bar{z}}(\overline{(I-P)(x)})$, $\forall x \in L^2(\mathbb{T})$; the last equality follows from $\bar{z}w_0 \in H^2$. So we have

$$\bar{z}w_0 \in \ker T_{|k|^2-1} \bigcap \ker T^*_{|k|^2-1}. \tag{3}$$

Since $w_0 \neq 0$, Lemma 2.1 and (3) show that $|k| \equiv 1$ and then (2) implies that

$$T_{\bar{f}k}h = kT_{\bar{f}}h, \quad \forall h \in M.$$

It follows that $T_\varphi|_M = kT_{\bar{\varphi}}|_M$, i.e., $\bar{k}T_\varphi|_M = T_{\bar{\varphi}}|_M \in H^2$. Lemma 2.2 implies that $T_\varphi(M)$ is a dense subset of M, we get $M \subseteq \{\psi \in H^2;\ \bar{k}^n\psi \in H^2,\ \forall n \geq 1\}$. Hence Lemma 2.3 shows that $M = 0$, a contradiction. □

Similarly, we can prove the following corollary.

Corollary 2.4 *Let $f \in H^\infty$ and $\varphi = f + \lambda\overline{f}, \lambda \in \mathbb{C}$. Then the following statements hold.*

(i) If $|\lambda| = 1$, then T_φ is normal;

(ii) If $|\lambda| \neq 1$, then the following statements are equivalent:

(a) f is not a constant function;

(b) T_φ is not a normal operator;

(c) T_φ is completely non-normal.

Proof (i) If $|\lambda| = 1$, there exists $l \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\lambda = e^{2il}$. So $\varphi = f + \lambda\overline{f} = e^{il}(e^{-il}f + e^{il}\overline{f}) = e^{il}(e^{-il}f + \overline{e^{-il}f})$. Thus T_φ is normal by Theorem (BH).

(ii) (a)⇒(c). Let M be defined as in above theorem. Since

$$\begin{aligned}\langle (T_\varphi^*T_\varphi - T_\varphi T_\varphi^*)h, h\rangle &= \langle (1-|\lambda|^2)(T_f^*T_f - T_fT_f^*)h, h\rangle \\ &= (1-|\lambda|^2)(||T_fh||^2 - ||T_f^*h||^2), \quad \forall h \in H^2,\end{aligned}$$

for every $h \in M$, we have $\|T_{\bar{f}}h\| = \|P(\bar{f}h)\| = \|T_fh\| = \|fh\|$, which implies that $\overline{f}h \in H^2$. It is easy to check that $T_\varphi^*h = \overline{\varphi}h \in M$, and $T_\varphi h = \varphi h \in M$. Therefore $(\overline{\lambda}T_\varphi - T_\varphi^*)h = (|\lambda|^2 - 1)\overline{f}h \in M$, i.e., $\overline{f}h \in M$. Considering condition (a) and Lemma 2.3, we have $M \subseteq \{\psi \in H^2;\ \bar{f}^n\psi \in H^2,\ \forall n \geq 1\} = \{0\}$. Hence (c) holds.

The rest of the proof is obvious. □

Proof of Theorem 1.2 Suppose (i) holds. Without loss of generality, assume $\alpha_0 = 0$ since $\lambda - T_\varphi$ and T_φ have the same reducing subspaces. We also assume $m > n$. If $m < n$, we shall consider T_φ^* instead.

Let $k = [\frac{n}{m-n}] + 1$, where $[\frac{n}{m-n}]$ is the maximum integer which is less than $\frac{n}{m-n}$. Then $k \geq 1$ and $km \geq (k+1)n$. Suppose M is a non-trivial reducing subspace of T_φ such that $P_MT_\varphi|_M$ is normal, then we claim that

$$(z^lH^2) \cap M \subseteq (z^{l+1}H^2) \cap M, \quad \forall l \geq km.$$

For every $f \in z^lH^2 \cap M$, we can write $f = z^lf_1$, with $f_1 \in H^2$. Denote $h_1 = \alpha_1 z + \cdots + \alpha_m z^m$, $h_2 = \overline{\alpha_{-1}}z + \cdots + \overline{\alpha_{-n}}z^n$, $g_1 = \overline{h}_1 z^m$ and $g_2 = \overline{h}_2 z^n$. Then we have

$$\overline{\varphi}^j f = (\overline{h}_1 + h_2)^j f = [\bar{z}^m(g_1 + z^m h_2)]^j z^l f_1 = z^{l-jm}(g_1 + z^m h_2)^j f_1, \tag{4}$$

and

$$\varphi^i f = (h_1 + \overline{h}_2)^i f = [\bar{z}^n(g_2 + z^n h_1)]^i z^l f_1 = z^{l-in}(g_2 + z^n h_1)^i f_1. \tag{5}$$

Since $l \geq km$, there exist integers $k_1 \geq 0$ and $0 \leq k_2 < m$ such that $l = (k+k_1)m + k_2$. Then $l \geq (k+k_1)m \geq (k+k_1+1)n$ and the following statements hold:

(iii) $\overline{\varphi}^j f \in H^2$, for $0 \leq j \leq k + k_1$;

(iv) $\varphi^i f \in H^2$, for $0 \leq i \leq (k+k_1)+1$.

Observe that $M \subseteq \mathrm{Ker}(T_\varphi^{*(k+k_1)+1} T_\varphi^{(k+k_1)+1} - T_\varphi^{(k+k_1)+1} T_\varphi^{*(k+k_1)+1})$,

$$\begin{aligned}\|T_\varphi^{(k+k_1)+1} f\| = \|T_\varphi^{*(k+k_1)+1} f\| &\leq \|\overline{\varphi}^{(k+k_1)+1} f\| \\ &= \|\varphi^{(k+k_1)+1} f\| = \|T_\varphi{}^{(k+k_1)+1} f\|.\end{aligned}$$

Therefore,

$$\|T_\varphi^{*(k+k_1)+1} f\| = \|\overline{\varphi}^{(k+k_1)+1} f\|,$$

which implies that $\overline{\varphi}^{(k+k_1)+1} f \in H^2$. So the statement (iii) can be replaced by

(iii)$'$ $\overline{\varphi}^j f \in H^2$, for $0 \leq j \leq k + k_1 + 1$.

Furthermore, a straightforward computation shows that

$$\begin{aligned}\overline{\varphi}^{m+1} f =& (\overline{h}_1 + h_2)^{m+1} f = \overline{h}_1^{m+1} f + C_{m+1}^1 \overline{h}_1^m h_2 f + C_{m+1}^2 \overline{h}_1^{m-1} h_2{}^2 f + \cdots + \\ & h_2{}^{m+1} f, \quad \forall m \in \mathbb{Z}^+.\end{aligned} \tag{6}$$

Combining with (iii)$'$, we have $\overline{h}_1^j f \in H^2$ for $0 \leq j \leq k + k_1 + 1$. Thus $z^{k_2 - m} g_1^{k+k_1+1} f_1 = z^{l-(k+k_1+1)m} g_1^{k+k_1+1} f_1 = \bar{z}^{(k+k_1+1)m} g_1^{k+k_1+1} f = \overline{h}_1^{k+k_1+1} f \in H^2$. Since $k_2 - m \leq -1$ and $g_1^{k+k_1+1}(0) = \overline{\alpha}_m^{k+k_1+1} \neq 0$, we have $\bar{z} f_1 \in H^2$, i.e., $f \in z^{l+1}H^2$. So we complete the proof of the claim.

Therefore, $(z^{km}H^2) \cap M \subseteq (\bigcap_{l=0}^{\infty} z^{km+l}H^2) = \{0\}$. It follows that $\mathrm{dim}M \leq \dim(H^2/z^{km}H^2) = km < +\infty$. Hence $\sigma_p(T_\varphi|_M) = \sigma(T_\varphi|_M) \neq \emptyset$. However, Lemma 2.2 shows that $\sigma_p(T_\varphi|_M) = \emptyset$, which induces a contradiction.

It remains to prove that if condition (ii) holds, then T_φ is completely non-normal. Let $\lambda = \frac{\alpha_{-n}}{\overline{\alpha}_n}$ and $\psi = \varphi - \lambda\overline{\varphi} = \sum_{k=-n+1}^{n} (\alpha_k - \lambda\overline{\alpha}_{-k}) z^k$. For this suppose, it is easy to see that if M is a reducing subspace of T_φ such that $P_M T_\varphi|_M$ is normal, then also is $P_M T_\psi|_M$. However, by (i) we have T_ψ is c.n.n. Hence $M = \{0\}$ as desired. $\square$

Remark The corollary 1.5 in [8] shows that if $\varphi = \sum_{k=-n}^{m} \alpha_k z^k$, then T_φ is normal if and only if $m = n$, $|\alpha_{-n}| = |\alpha_n|$, and

$$\overline{\alpha_n} \begin{pmatrix} \alpha_{-1} \\ \alpha_{-2} \\ \vdots \\ \vdots \\ \alpha_{-n} \end{pmatrix} = \alpha_{-n} \begin{pmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \\ \vdots \\ \vdots \\ \overline{\alpha_n} \end{pmatrix}.$$

It means that there exists $\varphi = \sum_{k=-n}^{m} \alpha_k z^k$ with $m = n$ and $|\alpha_{-n}| = |\alpha_n|$ such that T_φ is not normal. However, it is not clear whether T_φ is c.n.n.

3. Cyclicity of hyponormal Toeplitz operators

In this section, denote by $\sigma_{ap}(T)$, $\sigma_l(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$ and $\sigma_{lre}(T)$ the approximate point spectrum, the left spectrum, the right spectrum, the left essential spectrum, the right essential spectrum and the Wolf spectrum of T respectively. And let $m(\cdot)$ be the planar Lebesgue measure.

The research on operator's cyclicity is also an important part of operator theory. It has been investigated by a number of authors. In 1976, Deddens considered the problem of determining which subnormal operator has a cyclic adjoint. In 1998, Feldman [10] gave a complete answer to the question of Deddens. In [13], Wogen also considered the same problem for hyponormal operators. The following result was partly obtained by Clancey and Rogers in [4]. This problem still remains open.

Theorem (CR) *If T is a completely non-normal cohyponormal operator, such that $m(\sigma_r(T)) = 0$, then T has a dense set of cyclic vectors.*

In 1988, Cowen [6] proved that if $f, g \in H^2$ with $\varphi = f + \overline{g} \in L^\infty(\mathbb{T})$, then T_φ is hyponormal if and only if there exist a constant c and a function $k \in H^\infty$ with $\|k\|_\infty \leq 1$ such that $g = T_{\bar{k}} f + c$. Later the hyponormality of the Toeplitz operator has been widely discussed, see [8, 9, 11] for example. So it is not difficult to give some cohyponormal operators which are cyclic. Here we need some lemmas about the spectrum of completely non-normal hyponormal operators.

Lemma 3.1 *If $T \in \mathcal{B}(H)$ is a completely non-normal hyponormal operator, then $\sigma_l(T) = \sigma_{lre}(T) = \sigma_{le}(T)$.*

Proof Assume T is hyponormal and c.n.n., then $\lambda - T$ is hyponormal for every $\lambda \in \mathbb{C}$ since $(\lambda - T)^*(\lambda - T) - (\lambda - T)(\lambda - T)^* = T^*T - TT^*$. Let $M = \mathrm{Ker}(\lambda - T)$. Then M is a reducing subspace of $\lambda - T$ and $P_M T|_M = \lambda P_M I|_M$ is normal. By the assumption, we get $M = \{0\}$, which implies that $\sigma_p(T) = \emptyset$, i.e., $\sigma_{ap}(T) \subseteq \sigma_{lre}(T)$. On the other hand, it is obvious that $\sigma_{lre}(T) \subseteq \sigma_{le}(T) \subseteq \sigma_l(T) = \sigma_{ap}(T)$. Hence the proof is completed. $\Box$

Corollary 3.2 *Let $\varphi \in L^\infty(\mathbb{T})$ be as in Theorem 1.1 and $m(\varphi(\mathbb{T})) = 0$. Then $T_{\bar{\varphi}}$ has a dense set of cyclic vectors.*

Proof From Cowen's Theorem and the proof of Theorem 1.1, we see that T_φ is hyponormal. By Lemma 3.1 and Corollary 7.14 in [7], we have $\sigma_l(T_\varphi) = \sigma_{le}(T_\varphi) \subseteq \varphi(\mathbb{T})$ and $m(\sigma_r(T_{\overline{\varphi}})) = m(\sigma_l(T_\varphi)^*) = 0$, where $\sigma_l(T_\varphi)^* = \{\lambda; \bar{\lambda} \in \sigma_l(T_\varphi)\}$. The desired result is obvious by combining Theorem 1.1 with Theorem (CR). $\Box$

In the following, we concern about the Toeplitz operators with trigonometric polynomial symbols. Although the following lemma may be well known, we show the detail for readers' convenience.

Lemma 3.3 *If $\varphi = \sum_{k=-N}^{m} \alpha_k z^k$ with $\alpha_{-N}\alpha_m \neq 0$, then $m(\varphi(\mathbb{T})) = 0$.*

In order to prove above lemma, we define a map Γ from the space of complex-valued functions on $\mathbb{C}$ to the space of 2×2 matrix-valued functions as follows:

$$\Gamma:\ f\mapsto\begin{pmatrix} u & -v\\ v & u\end{pmatrix},$$

where $f(z)=u(x,\ y)+iv(x,\ y), z=x+iy$, and $u(x,\ y),\ v(x,\ y)$ are real-valued functions. The map Γ has the following properties.

1) $\Gamma(\bar f)=\begin{pmatrix} u & v\\ -v & u\end{pmatrix}$.

2) For every $a,b\in\mathbb{R}$,

(i) $\Gamma(af+bg)=a\Gamma(f)+b\Gamma(g)$;

(ii) $\Gamma((a+bi)fg)=\begin{pmatrix} a & -b\\ b & a\end{pmatrix}\Gamma(f)\Gamma(g)=\Gamma(a+bi)\Gamma(f)\Gamma(g)$;

(iii) $\Gamma(f)\Gamma(g)=\Gamma(g)\Gamma(f)$.

3) $|\Gamma(f)(x,\ y)|=|f(x,\ y)|^2$, $(x,\ y)\in\mathbb{R}^2$.

4) If $f=u+vi$ is analytic on some region, then

$$\Gamma(\frac{\partial f}{\partial z})=\Gamma(\frac{\partial f}{\partial x})=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix},\quad \Gamma(\frac{\partial \bar f}{\partial \bar z})=\begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y}\\ -\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}.$$

Proof of Lemma 3.3 Let $\varphi=f+\bar g$ where $f=\sum_{k=0}^m\alpha_k z^k$, $g=\sum_{k=0}^N\bar\alpha_{-k}z^k$ are polynomials. Write $f=u_1+iv_1$ and $g=u_2+iv_2$ where $u_i,\ v_i$ are real-valued harmonic functions. Let $J\varphi$ be the Jacobian of φ. A computation shows that

$$\begin{aligned}|J\varphi| &= \left|\begin{pmatrix} \frac{\partial}{\partial x}(u_1+u_2) & \frac{\partial}{\partial y}(u_1+u_2)\\ \frac{\partial}{\partial x}(v_1-v_2) & \frac{\partial}{\partial y}(v_1-v_2)\end{pmatrix}\right|\\ &= (\frac{\partial u_1}{\partial x})^2+(\frac{\partial u_1}{\partial y})^2-(\frac{\partial u_2}{\partial x})^2-(\frac{\partial u_2}{\partial y})^2\\ &= |\Gamma(\frac{\partial f}{\partial z})|-|\Gamma(\frac{\partial g}{\partial z})|=|\frac{\partial f}{\partial z}|^2-|\frac{\partial g}{\partial z}|^2.\end{aligned}$$

Note that $z^{N+m}(|\frac{\partial f}{\partial z}|^2-|\frac{\partial g}{\partial z}|^2)=\frac{\partial f}{\partial z}z^N(\sum_{k=1}^m k\bar\alpha_k z^{m-k+1})-\frac{\partial g}{\partial z}z^m(\sum_{k=1}^N k\alpha_{-k}z^{N-k+1})$, for every $z\in\mathbb{T}$. Since the right part of the equality is an analytic polynomial, it vanishes on finite points. Therefore for each $\delta\in(0,1)$, we can find an open set Ω_δ such that

$$\mathbb{T}\subseteq\Omega_\delta\subseteq\{\lambda\in\mathbb{C};1-\delta<|\lambda|<1+\delta\},$$

and the number of zeros of $|J\varphi|$ in Ω_δ is finite.

Let $E_\delta=\{(x,\ y)\in\Omega_\delta;\ |J\varphi(x+yi)|=0\}$. For each $(x,\ y)\in\Omega_\delta\backslash E_\delta$, there exists a neighborhood $U_{(x,\ y)}\subseteq\Omega_\delta$ such that $\varphi:\ U_{(x,\ y)}\to\varphi(U_{(x,\ y)})$ is a homeomorphism and $|J\varphi|$ has no zero points in $U_{(x,\ y)}$. Since $\{U_{(x,y)}\}$ is an open covering of the compact set $\varphi(\mathbb{T})\backslash\varphi(E_\delta)$, there exists a finite subfamily $\{U_1,U_2,\dots,U_N\}$ such that

$$\bigcup_{k=1}^N\varphi(U_k)\supseteq\varphi(\mathbb{T})\backslash\varphi(E_\delta).$$

By induction $\{U_1,U_2,\dots,U_N\}$ can be replaced by a subfamily such that no open set U_i is contained in the union of the others and such that the refined family has the same union as the original family. Write $I_j = U_j \backslash (\overline{\bigcup_{k=1}^{j-1} U_k})$. $\{I_j\}$ is a family of pairwise disjoint open sets. Hence,

$$\begin{aligned} m(\varphi(\mathbb{T})\backslash\varphi(E_\delta)) &\leqslant \sum_{k=1}^{N}\int_{\varphi(I_k)} 1\mathrm{d}\sigma + \int_{\varphi(\bigcup_{k=1}^{N} U_k \backslash \bigcup_{j=1}^{N} I_j)} 1\mathrm{d}\sigma \\ &= \sum_{k=1}^{N}\int_{I_k} |J\varphi|\mathrm{d}\sigma + \int_{\bigcup_{k=1}^{N} U_k \backslash \bigcup_{j=1}^{N} I_j} |J\varphi|\mathrm{d}\sigma \\ &\leqslant \|\,|\,J\varphi\,|\,\|_\infty \big(\sum_{k=1}^{N} m(I_k) + m(\Omega_\delta)\big) \\ &\leqslant 2\|\,|\,J\varphi\,|\,\|_\infty m(\Omega_\delta). \end{aligned}$$

Thus $0 \leqslant m(\varphi(\mathbb{T})) = \lim_{\delta\to 0} m(\varphi(\mathbb{T})\backslash\varphi(E_\delta)) \leqslant \lim_{\delta\to 0} 2\|\,|\,J\varphi\,|\,\|_\infty m(\Omega_\delta) = 0$ as desired. $\Box$

Theorem 3.4 *If $\varphi = \sum_{k=-N}^{m} \alpha_k z^k$ with $\alpha_{-N}\alpha_m \neq 0$ such that T_φ is hyponormal and non-normal, then $T_{\bar{\varphi}}$ has a dense set of cyclic vectors.*

Proof From Theorem 1.4 in [8] and Corollary 1.5 in [8], it is easy to see that if $m = n$ and $|\alpha_{-n}| = |\alpha_m|$, then the hyponormality and normality are equivalent. So Theorem 1.2 implies that T_φ is c.n.n. On the other hand, Lemmas 3.3 and 3.1 show that $m(\sigma_r(T_{\bar{\varphi}})) = m(\varphi(\mathbb{T})) = 0$. Now T_φ satisfies the assumption of Theorem (CR). The desired result is obvious. $\Box$

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