Journal of Mathematical Research & Exposition
Jul., 2011, Vol. 31, No. 4, pp. 742–748
DOI:10.3770/j.issn:1000-341X.2011.04.021
Http://jmre.dlut.edu.cn

Convergence of Composition Operators on Hardy-Smirnov Space

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Abstract We consider the convergence of composition operators on Hardy-Smirnov space over a simply connected domain properly contained in the complex plane. The convergence of the power of a composition operator is also considered. Our approach is a method from Joel H. Shapiro and Wayne Smith's celebrated work (Journal of Functional Analysis 205 (2003) 62-89). The resulting space is usually not the one obtained from the classical Hardy space of the unit disc by conformal mapping.

Keywords composition operators; convergence; Hardy-Smirnov space.

Document code A
MR(2010) Subject Classification 41A20; 65D05
Chinese Library Classification O241.5

1. Introduction

For a simply connected domain G that is properly contained in the complex plane, let τ be a Riemann map that takes the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ univalently onto G. Let Γ_r denote the τ-image of the circle $|z| = r$. For $0 < p < \infty$, let $\Lambda^p(G)$ be the collection of functions F holomorphic on G such that

$$\|F\|_{\Lambda^p(G)} = \Big(\sup_{0<r<1} \int_{\Gamma_r} |F(w)|^p |\mathrm{d}w|\Big)^{1/p} < \infty. \tag{1}$$

Following [4] and [14], we call these the Hardy-Smirnov spaces of G.

The classical Hardy space H^p $(0 < p < \infty)$ over the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the collection of functions analytic on the unit disc $\mathbb{U}$, satisfying

$$\|f\|_{H^p} = \Big(\sup_{0<r<1} \int_{\mathbb{T}} |f(r\xi)|^p \mathrm{d}m(\xi)\Big)^{1/p} < \infty, \tag{2}$$

where $\mathbb{T}$ is the unit circle $|z| = 1$, m is the normalized arc-length measure on $\mathbb{T}$.

If Φ is a function holomorphic on G with $\Phi(G) \subset G$, then Φ induces a linear composition operator (see [14]) C_Φ on the space $\mathrm{Hol}(G)$ of all functions holomorphic on G as follows

$$C_\Phi F = F \circ \Phi, \quad F \in \mathrm{Hol}(G).$$

Received November 20, 2009; Accepted January 13, 2011
Supported by the Natural Science Foundation of Yunnan Province (Grant No. 2009ZC013X) and Basic Research Foundation of Education Bureau of Yunnan Province (Grant No. 09Y0079).
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If G is the unit disc, various prospects of such operators have been extensively studied [3, 10–14].

In [14], inspired by an earlier result of Matache [6], Shapiro and Smith investigated Hardy-Smirnov spaces that support compact operators. Their celebrated work takes place on the space $\Lambda^p(G)$ which is defined by (1). From [4] and [14], we know that the map $C_\tau : F \to F \circ \tau$ is an isomorphism of $\Lambda^p(G)$ onto H^p if and only if both τ' and its reciprocal are bounded on $\mathbb{U}$. Either τ' or its reciprocal is unbounded on $\mathbb{U}$, our Hardy-Smirnov spaces are different from the conformally invariant ones. That is the point why we want to generalize the results on convergence of composition operators on the Hilbert Hardy space H^2 over the unit disc in [7].

In this paper, we will use the approach in [14] to consider the convergence of composition operators C_Φ on the Hardy-Smirnov spaces defined in (1).

Let us introduce some terminology from [14] that is proven to be quite useful and will be heavily used. For a simply connected domain G that is properly contained in the complex plane, let τ be a Riemann map that takes the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ univalently onto G. To every self-map Φ of G, the self-map φ of $\mathbb{U}$ is defined as follows

$$\varphi = \tau^{-1} \circ \Phi \circ \tau. \tag{3}$$

For each index $0 < p < \infty$, the operator V_p is defined as follows

$$(V_p F)(z) = \tau'(z)^{1/p} F(\tau(z)), \quad F \in \mathrm{Hol}(G),\ z \in \mathbb{U}. \tag{4}$$

The weighted operator

$$A_{\varphi,p} = V_p C_\Phi V_p^{-1}$$

maps H^p boundedly into itself if and only if C_Φ is bounded on $\Lambda^p(G)$. Since V_p establishes an isometric similarity between the two operators (for details see [14, p.66-67]), for a sequence of composition operators $\{C_{\Phi_n}\}$ induced by self-maps $\{\Phi_n\}$ of G on $\Lambda^p(G)$, its norm convergence is equivalent to norm convergence of the corresponding sequence of composition operators $\{A_{\varphi_n,p}\}$ induced by $\{\varphi_n = \tau^{-1} \circ \Phi_n \circ \tau\}$ on H^p. Thus for $f \in H^p$,

$$(A_{\varphi,p})f(z) = Q_\varphi(z)^{1/p} f(\varphi(z)), \text{ where } Q_\varphi(z) = \frac{\tau'(z)}{\tau'(\varphi(z))}, \quad z \in \mathbb{U}. \tag{5}$$

Using the operator $A_{\varphi,p}$, we can generalize the results on convergence of composition operators on the Hilbert Hardy space H^2 over the unit disc in [7].

Motivated by the work in [9], in [7], Matache considered convergent sequence of composition operators $\{C_{\varphi_n}\}$ induced by $\{\varphi_n\}$ that converges in some sense to φ. Matache supposed $\{\varphi_n\}$ and φ are bounded analytic self-maps of $\mathbb{U}$ endowed with the norm

$$\|f\|_\infty = \sup_{0<r<1} |f(z)|, \quad f \in \mathrm{Hol}(\mathbb{U}). \tag{6}$$

He pointed out that only the case

$$|\varphi(\xi)| < 1 \quad m\text{-a.e.} \tag{7}$$

on $\mathbb{T}$ is valuable for consideration.

Under the assumptions above, Matache proved that if C_φ is a Hilbert-Schmidt operator, then $\|C_{\varphi_n} - C_\varphi\|_{\mathrm{HS}} \to 0$ if and only if $\|C_{\varphi_n}\|_{\mathrm{HS}} \to \|C_\varphi\|_{\mathrm{HS}}$ and $\|\varphi_n - \varphi\|_{H^2} \to 0$. Let $\{\Phi_n\}$ be a sequence of self-maps of G whose counterpart $\{\varphi_n\}$ is defined in (3), where $\{\varphi_n\}$ and φ are bounded in the norm (6) and satisfy (7). Let $\{Q_{\varphi_n}(z)\}$ and $\{A_{\varphi_n,p}\}$ be defined in (5). In Section 2, we show that if

$$\int_{\mathbb{T}} \frac{|Q_\varphi|}{1-|\varphi|^2} \mathrm{d}m < \infty, \tag{8}$$

then with some restrictions on $\{Q_{\varphi_n}\}$, $\|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$ if and only if $\|A_{\varphi_n,2}\|_{\mathrm{HS}} \to \|A_{\varphi,2}\|_{\mathrm{HS}}$ and $\|Q_{\varphi_n}^{\frac{1}{2}} - Q_{\varphi}^{\frac{1}{2}}\|_{H^2} \to 0$.

Matache also considered the sequence $\{C_\varphi^n\}$ for non-inner φ. Matache proved that if φ has a fixed point b in $\mathbb{U}$, then $\|C_\varphi^n - C_b\|_{H^2} \to 0$. We will investigate the same problem in $\Lambda^p(G)$. We prove if τ' and its reciprocal are in H^2 and φ defined by (3) is non-inner in $\mathbb{U}$, Φ has a fixed point a in G; if $\|Q_{\varphi^{[n]}}\|_{H^2}$ is uniformly bounded, for the proper subclass $F \in \Lambda^2(G) \subset \Lambda^1(G)$, $\{\|C_\Phi^n F\|_{\Lambda^1(G)}\}$ converges.

2. Norm convergence of a sequence of composition operators

Recall that on any Hilbert space, the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ of an operator T is defined as

$$\|T\|_{HS}^2 = \sum_{n=0}^{\infty} \|Te_n\|^2,$$

where $\{e_n\}$ is an orthonormal basis [7, 11]. From the discussion in [7, p. 662], we know that $\|T\|_{\mathrm{HS}}$ is larger than or equals the operator norm $\|T\|$. Since V_p establishes an isometric similarity between the two operators (for details see [14, pp. 66–67]), we just need to investigate the case where $\|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$, which is equivalent to $\|C_{\Phi_n} - C_\Phi\|_{\mathrm{HS}(G)} \to 0$.

The main result of this section is as follows.

Theorem 1 *Let $\{\Phi_n\}$ be a sequence of self-maps of G and $\{\varphi_n\}$ be defined in (3), replacing $\{\Phi\}$ by $\{\Phi_n\}$. Let $\{Q_{\varphi_n}(z)\}$ and $\{A_{\varphi_n,2}\}$ be defined in (5). If $\{\varphi_n\}$ and φ are bounded in the norm (6), satisfying (7), $\{\varphi_n\} \to \varphi$ a.e. on $\mathbb{T}$, $Q_\varphi(z)$ satisfies (8), moreover, there is some function $\chi(\varphi)$ defined on $\mathbb{T}$ such that*

$$\frac{|Q_{\varphi_n}|}{1-|\varphi_n|^2} \le \chi(\varphi), \quad m\text{-a.e.} \tag{9}$$

and

$$\int_{\mathbb{T}} \chi(\varphi) \mathrm{d}m < \infty, \tag{10}$$

then $\|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$ if and only if $\|A_{\varphi_n,2}\|_{\mathrm{HS}} \to \|A_{\varphi,2}\|_{\mathrm{HS}}$ and $\|Q_{\varphi_n}^{\frac{1}{2}} - Q_{\varphi}^{\frac{1}{2}}\|_{H^2} \to 0$.

Proof As $\{1, z, z^2, z^3, \ldots\}$ is the standard basis of H^2, the direct calculation of Hilbert-Schmidt norm (see also [14, 2.2 Example]) shows that if $Q_\varphi(z)$ satisfies (8), then $A_{\varphi,2}$ is a Hilbert-Schmidt composition operator.

If $\|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$, then it is obvious that $\|A_{\varphi_n,2}\|_{\mathrm{HS}} \to \|A_{\varphi,2}\|_{\mathrm{HS}}$, and we have

$$\|Q_{\varphi_n}^{\frac{1}{2}} - Q_{\varphi}^{\frac{1}{2}}\|_{H^2} = \|A_{\varphi_n,2}(1) - A_{\varphi,2}(1)\|_{\mathrm{HS}} \le \|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0.$$

Conversely, if $\|A_{\varphi_n,2}\|_{\mathrm{HS}} \to \|A_{\varphi,2}\|_{\mathrm{HS}}$ and $\|Q_{\varphi_n}^{\frac{1}{2}} - Q_{\varphi}^{\frac{1}{2}}\|_{H^2} \to 0$, but $\|A_{\varphi_n,2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$ does not hold, we will get a contradiction. In this case, for some $\varepsilon_0 > 0$, there is a subsequence $\{A_{\varphi_{n_k},2}\}$ of $\{A_{\varphi_n,2}\}$ satisfying

$$\|A_{\varphi_{n_k},2} - A_{\varphi,2}\|_{\mathrm{HS}} \geq \varepsilon_0. \tag{11}$$

Since $\|Q_{\varphi_{n_k}}^{\frac{1}{2}} - Q_{\varphi}^{\frac{1}{2}}\|_{H^2} \to 0$, we can select a subsequence $\{Q_{\varphi_{m_k}}^{\frac{1}{2}}\}$ of $\{Q_{\varphi_{n_k}}^{\frac{1}{2}}\}$ such that $\{Q_{\varphi_{m_k}}^{\frac{1}{2}}\}$ converges a.e. to $Q_{\varphi}^{\frac{1}{2}}$. We have

$$\|A_{\varphi_{m_k},2} - A_{\varphi,2}\|_{\mathrm{HS}} = \int_{\mathbb{T}} \frac{|Q_{\varphi_{m_k}}|}{1-|\varphi_{m_k}|^2}\mathrm{d}m + \int_{\mathbb{T}} \frac{|Q_{\varphi}|}{1-|\varphi|^2}\mathrm{d}m - 2\Re \int_{\mathbb{T}} \frac{Q_{\varphi_{m_k}}^{\frac{1}{2}} \overline{Q_{\varphi}^{\frac{1}{2}}}}{1-\varphi\varphi_{m_k}}\mathrm{d}m. \tag{12}$$

By (9), we have

$$\frac{|Q_{\varphi_{m_k}}^{\frac{1}{2}} \overline{Q_{\varphi}^{\frac{1}{2}}}|}{1-|\varphi\varphi_{m_k}|} \leq \frac{|Q_{\varphi_{m_k}}^{\frac{1}{2}} Q_{\varphi}^{\frac{1}{2}}|}{(1-|\varphi_{m_k}|^2)^{\frac{1}{2}}(1-|\varphi|^2)^{\frac{1}{2}}} \leq \chi(\varphi) + \frac{|Q_{\varphi}|}{1-|\varphi|^2}, \quad m\text{-a.e.}$$

By the a.e. convergence and the dominated convergence theorem, combining (8) and (10), we have

$$\int_{\mathbb{T}} \frac{Q_{\varphi_{m_k}}^{\frac{1}{2}} \overline{Q_{\varphi}^{\frac{1}{2}}}}{1-|\varphi\varphi_{m_k}|}\mathrm{d}m \to \int_{\mathbb{T}} \frac{|Q_{\varphi}|}{1-|\varphi|^2}\mathrm{d}m, \tag{13}$$

and

$$\int_{\mathbb{T}} \frac{|Q_{\varphi_{m_k}}|}{1-|\varphi_{m_k}|^2}\mathrm{d}m \to \int_{\mathbb{T}} \frac{|Q_{\varphi}|}{1-|\varphi|^2}\mathrm{d}m. \tag{14}$$

From (12), (13) and (14), we have

$$\|A_{\varphi_{m_k},2} - A_{\varphi,2}\|_{\mathrm{HS}} \to 0$$

which contradicts (11).

3. Powers of composition operators

In this section we treat the convergence of the operator sequence $\{C_{\Phi}^n F\}$ for $F \in \Lambda^p(G)$ where $C_{\Phi}^n F = F(\Phi^{[n]})$ and $\Phi^{[n]} = \Phi \circ \cdots \circ \Phi$ is the n-fold iteration of Φ. From the reasoning in Section 2, we only need to investigate the corresponding sequence $\{A_{\varphi,p}^n\}$ in H^p, here

$$A_{\varphi,p}^n f(z) = Q_{\varphi^{[n]}}(z)^{1/p} f(\varphi^{[n]}(z)),\ Q_{\varphi^{[n]}}(z) = \frac{\tau'(z)}{\tau'(\varphi^{[n]}(z))}, \quad z \in \mathbb{U}, \tag{15}$$

where $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$ is the n-fold iteration of φ. Suppose Φ is a self-map of G satisfying $\Phi(G) \subset G$ and for some $a \in G, \Phi(a) = a$, then φ defined in (3) has a fixed point $b = \tau^{-1}(a)$ in $\mathbb{U}$ (see [14]).

Recall an inner function φ is an analytic self-map of $\mathbb{U}$ whose radial limit-function is unimodular m-a.e. on $\mathbb{T}$ (see [4, 5, 8]). From [8, p. 353, Exercise 6], we know that H^2 is a proper subclass of H^1.

The main results of this section are as follows.

Theorem 2 *Let Φ be a self-map of G satisfying $\Phi(G) \subset G$ and for some $a \in G$, $\Phi(a) = a$.*

If there is a Riemann map τ from G to $\mathbb{U}$ such that both τ' and its reciprocal are in H^2, $\{\|Q_{\varphi^{[n]}}\|_{H^2}\}$ is uniformly bounded, φ defined in (3) is bounded in the norm (6) and non-inner, then $\{\|C_\Phi^n F\|_{\Lambda^1(G)}\}$ converges for the proper subclass $F \in \Lambda^2(G) \subset \Lambda^1(G)$.

Theorem 3 *Let Φ be a self-map of G satisfying $\Phi(G) \subset G$ and for some $a \in G, \Phi(a) = a$. If there is a Riemann map τ from G to $\mathbb{U}$ such that τ' and its reciprocal are bounded on $\mathbb{T}$ in the norm (6) and non-inner, then for each $F \in \Lambda^p(G)$ and $0 < p < \infty$, the sequence $\{\|C_\Phi^n F\|_{\Lambda^p(G)}\}$ converges.*

In order to prove the theorems, we need the following lemmas.

Lemma 1 ([8, p. 339]) *Suppose $0 < p < \infty$, $f \in H^p$ and f is not equivalently zero, B is the Blaschke product formed with the zeros of f. Then there is a zero-free function $h \in H^2$ such that*

$$f = B \cdot h^{2/p}. \tag{17}$$

Lemma 2 ([14]) *If a composition operator C_Φ is bounded on $\Lambda^p(G)$ for some $0 < p < \infty$, then it is bounded for all such p. Actually, for arbitrary $0 < p < \infty$ and $0 < q < \infty$,*

$$\|A_{\varphi,p}\|^p = \|A_{\varphi,q}\|^q.$$

The following lemma is the main result of Section 3 in [7], here we cite it as a lemma for later use. For $f \in H^p$, let $C_a f = f(a)$. From [7] we know it is a bounded linear operator in H^p.

Lemma 3 ([7]) *Let φ be a self-map of $\mathbb{U}$ with a fixed point in $\mathbb{U}$ which is bounded in the norm defined in (6) and non-inner. Then for $f \in H^2$, $C_\varphi^n f(z) = f(\varphi^{[n]})(z)$ and $C_b f(z) = f(b)$, $\|C_\varphi^n - C_b\|_{H^2} \to 0$ as $n \to \infty$.*

Proof of Theorem 2 Since convergence of $\{\|C_\Phi^n F\|_{\Lambda^p(G)}\}$ is equivalent to the convergence of $\{\|A_{\varphi,p}^n f\|_{H^p}\}$, we just need to show $\|A_{\varphi,p}^n f - A_{b,p} f\|_{H^p} \to 0$ where $A_{b,p} f(z) = \frac{\tau'^{1/p}(z)}{\tau'^{1/p}(b)} f(b), b = \tau^{-1}(a)$. If

$$\|A_{\varphi,1}^n f - A_{b,1} f\|_{H^1} \to 0, \tag{18}$$

for the proper subclass $f \in H^2 \subset H^1$, then Theorem 2 follows. By the triangular inequality, for $f \in H^2 \subset H^1$ and $\|f\|_{H^1} = 1$

$$\begin{aligned}\|A_{\varphi,1}^n - A_{b,1}\|_{H^1} &= \|Q_{\varphi^{[n]}} f(\varphi^{[n]}) - \tau'/\tau'(b) \cdot f(b)\|_{H^1} \\ &\le \|Q_{\varphi^{[n]}} f(\varphi^{[n]}) - Q_{\varphi^{[n]}} f(b)\|_{H^1} + \|Q_{\varphi^{[n]}} f(b) - \tau'/\tau'(b) \cdot f(b)\|_{H^1}.\end{aligned}$$

Then we just need to prove

$$\|Q_{\varphi^{[n]}} f(\varphi^{[n]}) - Q_{\varphi^{[n]}} f(b)\|_{H^1} \to 0 \tag{19}$$

and

$$\|Q_{\varphi^{[n]}} f(b) - \tau'/\tau'(b) \cdot f(b)\|_{H^1} \to 0. \tag{20}$$

By triangular inequality and the decomposition $f = B \cdot h$ for $f \in H^2 \subset H^1$ where $h \in H^2$ is the decomposition in Lemma 1,

$$\begin{aligned}&\|Q_{\varphi^{[n]}}f(\varphi^{[n]})-Q_{\varphi^{[n]}}f(b)\|_{H^1}=\|Q_{\varphi^{[n]}}B(\varphi^{[n]})h(\varphi^{[n]})-Q_{\varphi^{[n]}}B(b)h(b)\|_{H^1}\\&\quad\le\|Q_{\varphi^{[n]}}B(\varphi^{[n]})h(\varphi^{[n]})-Q_{\varphi^{[n]}}B(\varphi^{[n]})h(b)\|_{H^1}+\|Q_{\varphi^{[n]}}B(\varphi^{[n]})h(b)-Q_{\varphi^{[n]}}B(b)h(b)\|_{H^1}.\end{aligned}$$

By the boundedness of $B(z)$, we have

$$\|Q_{\varphi^{[n]}}f(\varphi^{[n]})-Q_{\varphi^{[n]}}f(b)\|_{H^1}\le\|Q_{\varphi^{[n]}}(h(\varphi^{[n]})-h(b))\|_{H^1}+|h(b)|\cdot\|Q_{\varphi^{[n]}}(B(\varphi^{[n]})-B(b))\|_{H^1}. \tag{21}$$

Since $\{\|Q_{\varphi^{[n]}}\|_{H^2}\}$ is uniformly bounded, there exists some $M>0$, such that $\|Q_{\varphi^{[n]}}\|_{H^2}\le M$. By the Hölder's inequality, we have

$$\begin{aligned}\|Q_{\varphi^{[n]}}(h(\varphi^{[n]})-h(b))\|_{H^1}&\le\|Q_{\varphi^{[n]}}\|_{H^2}^{1/2}\cdot\|h(\varphi^{[n]})-h(b)\|_{H^2}^{1/2}\\&\le M^{1/2}\cdot\|h(\varphi^{[n]})-h(b)\|_{H^2}^{1/2}.\end{aligned}$$

From Lemma 3 and $h\in H^2$, it follows

$$\|h(\varphi^{[n]})-h(b)\|_{H^2}=\|C_\varphi^n h-C_b h\|_{H^2}\le\|h\|_{H^2}\cdot\|C_\varphi^n-C_b\|_{H^2}\to 0.$$

From the boundedness of

$$\|1/\tau'(\varphi^{[n]})\|_{H^2}=\|C_\varphi^n(1/\tau')\|_{H^2},$$

we have

$$\|Q_{\varphi^{[n]}}(h(\varphi^{[n]})-h(b))\|_{H^1}\to 0. \tag{22}$$

Repeating the above reasoning gives

$$\begin{aligned}\|Q_{\varphi^{[n]}}(B(\varphi^{[n]})-B(b))\|_{H^1}&\le\|Q_{\varphi^{[n]}}\|_{H^2}^{1/2}\cdot\|(B(\varphi^{[n]})-B(b))\|_{H^2}^{1/2}\\&\le M^{1/2}\cdot\|(B(\varphi^{[n]})-B(b))\|_{H^2}^{1/2}.\end{aligned}$$

By the boundedness of $B(z)$, from [1] and [11], we know that $B(\varphi^{[n]})$ converges to $B(b)$ a.e.. Thus applying the bounded convergence theorem yields

$$\|Q_{\varphi^{[n]}}(B(\varphi^{[n]})-B(b))\|_{H^1}\to 0. \tag{23}$$

From (21), (22) and (23), we can get (19). By the Hölder's inequality, we have

$$\|Q_{\varphi^{[n]}}f(b)-\tau'/\tau'(b)\cdot f(b)\|_{H^1}\le|f(b)|\cdot\|\tau'\|_{H^2}^{1/2}\cdot\|1/\tau'(\varphi^{[n]})-1/\tau'(b)\|_{H^2}^{1/2}.$$

By Lemma 3, we get (20).

Remark In the proof of Theorem 2, the boundedness of Blaschke product is applied. We refer to [4], [5] and [8] for more details.

Proof of Theorem 3 Since convergence of $\{\|C_\Phi^n F\|_{\Lambda^p(G)}\}$ is equivalent to the convergence of $\{\|A_{\varphi,p}^n f\|_{H^p}\}$, we just need to prove the convergence of the latter one. Denote $A_{b,p}f(z)=\frac{\tau'^{1/p}(z)}{\tau'^{1/p}(b)}f(b)$, $b=\tau^{-1}(a)$, it is obvious that $A_{b,p}$ is bounded by the supposition of Theorem 3, and we just need to prove $\|A_{\varphi,p}^n-A_{b,p}\|_{H^p}\to 0$. By Lemma 2, we only need to prove the case where

$p = 2$. By the boundedness of τ' and its reciprocal we know that the operator $Tf = \tau' f$ and its inverse operator $T^{-1}f = \frac{1}{\tau'}f$ are both bounded in H^2. Taking $f \in H^2$ and $\|f\|_{H^2} = 1$ gives

$$\begin{aligned}\|A^n_{\varphi,2} - A_{b,2}\|_{H^2} &= \|A^n_{\varphi,2}f - A_{b,2}f\|_{H^2} = \|TC^n_\varphi T^{-1}f - TC_bT^{-1}f\|_{H^2} \\ &\le \|T\|_{H^2} \cdot \|T^{-1}\|_{H^2} \cdot \|C^n_\varphi - C_b\|_{H^2}.\end{aligned}$$

Thus by Lemma 3, we have $\|A^n_{\varphi,2} - A_{b,2}\|_{H^2} \to 0$.

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