Upper Locating-Domination Numbers of Cycles

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Abstract A set \(D\) of vertices in a graph \(G = (V, E)\) is a locating-dominating set (LDS) if for every two vertices \(u, v\) of \(V \setminus D\) the sets \(N(u) \cap D\) and \(N(v) \cap D\) are non-empty and different. The locating-domination number \(\gamma_L(G)\) is the minimum cardinality of an LDS of \(G\), and the upper-locating domination number \(\Gamma_L(G)\) is the maximum cardinality of a minimal LDS of \(G\). In the present paper, methods for determining the exact values of the upper locating-domination numbers of cycles are provided.

Keywords locating-domination number; upper locating-domination number; cycle.

1. Introduction

All graphs considered in this paper are finite simple graphs, that is, undirected graphs without loops or multiple edges. We in general follow [4] for notation and graph theory terminology. Let \(G = (V, E)\) be a simple graph with vertex set \(V\) and edge set \(E\). For a vertex \(v \in V\), the open neighborhood \(N(v)\) of \(v\) consists of the vertices adjacent to \(v\); the closed neighborhood of \(v\) is \(N[v] = N(v) \cup \{v\}\). Also, let \(d_G(v) = |N(v)|\) be the degree of \(v\) and \(\delta(G)\) denote the minimum degree of graph \(G\).

A set \(D \subseteq V\) is a dominating set if every vertex of \(V \setminus D\) has at least one neighbor in \(D\). The domination number \(\gamma(G)\) is the minimum cardinality of a dominating set in \(G\). A dominating set \(D \subseteq V\) is a locating-dominating set (LDS) if every two vertices \(u, v\) of \(V \setminus D\) satisfy \(N(u) \cap D \neq N(v) \cap D\). The locating-domination number \(\gamma_L(G)\) is the minimum cardinality of an LDS of \(G\), and the upper locating-domination number, denoted by \(\Gamma_L(G)\), is the maximum cardinality of a minimal LDS of \(G\). A minimal LDS with maximum cardinality is called a \(\Gamma_L(G)\)-set. Locating domination was introduced by Slater [5, 6]. For further studies on locating-domination we refer to [1], [2] and [3].

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So far as we know, no work has been done on the upper locating-domination number, except for the recent work by Mustapha Chellali, et al [3], in which, the authors presented the exact value of $\Gamma_L(G)$ for $G$ a path. In this paper, we determine the exact values of $\Gamma_L(G)$ for $G$ a cycle.

2. Upper locating-domination numbers of cycles

The upper locating-domination number of any a path has been given as follows.

**Theorem 1 ([3])** For every path $P_n$,

$$\Gamma_L(P_n) = \begin{cases} 
4k, & \text{if } n = 7k; \\
4k + 1, & \text{if } n = 7k + 1 \text{ or } n = 7k + 2; \\
4k + 2, & \text{if } n = 7k + 3 \text{ or } n = 7k + 4; \\
4k + 3, & \text{if } n = 7k + 5; \\
4k + 4, & \text{if } n = 7k + 6.
\end{cases}$$

On the basis of Theorem 1, we compute the values of upper locating-domination numbers of cycles, which is shown as follows.

**Theorem 2** For every cycle $C_n$ with $n \geq 4$,

$$\Gamma_L(C_n) = \begin{cases} 
4k, & \text{if } n = 7k \text{ or } n = 7k + 1; \\
4k + 1, & \text{if } n = 7k + 2 \text{ or } n = 7k + 3; \\
4k + 2, & \text{if } n = 7k + 4 \text{ or } n = 7k + 5; \\
4k + 3, & \text{if } n = 7k + 6.
\end{cases}$$

To prove Theorem 2, we first give two lemmas as follows.

**Lemma 3** For every cycle $C_n$ with $n \geq 4$,

$$\Gamma_L(C_n) \geq \begin{cases} 
4k, & \text{if } n = 7k \text{ or } n = 7k + 1; \\
4k + 1, & \text{if } n = 7k + 2 \text{ or } n = 7k + 3; \\
4k + 2, & \text{if } n = 7k + 4 \text{ or } n = 7k + 5; \\
4k + 3, & \text{if } n = 7k + 6.
\end{cases}$$

**Proof** Suppose $C_n = v_1v_2 \cdots v_nv_1$. When $n = 7k$ or $n = 7k + 1$, let

$$D = \bigcup_{i=0}^{k-1}\{v_{7i+1}, v_{7i+2}, v_{7i+5}, v_{7i+6}\}.$$ 

First, we can easily find that $D$ is an LDS of $C_n$ and $|D| = 4k$. Also, $D$ is minimal. In fact, for any a vertex $v \in D$, $D \setminus \{v\}$ is no longer an LDS of $C_n$. This means that $\Gamma_L(C_n) \geq 4k$.

When $n = 7k + 2$ or $n = 7k + 3$, let

$$D = \{v_{7k+1}\} \cup \bigcup_{i=0}^{k-1}\{v_{7i+1}, v_{7i+2}, v_{7i+5}, v_{7i+6}\}.$$
It is not hard for us to find that $D$ is a minimal LDS, and $|D| = 4k + 1$, which means that $\Gamma_L(C_n) \geq 4k + 1$.

When $n = 7k + 4$, let

$$D = \{v_1, v_2, v_5, v_6, v_9, v_{10}\} \cup \bigcup_{i=1}^{k-1}\{v_{7i+5}, v_{7i+6}, v_{7i+9}, v_{7i+10}\}.$$  

It is not hard for us to verify that $D$ is a minimal LDS, and $|D| = 4k + 2$, which means that $\Gamma_L(C_n) \geq 4k + 2$.

When $n = 7k + 5$, we distinguish two cases. If $k = 0$, it can be easily verified that $\Gamma_L(C_5) = 2 \geq 4 \cdot 0 + 2$. If $k \geq 1$, let

$$D = \{v_1, v_2, v_5, v_6, v_9, v_{10}\} \cup \bigcup_{i=1}^{k-1}\{v_{7i+6}, v_{7i+7}, v_{7i+10}, v_{7i+11}\}.$$  

One can easily verify that $D$ is a minimal LDS, and $|D| = 4k + 2$, which means that $\Gamma_L(C_n) \geq 4k + 2$.

When $n = 7k + 6$, we distinguish two cases. If $k = 0$, it can be easily verified that $\Gamma_L(C_6) = 3 \geq 4 \cdot 0 + 3$. If $k \geq 1$, let

$$D = \{v_1, v_2, v_5, v_6, v_9, v_{12}\} \cup \bigcup_{i=2}^{k}\{v_{7i}, v_{7i+1}, v_{7i+4}, v_{7i+5}\}.$$  

It is not difficult to find that $D$ is a minimal LDS, and $|D| = 4k + 3$. This implies that $\Gamma_L(C_n) \geq 4k + 3$. We have finished the proof of Lemma 3. □

**Lemma 4** For every cycle $C_n$ with $n \geq 4$, $\Gamma_L(C_n) \leq \Gamma_L(P_{n-1})$.

**Proof** First we can easily verify that the conclusion of Lemma 4 holds for $n \leq 12$. In fact, the values of $\Gamma_L(C_n)$ for $4 \leq n \leq 12$ are as follows.

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_L(C_n)$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

So we may assume that $n \geq 13$ in what follows. We claim that every $\Gamma_L(C_n)$-set contains two consecutive vertices of $C_n$. For otherwise, suppose to the contrary that there exists a $\Gamma_L(C_n)$-set $D$ such that any two vertices of $D$ are not consecutive. Then $|D| \leq \left\lfloor \frac{n}{4} \right\rfloor$. Noticing $n \geq 13$, we can prove, by some simple computations, that

$$|D| \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \begin{cases} \left\lfloor \frac{7k+1}{2} \right\rfloor < 4k, & \text{if } n = 7k \text{ or } n = 7k + 1; \\ \left\lfloor \frac{7k+3}{2} \right\rfloor < 4k + 1, & \text{if } n = 7k + 2 \text{ or } n = 7k + 3; \\ \left\lfloor \frac{7k+5}{2} \right\rfloor < 4k + 2, & \text{if } n = 7k + 4 \text{ or } n = 7k + 5; \\ \left\lfloor \frac{7k+6}{2} \right\rfloor < 4k + 3, & \text{if } n = 7k + 6. \end{cases}$$

But this contradicts Lemma 3. Suppose $D$ is a $\Gamma_L(C_n)$-set, and assume $v_1, v_2 \in D$ without loss of generality. Noticing that no three consecutive vertices are in $D$ by the minimality of $D$, we have $v_n \notin D$. Let $P_{n-1}$ be the path resulting from $C_n$ by removing the vertex $v_n$. Then $D$ is a minimal LDS of $P_{n-1}$, and therefore $\Gamma_L(C_n) = |D| \leq \Gamma_L(P_{n-1})$ as desired. □
Proof of Theorem 2  By Theorem 1,
\[
\Gamma_L(P_{n-1}) = \begin{cases} 
4k, & \text{if } n = 7k \text{ or } n = 7k + 1; \\
4k + 1, & \text{if } n = 7k + 2 \text{ or } n = 7k + 3; \\
4k + 2, & \text{if } n = 7k + 4 \text{ or } n = 7k + 5; \\
4k + 3, & \text{if } n = 7k + 6.
\end{cases}
\]

Then, Theorem 2 follows from Lemmas 3 and 4. □

References