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Morita Equivalences Induced by Two-Sided Group Relative Hopf-Module

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Abstract Let H be a Hopf π-coalgebra over a commutative ring k with bijective antipode S, and A and B right π-H-comodulelike algebras. We show that the pair of adjoint functors $(F_3 = A\otimes B^{op}\otimes_{A\Box_H B^{op}} -, G_3 = (-)^{coH})$ between the categories ${}_{A\Box_H B^{op}}\mathcal{M}$ and ${}_A\mathcal{M}_B^{\pi-H}$ is a pair of inverse equivalences, when A is a faithfully flat π-H-Galois extension. Furthermore, the categories $\underline{\mathbf{Morita}}^{\pi-H}(A,B)$ and $\underline{\mathbf{Morita}}^{\Box_{\pi-H}}(A^{coH}, B^{coH})$ are equivalent, if A and B are faithfully flat π-H-Galois extensions.

Keywords Hopf group Galois extension; Morita equivalence; group relative Hopf-module.

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1. Introduction

As a generalization of ordinary Hopf algebras [1], Hopf group-colgebras were studied in the work of Turaev [2] related to homotopy quantum field theories. Let us note that there exists a symmetric monoidal category, the so called Turaev category, the Hopf algebras in which are the same as Hopf group-coalgebras [3]. A purely algebraic study of Hopf group-coalgebras can be found in the references [4–6].

In the theory of the classic Hopf algebras, Caenepeel et al. proved the following results: Let A and B be right faithfully flat H-Galois extensions of A^{coH} and B^{coH}. Then the categories $\underline{\mathbf{Morita}}^{H}(A,B)$ and $\underline{\mathbf{Morita}}^{\Box_H}(A^{coH}, B^{coH})$ are equivalent [7]. It is natural to ask whether or not there exists an analogue of the above results in the setting of Hopf π-coalgebras. This becomes a motivation of our paper.

This paper is organized as follows.

In Section 1, we recall some basic definitions and results related to relative group Hopf modules and group corings. In Section 2, we get a Structure Theorem for two-sided group relative Hopf modules (cf. Theorem 2.2), which is also a main tool used during the rest of the paper. Furthemore, the compatibility of the category equivalence with the Hom and tensor

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functors has been also investigated. In Section 3, we introduce the notion of π-H-Morita contexts and show that if A and B are right faithfully flat π-H-Galois extensions of A^{coH} and B^{coH}, then the categories $\underline{\textbf{Morita}}^{\pi-H}(A,B)$ and $\underline{\textbf{Morita}}^{\square_{\pi-H}}(A^{coH},B^{coH})$ are equivalent (see Theorem 3.6). The main results we get generalize the results of Caenepeel et al. [7] to the setting of Hopf π-coalgebras.

2. Preliminaries

Throughout this paper we will adopt the following notational conventions. k denotes a commutative ring. We will work over k. Let π be a discrete group with unit 1, $H=(\{H_\alpha,m_\alpha,1_\alpha\}_{\alpha\in\pi}, \alpha,\varepsilon,S)$ a Hopf π-coalgebra [4] with bijective antipode S, and $\otimes$ means $\otimes_k$. For an object M in a category, M will also denote the identity morphism on M.

Let M be a right π-H-comodulelike object with structure maps $\rho^M=\{\rho^M_\alpha\}_{\alpha\in\pi}$. The coinvariants of H on M are the elements of the k-module

$$M^{coH}=\{m\in M \mid \rho^M_\alpha(m)=m\otimes 1_\alpha,\ \text{for all } \alpha\in\pi\}.$$

M^{coH} is called a π-coinvariant submodule of M. It is easy to see that M^{coH} is a right π-H-comodulelike object. Similarly, we may define the coinvariants N^{coH} of H on left π-H-comodulelike object N.

In particular, let A be a right π-H-comodulelike algebra. The coinvariant of H on A is $A^{coH}=\{a\in A\mid \rho^A_\alpha(a)=a\otimes 1_\alpha,\ \text{for all } \alpha\in\pi\}$. It is easy to see that A^{coH} is a subalgebra of A and (A,ρ^A_1) is an ordinary right H_1-comodule algebra.

A right relative π-(H,A)-Hopf module M is both a right A-module and π-H-comodulelike object such that the following relations hold:

$$\rho^M_\alpha(ma)=m_{[0]}a_{[0]}\otimes m_{[1,\alpha]}a_{[1,\alpha]},\quad \text{for all}\ \ \alpha\in\pi, m\in M, a\in A.$$

$\mathcal{M}^{\pi-H}_A$ denotes the category of right relative π-(H,A)-Hopf module, where the morphisms are both right A-linear maps and π-H-comodulelike maps. Similarly, the category of left realative π-(H,A)-Hopf module is denoted by ${}_A\mathcal{M}^{\pi-H}$.

Recall from [8] that a π-A-coring $\mathcal{C}$ is a family $\mathcal{C}=\{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ of A-bimodules together with A-bimodule maps

$$\Delta=\{\Delta_{\alpha,\beta}:\mathcal{C}_{\alpha\beta}\longrightarrow\mathcal{C}_\alpha\otimes_A\mathcal{C}_\beta\}_{\alpha,\beta\in\pi}\quad\text{and}\quad \varepsilon:\mathcal{C}_1\longrightarrow A$$

satisfying the coassociativity in the sense that, for any α, β, $\gamma\in\pi$,

$$(\Delta_{\alpha,\beta}\otimes_A\mathcal{C}_\gamma)\Delta_{\alpha\beta,\gamma}=(\mathcal{C}_\alpha\otimes_A\Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

and the counit properties in the sense that, for all $\alpha\in\pi$,

$$(\mathcal{C}_\alpha\otimes_A\varepsilon)\Delta_{\alpha,1}=\mathcal{C}_\alpha=(\varepsilon\otimes_A\mathcal{C}_\alpha)\Delta_{1,\alpha}.$$

Let A be a right π-H-comodulelike algebra. Then $\mathcal{C}=A\otimes H=(A\otimes H_\alpha)_{\alpha\in\pi}$ is a π-A-coring (group coring).

Assume that A is a right π-H-comodulelike algebra, and B is a left π-H-comodulelike algebra, define the cotensor product

$$A\Box_H B=\{\sum_i a_i\otimes b_i\in A\otimes B\mid \sum_i \rho^A_\alpha(a_i)\otimes b_i=\sum_i a_i\otimes\lambda^B_\alpha(b_i),\forall\alpha\in\pi\}.$$

If H is cocommutative, then $A\Box_H B$ is also a right (or left) π-H-comodulelike algebra with the diagonal comodulelike structure.

Theorem 1.1 *Let A be a right π-H-comodulelike algebra. We have a pair of adjoint functors (F_1,G_1) between the categories ${}_{A^{coH}}\mathcal{M}$ and ${}_A\mathcal{M}^{\pi-H}$. For any $N\in{}_{A^{coH}}\mathcal{M}$, and $M\in{}_A\mathcal{M}^{\pi-H}$, $F_1(N)=A\otimes_{A^{coH}}N$, $G_1(M)=M^{coH}$. The unit and counit of the adjunction (F_1,G_1) are given by the formulas*

$$\eta_{1,N}:N\to(A\otimes_{A^{coH}}N)^{coH},\eta_{1,N}(n)=1_A\otimes n;$$
$$\varepsilon_{1,M}:A\otimes_{A^{coH}}M^{coH}\to M,\varepsilon_{1,M}(a\otimes_{A^{coH}}m)=am.$$

Consider the morphism of group corings

$$can:(A\otimes_T A)\langle\pi\rangle\longrightarrow A\otimes H,\quad can_\alpha(a\otimes b)=a_{[0]}b\otimes a_{[1,\,\alpha]}.$$

Then the following assertions are equivalent:

(1) can is an isomorphism of group coring and A is faithfully flat as a right A^{coH}-module;

(2) (F_1,G_1) is a pair of inverse equivalences between the categories ${}_{A^{coH}}\mathcal{M}$ and ${}_A\mathcal{M}^{\pi-H}$ and A is flat as a right A^{coH}-module.

If the above equivalent conditions hold, then we call A a faithfully flat group Galois extension (simply π-H-Galois extension).

Proof From Theorem 1.1 of [7], we can infer that A is a left π-H^{cop}-comodulelike algebra, so, (2) is equivalent to flatness of $A\in\mathcal{M}_{A^{coH}}\cong\mathcal{M}_{A^{coH^{cop}}}$ and equivalence between the categories ${}_{A^{coH}}\mathcal{M}$ and ${}^{\pi-H^{cop}}_{A}\mathcal{M}\cong{}_A\mathcal{M}^{\pi-H}$. $\square$

For a concise treatment of corings and their applications, we refer to [9, 10].

3. Two-sided relative group Hopf module

Under our assumption on H, for any $\alpha\in\pi$, set $(H\otimes H^{cop})_\alpha=H_\alpha\otimes H^{cop}_\alpha=H_\alpha\otimes H_{\alpha^{-1}}$. It is clear that $H\otimes H^{cop}=\{H_\alpha\otimes H^{cop}_\alpha\}_{\alpha\in\pi}$ is a Hopf π-coalgebra with the comultiplication $\Delta\otimes\Delta^{cop}$ and counit $\varepsilon\otimes\varepsilon$. It can be verified that H is a left π-$H\otimes H^{cop}$-module coalgebra with a module structure decomposition $\psi=\{\psi_\alpha:H_\alpha\otimes H_{\alpha^{-1}}\otimes H_\alpha\to H_\alpha,\psi_\alpha(l\otimes m\otimes h)=lhS_{\alpha^{-1}}(m)\}$.

Let A be a right π-H-comodulelike algebra and B a left π-H-comodulelike algebra. Then $A\otimes B$ is a right π-$H\otimes H^{cop}$-comodulelike algebra, with the π-comodule structure maps $\rho^{A\otimes B}=\{\rho^{A\otimes B}_\alpha:A\otimes B\to A\otimes B\otimes H_\alpha\otimes H_{\alpha^{-1}}\}_{\alpha\in\pi}$ defined by

$$\rho^{A\otimes B}_\alpha(a\otimes b)=a_{[0]}\otimes b_{[0]}\otimes a_{[1,\alpha]}\otimes b_{[-1,\alpha^{-1}]}.$$

Then by [11], $(H\otimes H^{cop},A\otimes B,H)$ is a left-right π-Doi-Hopf datum, and the category of π-Doi-Hopf modules is denoted by ${}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$. It is straightforward to verify that $A\otimes B$

is an object of ${}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$, with comodulelike structure maps

$$\rho_\alpha^{A\otimes B}(a\otimes b)=a_{[0]}\otimes b_{[0]}\otimes a_{[1,\alpha]}S_{\alpha^{-1}}(b_{[-1,\alpha^{-1}]}).$$

Proposition 2.1 *With the notation as above, we have a pair of adjoint functors ($F_2=A\otimes B\otimes_{A\Box_H B}-, G_2=(-)^{coH}$) between the categories ${}_{A\Box_H B}\mathcal{M}$ and ${}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$. Furthermore, if A is a faithfully flat π-H-Galois extension, then (F_2,G_2) is a pair of inverse equivalences between the categories ${}_{A\Box_H B}\mathcal{M}$ and ${}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$.*

Proof We only prove the second part of the conclusion. Firstly, we describe the unit of this adjunction:

$$\eta_{2,M}:M\longrightarrow(A\otimes B\otimes_{A\Box_H B}M)^{coH},\quad \eta_{2,M}(m)=1_A\otimes 1_B\otimes_{A\Box_H B}M.$$

The counit ε_2 is the following:

$$\varepsilon_{2,N}:A\otimes B\otimes_{A\Box_H B}N^{coH}\longrightarrow N,\quad \varepsilon_{2,N}(a\otimes b\otimes_{A\Box_H B}n)=(a\otimes b)n.$$

Take $M\in{}_{A\Box_H B}\mathcal{M}$. We can check that $\iota: A^{coH}\to A\Box_H B, \iota(a)=a\otimes 1_B$ is an algebra map, and then M is a left A^{coH}-module by restriction of scalars. Consider the composite of some isomorphic maps:

$$\begin{aligned}g_M: A\otimes_{A^{coH}}M&\cong A\otimes_{A^{coH}}(A\Box_H B)\otimes_{A\Box_H B}M\\ &\longrightarrow A\otimes B\otimes_{A\Box_H B}M.\end{aligned}$$

It is easy to see that $g_M(a\otimes_{A^{coH}}m)=a\otimes 1_B\otimes_{A\Box_H B}m$, and g_M is a right π-H-comodulelike map. It follows that g_M is restricted to an isomorphism

$$(A\otimes_{A^{coH}}M)^{coH}\longrightarrow(A\otimes B\otimes_{A\Box_H B}M)^{coH}.$$

It is observed that $\eta_{2,M}=g_M^{coH}\circ\eta_{1,M}$. It follows immediately from Theorem 1.1 that $\eta_{1,M}$ is an isomorphism, furthermore, $\eta_{2,M}$ is also an isomorphism.

Take $N\in{}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$, then N is both a left A-module by $a\cdot n=(a\otimes 1_B)n$, and a π-(H,A)-relative Hopf module, since

$$\begin{aligned}\rho_\alpha(a\cdot n)=\rho_\alpha((a\otimes 1_B)n)&=(a_{[0]}\otimes 1_B)n_{[0]}\otimes a_{[1,\alpha]}n_{[1,\alpha]}S_{\alpha^{-1}}(1_{\alpha^{-1}})\\ &=a_{[0]}n_{[0]}\otimes a_{[1,\alpha]}n_{[1,\alpha]}.\end{aligned}$$

It is then to see that $\varepsilon_{1,N}=\varepsilon_{2,N}\circ g_{N^{coH}}$. Since A is a faithfully flat π-H-Galois extension, $\varepsilon_{1,N}$ is an isomorphism, and this implies that $\varepsilon_{2,N}$ is also an isomorphism. $\Box$

In what follows, we always assume that H is a finite type Hopf π-coalgebra [4, 5], and A and B are right π-H-comodulelike algebras.

A two-sided relative π-(H,A,B)-Hopf module M is both a left A-module, right B-module and π-H-comodulelike object such that the following relation holds:

$$\rho_\alpha^M(amb)=a_{[0]}m_{[0]}b_{[0]}\otimes a_{[1,\alpha]}m_{[1,\alpha]}b_{[1,\alpha]},\quad\text{for all}\quad \alpha\in\pi, m\in M, a\in A, b\in B.$$

We denote the category of two-sided relative π-(H,A,B)-Hopf module with A-linear, B-linear, π-H-comodulelike maps by ${}_A\mathcal{M}_B^{\pi-H}$.

Observe that B^{op} is a left π-H-comodulelike algebra, with the left π-H-comodule maps $\lambda^{B^{op}} = \{\lambda_\alpha^{B^{op}}\}_{\alpha\in\pi}$, $\lambda_\alpha^{B^{op}}(b) = S_\alpha^{-1}(b_{[1,\alpha^{-1}]}) \otimes b_{[0]}$. In particular, $A \otimes B^{op}$ is a right π-$H \otimes H^{cop}$-comodulelike algebra with the comodule structure maps

$$\rho_\alpha^{A\otimes B^{op}}(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1,\alpha]} \otimes S_{\alpha^{-1}}^{-1}(b_{[1,\alpha]}), \quad \forall \alpha \in \pi.$$

Furthermore, $A \otimes B^{op}$ is a right π-H-comodulelike algebra with the comodulelike structure maps $\rho = \{\rho_\alpha\}_{\alpha\in\pi}$, $\rho_\alpha(a \otimes b) = a_{[0]} \otimes b_{[0]} \otimes a_{[1,\alpha]}b_{[1,\alpha]}$. So we have $(A \otimes B^{op})^{coH} = A\Box_H B^{op}$.

It is straightforward to verify that the category ${}_{A\otimes B}\mathcal{M}^{\pi-H}(H\otimes H^{cop})$ of left-right π-Doi-Hopf modules is isomorphic to the category of two-sided relative π-(H, A, B)-Hopf module ${}_A\mathcal{M}_B^{\pi-H}$.

Applying Proposition 2.1, we immediately obtain the Structure Theorem for two-sided relative π-(H, A, B)-Hopf modules.

Theorem 2.2 *With the notations as above, we have a pair of adjoint functors* $(F_3 = A \otimes B^{op} \otimes_{A\Box_H B^{op}} -, G_3 = (-)^{coH})$ *between the categories* ${}_{A\Box_H B^{op}}\mathcal{M}$ *and* ${}_A\mathcal{M}_B^{\pi-H}$. *If A is a faithfully flat π-H-Galois extension, then (F_3, G_3) is a pair of inverse equivalences.*

Proposition 2.3 *Let A, B, C be right π-H-comodulelike algebras. If $M \in {}_A\mathcal{M}_B^{\pi-H}$ and $N \in {}_B\mathcal{M}_C^{\pi-H}$, then $M \otimes_B N \in {}_A\mathcal{M}_C^{\pi-H}$. When A and B are faithfully flat π-H-Galois extensions, the map*

$$f : M^{coH} \otimes_{B^{coH}} N^{coH} \longrightarrow (M \otimes_B N)^{coH}, \quad f(m \otimes_{B^{coH}} n) = m \otimes_B n$$

is an isomorphism. Consequently, $M^{coH} \otimes_{B^{coH}} N^{coH}$ is a left $A\Box_H C^{op}$-module.

Proof We define the right π-H-comodulelike structure maps on $M \otimes_B N$ as $\rho_\alpha^{M\otimes_B N}(m \otimes_B n) = m_{[0]} \otimes_B n_{[0]} \otimes m_{[1,\alpha]}n_{[1,\alpha]}$, for all $\alpha \in \pi, m \in M$ and $n \in N$, and the (A, C)-bimodule action on $M \otimes_B N$ is natural. It is easy to check $M \otimes_B N \in {}_A\mathcal{M}_C^{\pi-H}$ with the given action and coaction.

It follows from Theorem 1.1 that

$$\varepsilon_{1,M} : A \otimes_{A^{coH}} M^{coH} \to M, \quad \text{and} \quad \varepsilon_{1,N} : B \otimes_{B^{coH}} N^{coH} \to N$$

are isomorphism. Let g be the composite of the following isomorphic maps

$$\varepsilon_{1,M} \otimes_{B^{coH}} N^{coH} : A \otimes_{A^{coH}} M^{coH} \otimes_{B^{coH}} N^{coH} \longrightarrow M \otimes_{B^{coH}} N^{coH},$$

and

$$M \otimes_B \varepsilon_{1,N} : M \otimes_{B^{coH}} N^{coH} \longrightarrow M \otimes_B N.$$

g is bijective with the formula

$$g(a \otimes_{A^{coH}} m \otimes_{B^{coH}} n) = am \otimes_B n,$$

for all $a \in A, m \in M^{coH}$ and $n \in N^{coH}$. It is easy to verify that g is an isomorphism in ${}_A\mathcal{M}^{\pi-H}$. Then g restricts to an isomorphism

$$g^{coH} : (A \otimes_{A^{coH}} M^{coH} \otimes_{B^{coH}} N^{coH})^{coH} \longrightarrow (M \otimes_B N)^{coH}.$$

The map f is the composition of g^{coH} and the isomorphism

$$\eta_{1,M^{coH}\otimes_{B^{coH}}N^{coH}} : M^{coH} \otimes_{B^{coH}} N^{coH} \longrightarrow (A \otimes_{A^{coH}} M^{coH} \otimes_{B^{coH}} N^{coH})^{coH}.$$

Finally, the left $A\Box_H C^{op}$-action on $(M \otimes_B N)^{coH}$ can be transported using f to $M^{coH} \otimes_{B^{coH}} N^{coH}$. $\Box$

In what follows, let $M, N \in {}_A\mathcal{M}^{\pi-H}$. We define the maps

$$\rho_\alpha : {}_A\mathrm{Hom}(M,N) \longrightarrow {}_A\mathrm{Hom}(M,N) \otimes H_\alpha, \rho_\alpha(f) = f_{[0]} \otimes f_{[1,\alpha]}$$

satisfying the relation

$$f_{[0]}(m) \otimes f_{[1,\alpha]} = f(m_{[0]})_{[0]} \otimes S_\alpha^{-1}(m_{[1,\alpha^{-1}]}) f(m_{[0]})_{[1,\alpha]},$$

for all $f \in {}_A\mathrm{Hom}(M,N)$, and $\alpha, \beta \in \pi$. A straightforward calculation shows that ${}_A\mathrm{Hom}(M,N)$ is a right π-H-comodulelike object with the π-H-comodulelike structure maps $\rho = \{\rho_\alpha\}_{\alpha\in\pi}$.

Proposition 2.4 *Let A, B, C be right π-H-comodulelike algebras. If $M \in {}_A\mathcal{M}_B^{\pi-H}$ and $N \in {}_A\mathcal{M}_C^{\pi-H}$, then ${}_A\mathrm{Hom}(M,N) \in {}_B\mathcal{M}_C^{\pi-H}$, and we have a map $\psi : {}_A\mathrm{Hom}(M,N)^{coH} \to {}_{A^{coH}}\mathrm{Hom}(M^{coH}, N^{coH}), \psi(f) = f^{coH}$, where f^{coH} is a restriction to M^{coH}. If A is a faithfully flat π-H-Galois extension, then ψ is an isomorphism of left $B\Box_H C^{op}$-modules.*

Proof We firstly define the (B,C)-bimodule structure on ${}_A\mathrm{Hom}(M,N)$ as $(b \cdot g \cdot c) = g(mb)c$ for any $g \in {}_A\mathrm{Hom}(M,N)$, $b \in B, c \in C$. It is clear that $b \cdot g \cdot c$ is left A-linear. The right π-H-comodule is defined as above. Secondly, one can show that ${}_A\mathrm{Hom}(M,N) \in {}_B\mathcal{M}_C^{\pi-H}$ by a tedious computation.

ψ is well-defined, by the definition of $\rho_\alpha(f)$, it follows that $f(m) \otimes 1_\alpha = f(m)_{[0]} \otimes f(m)_{[1,\alpha]}$, for any $f \in {}_A\mathrm{Hom}(M,N)^{coH}, m \in M^{coH}$, so $f(m) \in N^{coH}$.

If A is faithfully flat π-H-Galois extension, then we define the inverse of ψ as $\varphi(f) = \varepsilon_{1,N} \circ (A \otimes f) \circ \varepsilon_{1,M}^{-1}$ and this completes the whole proof. $\Box$

Remark 2.5 In particular, set $M = N$, it is straightforward to verify that ${}_A\mathrm{End}(M)^{op}$ is a right π-H-comodule algebra.

Proposition 2.6 *Let A, B, C be right π-H-comodulelike algebras. Consider $M \in {}_A\mathcal{M}_B^{\pi-H}$ and $N \in {}_A\mathcal{M}_C^{\pi-H}$, then the evaluation map $\varphi : M \otimes_B {}_A\mathrm{Hom}(M,N) \to N, \varphi(m \otimes f) = f(m)$ is a morphism of the category ${}_A\mathcal{M}_C^{\pi-H}$. If A and B are faithfully flat π-H-Galois extensions, then the evaluation map*

$$\varphi^{coH} : M^{coH} \otimes_{B^{coH}} {}_{A^{coH}}\mathrm{Hom}(M^{coH}, N^{coH}) \longrightarrow N^{coH}$$

is left $A\Box_H C^{op}$-linear.

Proof We only show that φ is a right π-H-comodulelike map. In fact, for any $f \in {}_A\mathrm{Hom}(M,N)$, and $m \in M$,

$$\begin{aligned}(\varphi \otimes H_\alpha)(\rho_\alpha(m \otimes_B f)) &= (\varphi \otimes H_\alpha)(m_{[0]} \otimes_B f_{[0]} \otimes m_{[1,\alpha]} \otimes f_{[1,\alpha]})\\ &= f(m_{[0]})_{[0]} \otimes m_{[2,\alpha]} S_\alpha^{-1}(m_{[1,\alpha^{-1}]}) f(m_{[0]})_{[1,\alpha]}\\ &= f(m)_{[0]} \otimes f(m)_{[1,\alpha]} = \rho_\alpha(\varphi(m \otimes_B f)).\end{aligned}$$

The second statement follows from Propositions 2.3, 2.4 and Remark 2.5. $\Box$

Proposition 2.7 *Take $M \in {}_A\mathcal{M}_B^{\pi-H}$, then the map $\phi : B \to {}_A\mathrm{End}(M), \phi(b)(m) = mb$ is a morphism in ${}_B\mathcal{M}_B^{\pi-H}$. In particular, ϕ is also an algebra map between B and ${}_A\mathrm{End}(M)^{op}$. If A and B are faithfully flat π-H-Galois extensions, then the map $\phi^{coH} : B^{coH} \to {}_A\mathrm{End}(M)^{coH} \cong {}_{A^{coH}}\mathrm{End}(M^{coH})$ is left $B\square_H B^{op}$-linear.*

Proof One can check ϕ is a B-bimodule map and also an algebra map between B and ${}_A\mathrm{End}(M)^{op}$ by a straightforward computation. ϕ is also a right π-H-comodulelike map, since

$$\begin{aligned}\phi(b)_{[0]}(m) \otimes \phi(b)_{[1,\alpha]} &= \phi(b)(m_{[0]})_{[0]} \otimes S_\alpha^{-1}(m_{[1,\alpha^{-1}]})\phi(b)(m_{[0]})_{[1,\alpha]}\\ &= \phi(b_{[0]})(m) \otimes b_{[1,\alpha]},\end{aligned}$$

for any $m \in M$, $b \in B$. □

Applying Propositions 2.6 and 2.7, we can obtain the second statement.

Proposition 2.8 *Let A, B, C be right π-H-comodulelike algebras, and consider $M \in {}_A\mathcal{M}_B^{\pi-H}, N \in {}_A\mathcal{M}_C^{\pi-H}, A \in {}_A\mathcal{M}_A^{\pi-H}$, then the map $\xi : {}_A\mathrm{Hom}(M, A) \otimes_A N \to {}_A\mathrm{Hom}(M, N), \xi(f \otimes n)(m) = f(m)n$ is a morphism in ${}_B\mathcal{M}_C^{\pi-H}$. If A and B are faithfully flat π-H-Galois extensions, then the map $\xi^{coH} : {}_{A^{coH}}\mathrm{Hom}(M^{coH}, A^{coH}) \otimes_{A^{coH}} N^{coH} \to {}_{A^{coH}}\mathrm{Hom}(M^{coH}, N^{coH})$ is left $B\square_H C^{op}$-linear.*

Proof We have to show that ξ is a right π-H-comodulelike map. In fact, it suffices to compute that the relation

$$\begin{aligned}\xi(f_{[0]} \otimes n_{[0]})(m) \otimes f_{[1,\alpha]}n_{[1,\alpha]} &= f_{[0]}(m)n_{[0]} \otimes f_{[1,\alpha]}n_{[1,\alpha]}\\ &= f(m_{[0]})_{[0]}n_{[0]} \otimes S_\alpha^{-1}(m_{[1,\alpha^{-1}]})f(m_{[0]})_{[1,\alpha]}n_{[1,\alpha]}\\ &= \xi(f \otimes n)_{[0]}(m) \otimes \xi(f \otimes n)_{[1,\alpha]}\end{aligned}$$

holds, for any $f \in {}_A\mathrm{Hom}(M, A)$, $n \in N, m \in M$.

The rest of the proof is similar to Proposition 2.7. □

3. Generalized Morita equivalences

In this section, we introduce the group Morita context and study the generalized Morita equivalences induced by two-sided relative π-(H, A, B)-Hopf module.

Definition 3.1 *Let A and B be right π-H-comodulelike algebras. Consider $M \in {}_A\mathcal{M}_B^{\pi-H}$, $N \in {}_B\mathcal{M}_A^{\pi-H}, A \in {}_A\mathcal{M}_A^{\pi-H}$ and $B \in {}_B\mathcal{M}_B^{\pi-H}$, then a group Morita context over H (simply π-H-Morita context) connecting A and B is a Morita context (A, B, M, N, τ, μ) of objects defined above such that $\tau : M \otimes_B N \to A$ is a morphism in ${}_A\mathcal{M}_A^{\pi-H}$ and $\mu : N \otimes_A M \to B$ is a morphism in ${}_B\mathcal{M}_B^{\pi-H}$.*

A morphism between two π-H-Morita context (A, B, M, N, τ, μ) and $(A', B', M', N', \tau', \mu')$ consists of a fourtuple (f, g, s, t), where $f : A \to A', g : B \to B'$ are π-H-comodulelike algebra maps, $s : M \to M'$ is a morphism in ${}_A\mathcal{M}_B^{\pi-H}$ and $t : N \to N'$ is a morphism in ${}_B\mathcal{M}_A^{\pi-H}$ satisfying the relations $\tau' \circ (s \otimes t) = f\tau$ and $\mu' \circ (t \otimes s) = g\mu$. We use the notation $\underline{\mathbf{Morita}}^{\pi-H}(A, B)$ for the subcategory of the category of π-H-Morita context, consisting of π-H-Morita context connecting A and B, and morphisms with the identity of A and B as the underlying algebra maps.

Example 3.2 Let A be a right π-H-comodulelike algebra. Take $P \in {}_A\mathcal{M}^{\pi-H}$, then through Remark 2.5, $B = {}_A\mathrm{End}(P)^{op}$ is a right π-H-comodulelike algebra. Therefore one can show that $P \in {}_A\mathcal{M}_B^{\pi-H}$ with the right B-module given by $p \cdot f = f(p)$, for all $f \in B, p \in P$ and $Q = {}_A\mathrm{Hom}(P, A) \in {}_B\mathcal{M}_A^{\pi-H}$ by Proposition 2.4. Define the map

$$\tau : P \otimes_B {}_A\mathrm{Hom}(P, A) \to A, \tau(p \otimes f) = f(p),$$

and the map

$$\mu : {}_A\mathrm{Hom}(P, A) \otimes_A P \to {}_A\mathrm{End}(P)^{op}, \mu(f \otimes p)(x) = f(x)p.$$

It can be verified that the sextuple (A, B, P, Q, τ, μ) is a π-H-Morita context by a tedious and straightforward computation. In this case, we call it the π-H-Morita context associated with $P \in {}_A\mathcal{M}^{\pi-H}$.

Remark 3.3 1) The π-H-Morita context associated to P is strict if and only if P is a left A-progenerator.

2) If π-H-Morita context (A, B, P, Q, τ, μ) is strict, then $(F_4 = P \otimes_B -, G_4 = N \otimes_A -)$ between the categories ${}_A\mathcal{M}^{\pi-H}$ and ${}_B\mathcal{M}^{\pi-H}$ is a pair of inverse equivalences.

Proposition 3.4 *If (A, B, M, N, τ, μ) is a strict π-H-Morita context, then the π-H-Morita context is isomorphic to the π-H-Morita context associated to $M \in {}_A\mathcal{M}^{\pi-H}$.*

Proof Firstly, by [12, Theorem 3.5] and Remark 2.5, it follows that $g : B \to {}_A\mathrm{End}(M)^{op}$ is an isomorphism of π-H-comodulelike algebra and $t : N \to {}_A\mathrm{Hom}(M, A)$, $t(n)(m) = \tau(m \otimes n)$ is an isomorphism of (B, A)-bimodules.

Secondly, one can show that t is a π-H-comodulelike map and (A, g, M, t) is an isomorphism of π-H-Morita context by a tedious computation. $\square$

Definition 3.5. *Let A and B be faithfully flat π-H-Galois extensions of A^{coH} and B^{coH}. A $\square_{\pi-H}$-Morita context between A^{coH} and B^{coH} is a Morita context $(A^{coH}, B^{coH}, P, Q, \tau', \mu')$ satisfying the following conditions:*

(M1) P is a left $A\square_H B^{op}$-module;

(M2) Q is a left $B\square_H A^{op}$-module;

(M3) $\tau' : P \otimes_{B^{coH}} Q \to A^{coH}$ is left $A\square_H A^{op}$-linear;

(M4) $\mu' : Q \otimes_{A^{coH}} P \to B^{coH}$ is left $B\square_H B^{op}$-linear.

A morphism between two $\square_{\pi-H}$-Morita contexts connecting A^{coH} and B^{coH} is a morphism between Morita contexts of the form $(A^{coH}, B^{coH}, \alpha, \beta)$, where α is left $A\square_H B^{op}$-linear and β is left $B\square_H A^{op}$-linear. The category of $\square_{\pi-H}$-Morita contexts connecting A^{coH} and B^{coH} will be denoted by $\underline{\mathbf{Morita}}^{\square_{\pi-H}}(A^{coH}, B^{coH})$.

In what follows, we will give the main result of this paper.

Theorem 3.6 *Let A and B be faithfully flat π-H-Galois extensions of A^{coH} and B^{coH}. Then* *the categories* $\underline{\mathbf{Morita}}^{\pi-H}(A,B)$ *and* $\underline{\mathbf{Morita}}^{\Box_{\pi-H}}(A^{coH},B^{coH})$ *are equivalent. The equivalence functors send strict contexts to strict contexts.*

Proof Let (A,B,P,Q,τ,μ) be a π-H-Morita context. Then $P^{coH}\in{}_{A\Box_H B^{op}}\mathcal{M}$ and $Q^{coH}\in{}_{B\Box_H A^{op}}\mathcal{M}$ by Theorem 2.2. It follows from Proposition 2.3 that we have a left $A\Box_H A^{op}$-linear map

$$\tau'=\tau^{coH}\circ f: P^{coH}\otimes_{B^{coH}}Q^{coH}\xrightarrow{f}(P\otimes_B Q)^{coH}\xrightarrow{\tau^{coH}}A^{coH}$$

and a left $B\Box_H B^{op}$-linear map

$$\mu'=\mu^{coH}\circ f: Q^{coH}\otimes_{A^{coH}}P^{coH}\xrightarrow{f}(Q\otimes_B P)^{coH}\xrightarrow{\mu^{coH}}B^{coH},$$

where f is defined in Proposition 2.3. By Proposition 2.3, $(P\otimes_B Q)^{coH}\otimes_{A^{coH}}P^{coH}\overset{f'}{\cong}(P\otimes_B Q\otimes_A P)^{coH}$ and $P^{coH}\otimes_{B^{coH}}(Q\otimes_A P)^{coH}\overset{f''}{\cong}(P\otimes_B Q\otimes_A P)^{coH}$, it follows that $f'\circ(f\otimes_{A^{coH}}P^{coH})=f'\circ(M^{coH}\otimes_{B^{coH}}f)$.

Since $\tau\otimes_A P=P\otimes_B\mu$, it follows that

$$\tau'\otimes_{A^{coH}}P^{coH}=P^{coH}\otimes_{B^{coH}}\mu'.$$

Similarly, we can obtain

$$\mu'\otimes_{B^{coH}}Q^{coH}=Q^{coH}\otimes_{B^{coH}}\tau'.$$

Therefore, it follows that $(A^{coH},B^{coH},P^{coH},Q^{coH},\tau',\mu')$ is a $\Box_{\pi-H}$-Morita context. If $(A,\ B,\ P,\ Q,\ \tau,\ \mu)$ is strict, then it follows that $(A^{coH},B^{coH},P^{coH},Q^{coH},\tau',\mu')$ is strict by the definition of τ' and μ'.

Conversely, let $(A^{coH},B^{coH},M,N,\tau',\mu')$ be a $\Box_{\pi-H}$-Morita context. Then set $P=(A\otimes B^{op})\otimes_{A\Box_H B^{op}}M$ and $Q=(B\otimes A^{op})\otimes_{B\Box_H A^{op}}N$. From Theorem 2.2, we can obtain $P\in{}_A\mathcal{M}_B^{\pi-H}$, $Q\in{}_B\mathcal{M}_A^{\pi-H}$ and $A\overset{\iota_A}{\cong}(A\otimes A^{op})\otimes_{A\Box_H A^{op}}A^{coH}, B\overset{\iota_B}{\cong}(B\otimes B^{op})\otimes_{B\Box_H B^{op}}B^{coH}$. Define $\tau:P\otimes_B Q\longrightarrow A,\tau=\iota_A\circ F(\tau')\circ h$, and $\mu:Q\otimes_A P\longrightarrow B,\mu=\iota_B\circ F(\mu')\circ h$. By Theorem 2.2 and the definition of τ,μ, one can deduce that (A,B,P,Q,τ,μ) is a π-H-Morita context. $\Box$

Proposition 3.7 *Assume that A and B are faithfully flat π-H-Galois extensions of A^{coH} and B^{coH}, and let $(A^{coH},B^{coH},M,N,\tau',\mu')$ be a strict Morita context. If M has a left $A\Box_H B^{op}$-module structure, then there exists a unique left $B\Box_H A^{op}$-module structure on N such that $(A^{coH},B^{coH},M,N,\tau',\mu')$ is a strict $\Box_{\pi-H}$-Morita context.*

Proof One can see that $\hat{M}=A\otimes B^{op}\otimes_{A\Box_H B^{op}}M\in{}_A\mathcal{M}_B^{\pi-H}$ and $\phi^{coH}:B^{coH}\to{}_A\mathrm{End}(\hat{M})^{coH}\cong{}_{A^{coH}}\mathrm{End}(\hat{M}^{coH})$ is an isomorphism as left $B\Box_H B^{op}$-module by Proposition 2.7 and the strict property of the Morita context. So $\phi:B\to{}_A\mathrm{End}(\hat{M})$ is an isomorphism in ${}_B\mathcal{M}_B^{\pi-H}$ by the faithful flatness of B and $\hat{M}$ is a progenerator by the strict property of the Morita context. Set $\hat{N}={}_A\mathrm{Hom}(\hat{M},A)$, it follows that $\hat{N}^{coH}\cong{}_{A^{coH}}\mathrm{Hom}(M,A^{coH})$ as left $B\Box_H A^{op}$-modules and N and ${}_{A^{coH}}\mathrm{Hom}(M,A^{coH})$ are canonically isomorphism as (B^{coH},A^{coH})-bimodule. Using the isomorphism, the left $B\Box_H A^{op}$-module structure can be transported to N. The corresponding $\Box_{\pi-H}$-Morita context from Theorem 3.6 is canonically isomorphic to $(A^{coH},B^{coH},M,N,\tau',\mu')$, this completes the proof. $\Box$

Recall that M is a left A-progenerator if M is finitely generated projective and its trace ideal is the whole ring. If this property holds in ${}_A\mathcal{M}^{\pi-H}$, then we call M a π-H-progenerator.

The following result is easy to obtain.

Corollary 3.8 *Assume that A and B are faithfully flat π-H-Galois extensions of A^{coH} and B^{coH}. If π-H-Morita context (A, B, P, Q, τ, μ) is strict, the P is a π-H-progenerator.*

Proposition 3.9 *If A is a faithfully flat Galois extension of A^{coH} and $P \in {}_A\mathcal{M}^{\pi-H}$ is a left A-progenerator, then $B = {}_A\mathrm{End}(P)^{op}$ is a faithfully flat π-H-Galois extension of B^{coH} if and only if P is a π-H-progenerator.*

Proof By Proposition 3.4, the π-H-Morita context $(A, B, P, Q = {}_A\mathrm{Hom}(P, A), \tau, \mu)$ defined in Example 3.2 is strict.

Necessity. It immediately follows from Corollary 3.8.

Sufficiency. By Example 3.2, $P \in {}_A\mathcal{M}_B^{\pi-H}$ and $P_1 = P^{coH} \in {}_{A^{coH}}\mathcal{M}_{B^{coH}}$. From Theorem 1.1, it follows that P_1 is a left A^{coH}-progenerator. One can get $B^{coH} \cong {}_{A^{coH}}\mathrm{End}(P_1)^{op}$ and $Q^{coH} \cong {}_{A^{coH}}\mathrm{Hom}(P_1, B^{coH})$ by Proposition 2.4. The Morita context $(A^{coH}, B^{coH} \cong {}_{A^{coH}}\mathrm{End}(P_1)^{op}, P_1, {}_{A^{coH}}\mathrm{Hom}(P_1, B^{coH}), \tau', \mu')$ associated to $P_1 \in {}_{A^{coH}}\mathcal{M}$ is strict. From the fact that the three functors $P_1 \otimes_{B^{coH}} -, A \otimes_{A^{coH}} -$ and $P \otimes_B -$ are equivalent functors, it follows that $B \otimes_{B^{coH}} -$ is also an equivalence. Furthermore, it follows that B is a faithfully flat Galois extension of B^{coH}.

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