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A Characterization Theorem of the Differential of Functions Valued in B-Valued Generalized Functionals

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Abstract Banach space-valued generalized functionals of white noise form an important part of vector-valued generalized functionals of white noise. In this paper, we discuss the differential of abstract function valued in B-valued generalized functional space. A characterized theorem is obtained by using their S-transform.

Keywords white noise analysis; B-valued generalized functionals; differential; S-transform.

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1. Introduction

Scalar-valued functionals of white noise have been investigated considerably [1–3]. However vector-valued functionals of white noise, as was pointed out in [4], would play a more important role in applications of white noise analysis to many research fields. Hence it is of interest to make studies on vector valued generalized functionals of white noise, which make up a considerable part of vector-valued functionals of white noise.

In [5] and [6], the authors actually gave analytic characterizations of $\mathcal{G}^*$-valued generalized functionals of white noise, where $\mathcal{G}$ is a standard countable Hilbert space constructed from a separable Hilbert space and a positive self-adjoint operator in it. In [7], the authors introduced the differential of generalized operators-valued function. In [8] and [9], Wang and Huang characterized Banach space-valued generalized functionals of white noise via their S-transforms and moments, respectively. In this paper, we discuss the differential of abstract function valued in Banach space-valued generalized functionals of white noise.

2. Preliminaries

In this section we briefly recall some general notions, notations and facts. Let $\mathbf{R}$ be the real field and $\mathbf{C}$ the complex field. In the following, we assume that $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$ is given. For a topological vector space V over $\mathbf{K}$, we denote by V^* the usual topological dual of V. Unless specified otherwise, the canonical bilinear form on $V^* \times V$ is written as $\langle \cdot, \cdot \rangle_{V^* \times V}$. Let V_1 and V_2

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be locally convex topological vector space over $\mathbf{K}$. We denote by $\mathcal{L}[V_1, V_2]$ the space of continuous linear operators form V_1 to V_2. For $T \in \mathcal{L}[V_1, V_2]$, the dual of T is written as T^*, which satisfies

$$\langle f, Tx\rangle_{V_2{}^* \times V_2} = \langle T^* f, x\rangle_{V_1{}^* \times V_1}, \quad x \in V_1, f \in V_2{}^*.$$

We note that if $T \in \mathcal{L}[V_1, V_2]$, then $T^* \in \mathcal{L}[V_2^*, V_1^*]$.

Denote by H the Hilbert space $L^2(\mathbf{R}^m)$ with inner product $\langle \cdot, \cdot\rangle$ and norm $|\cdot|_0$. Let $E = S(\mathbf{R}^m)$ be the Schwartz test function space. Then E is a countably Hilbert nuclear space with stand norms $\{|\cdot|_p | p \geq 0\}$ and $E \subset H \subset E^*$constitutes a Gel'fand triple. The canonical bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot\rangle$ which is consistent with inner product in H.

Let $(E)_{\mathbf{C}} \subset (L^2)_{\mathbf{C}} \subset (E)^*_{\mathbf{C}}$ be the canonical framework of white noise analysis associated with $E_{\mathbf{C}} \subset H \subset E_{\mathbf{C}^*}$. The canonical bilinear form on $(E)^*_{\mathbf{C}} \times (E)_{\mathbf{C}}$ is denoted by $\langle\langle \cdot, \cdot\rangle\rangle$. For $\xi \in E_{\mathbf{C}}$, let $\mathcal{E}(\xi) = e^{\langle \cdot, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle}$ be the exponential vector corresponding to ξ. Then $\{\mathcal{E}(\xi) | \xi \in E_{\mathbf{C}}\}$ is a total subset of $(E)_{\mathbf{C}}$.

Let X be a complex Banach space over $\mathbf{K}$ with norm $\|\cdot\|_X$. By an X-valued generalized functional we mean a continuous linear mapping form $(E)_{\mathbf{C}}$ to X. As usual, we denote by $\mathcal{L} = \mathcal{L}[(E)_{\mathbf{C}}, X]$ the space of X-valued generalized functionals. For $T \in \mathcal{L}[(E)_{\mathbf{C}}, X]$, its S-transform $\widehat{T}$ is defined by $\widehat{T}(\xi) = T \circ \mathcal{E}(\xi)$, $\xi \in E_{\mathbf{C}}$, where $T \circ \mathcal{E}(\xi)$ is the composition of T and $\mathcal{E}(\xi)$.

Let $G : E_{\mathbf{C}} \to X$ be an abstract function that satisfies the following two conditions:

(1) There exist $p \geq 0$ and $a > 0$, $M \geq 0$ such that

$$\|G(\xi)\|_X \leq M e^{a|\xi|_p^2}, \quad \xi \in E_{\mathbf{C}}.$$

(2) For any ξ, $\eta \in E_{\mathbf{C}}$ and $f \in X^*$, the complex function $z \to \langle f, G(\xi + z\eta)\rangle_{X^* \times X}$ is an entire function on $\mathbf{C}$.

Then G is said to be an abstract U-functional.

We denote by $\mathcal{U}$ the linear space of abstract U-functionals.

Theorem 1 ([9]) *Let $T \in \mathcal{L}[(E)_{\mathbf{C}}, X]$ be an X-valued generalized functional. Then $\widehat{T} = T \circ \mathcal{E}(\xi) \in \mathcal{U}$. Conversely, if $G \in \mathcal{U}$ is an abstract U-functional. Then there exists a unique X-valued generalized functional $T \in \mathcal{L}[(E)_{\mathbf{C}}, X]$ such that $G = T \circ \mathcal{E}(\xi)$. Moreover for $q \geq p$ with $2e^2 a\|A^{-(q-p)}\|^2_{HS} < 1$, we have*

$$\|T\varphi\|_X \leq M(1 - 2e^2 a\|A^{-(q-p)}\|^2_{HS})^{-\frac{1}{2}} \|\varphi\|_q, \varphi \in (E)_{\mathbf{C}}.$$

3. Main results

Let $\Omega \subset \mathbf{R}^d$ be an open set, and $T(\cdot) : \Omega \to \mathcal{L}[(E)_{\mathbf{C}}, X]$ be a weakly measurable function valued in X-valued generalized functionals, that is, for any $f \in X^*$ and $\varphi \in (E)_{\mathbf{C}}$, $\langle f, T(\cdot)\varphi\rangle_{X^* \times X}$ is Borel measurable.

Definition 1 *Let $T \in \mathcal{L}[(E)_{\mathbf{C}}, X]$ and $\{T_n\}_{n \geq 1} \in \mathcal{L}[(E)_{\mathbf{C}}, X]$. The sequence $\{T_n\}_{n \geq 1}$ is said to* *converge weakly to T if for each $f \in X^*$ and $\varphi \in (E)_{\mathbf{C}}$, we have*

$$\lim_{n\to\infty} \langle f, T_n\varphi\rangle_{X^*\times X} = \langle f, T\varphi\rangle_{X^*\times X}.$$

Denote by $T = w - \lim_{n\to\infty} T_n$.

Definition 2 *Let $T(\cdot) : \Omega \to \mathcal{L}[(E)_{\mathbf{C}}, X]$ be a given mapping. $T(\cdot)$ is called weakly continuous at $x_0 \in \Omega$ if for each $f \in X^*$ and $\varphi \in (E)_{\mathbf{C}}$, we have*

$$\lim_{x\to x_0} \langle f, T(x)\varphi\rangle_{X^*\times X} = \langle f, T(x_0)\varphi\rangle_{X^*\times X}.$$

Now we begin to state and prove our main result.

Definition 3 *Let $T(\cdot) : \Omega \to \mathcal{L}[(E)_{\mathbf{C}}, X]$ be a given mapping, $\{e_k\}_{k=1,2,\ldots,d}$ is an orthonormal basis of $\mathbf{R}^d$, $x_0 \in \Omega$. If there exists an X-valued generalized functional $(D_kT)(x_0)$, such that*

$$w - \lim_{h\to 0} \frac{T(x_0 + he_k) - T(x_0)}{h} = (D_kT)(x_0),$$

then $(D_kT)(x_0)$ is called the weak partial derivative of $T(\cdot)$ at x_0 in the direction of k.

If the weak partial derivatives of $T(\cdot)$ in all directions exist at each point in Ω, $T(\cdot)$ is said to be weakly differentiable in Ω. And if for each $k = 1, 2, \ldots, d$, $x \to D_kT(x)$ is weakly continuous from Ω to $\mathcal{L}[(E)_{\mathbf{C}}, X]$, then $T(\cdot)$ is said to be weakly continuously differentiable. We denote by $C^1_w(\Omega, \mathcal{L})$ the space of functions that are weakly continuously differentiable from Ω to $\mathcal{L}[(E)_{\mathbf{C}}, X]$.

From the Definition 3 we have the following conclusion.

Theorem 2 *If $T(\cdot) \in C^1_w(\Omega, \mathcal{L})$, then for any $f \in X^*$ and $\varphi \in (E)_{\mathbf{C}}$, the function $x \to \langle f, T(x)\varphi\rangle_{X^*\times X}$ is weakly continuously differentiable.*

The nth weak partial derivative of $T(\cdot)$ at x in the direction of k, denoted by $D^n_kT(x)$, can be defined inductively.

Definition 4 *For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_d \le n$, and $x \in \Omega$, if the weak partial derivative $D^\alpha_T(x) = D^{\alpha_1}_1 D^{\alpha_2}_2 \cdots D^{\alpha_d}_d T(x)$ exists and is continuous, then $T(\cdot) : \Omega \to \mathcal{L}[(E)_{\mathbf{C}}, X]$ is said to be n-times weakly continuously differentiable.*

We denote by $C^n_w(\Omega, \mathcal{L})$ the space of functions that are n-times weakly continuously differentiable, $C^0_w(\Omega, \mathcal{L}) = C_w(\Omega, \mathcal{L})$ be the space of functions that are weakly continuously differentiable, and $C^\infty_w(\Omega, \mathcal{L}) = \bigcap_n C^n_w(\Omega, \mathcal{L})$.

If V is an abstract $\mathcal{U}$-functional, we denote by $S^{-1}V$ the X-valued generalized functional corresponding to V by its S-transform.

Definition 5 *Let V be a mapping from Ω to $\mathcal{U}$. If for any $x \in \Omega$, there exists a neighborhood $\Omega_x \subset \Omega$ of x and $p \in \mathbf{R}$, $a > 0$, $M \ge 0$, such that*

$$\|V(x_0)(\xi)\|_X \le Me^{a|\xi|^2_p}$$

for any $x_0 \in \Omega_x, \xi \in E_{\mathbf{C}}$. Then V is said to satisfy a locally uniform condition.

We denote by $G^2_{lu}(\Omega, \mathcal{U})$ the space of functions that satisfy the locally uniform condition. If for any $x \in \Omega$, $\Omega_x = \Omega$, then V is said to satisfy a uniform condition. We denote by $G^2_u(\Omega, \mathcal{U})$

the space of functions that satisfy the uniform condition.

Theorem 3 *Let $V \in G_u^2(\Omega,\mathcal{U})$. Then there exists $p \geq 0$, such that $\{T(x) = S^{-1}V(x), x \in \Omega\} \subset \mathcal{L}[(E_p)_{\mathbf{C}}, X]$, moreover it is a bounded subset of $\mathcal{L}[(E_p)_{\mathbf{C}}, X]$.*

Proof $V \in G_u^2(\Omega,\mathcal{U})$ implies that for any $x \in \Omega$, there exist $p \in \mathbf{R}$ and $a > 0$, $M \geq 0$, such that

$$\|V(x)(\xi)\|_X \leq Me^{a|\xi|_p^2}.$$

Moreover, $T(x) = S^{-1}V(x)$, that is, $V(x) = ST(x) = T(x) \circ \mathcal{E}$. Then it is an immediate consequence of Theorem 1. □

Theorem 4 *Let $V \in G_{lu}^2(\Omega,\mathcal{U})$ satisfy the condition: for each $\xi \in E_{\mathbf{C}}$, $x \to V(x)(\xi)$ is weakly continuous. Then $T(\cdot) \in C_w(\Omega, \mathcal{L})$.*

Proof Let $x \in \Omega$. Since $V \in G_{lu}^2(\Omega,\mathcal{U})$, for the field Ω_x of x, $V \in G_u^2(\Omega_x,\mathcal{U})$. By Theorem 3, there exists $p \geq 0$, such that X-valued generalized functionals $\{T(x_0) : x_0 \in \Omega_x\} \subset \mathcal{L}[(E_p)_{\mathbf{C}}, X]$, moreover it is a bounded subset of $\mathcal{L}[(E_p)_{\mathbf{C}}, X]$. Suppose $M \geq 0$ is bounded, that is,

$$\|T(x_0)\varphi\|_X \leq M\|\varphi\|_p, \quad \varphi \in (E)_{\mathbf{C}}.$$

Since $x \to V(x)(\xi)$ is weakly continuous, for any $\psi \in \{\mathbf{E}(\xi)|\xi \in E_{\mathbf{C}}\}$ and $f \in X^*$, $\langle f, T(x)\psi\rangle_{X^*\times X}$ is continuous, that is, for any $\varepsilon > 0$, there exists $\delta > 0$, when $\{x_0||x-x_0| < \delta\} \subset \Omega_x$, and $|x - x_0| < \delta$, we have

$$|\langle f, T(x)\psi\rangle_{X^*\times X} - \langle f, T(x_0)\psi\rangle_{X^*\times X}| < \varepsilon/2, \quad f \in X^*.$$

Because $\{\mathcal{E}(\xi)|\xi \in E_{\mathbf{C}}\}$ is total in $(E)_{\mathbf{C}}$, for any $\varphi \in (E)_{\mathbf{C}}$, there exists $\psi \in \{\mathcal{E}(\xi)|\xi \in E_{\mathbf{C}}\}$, such that

$$\|\varphi - \psi\|_p \leq \varepsilon/4M.$$

We then come to

$$\begin{aligned}
&|\langle f, T(x)\varphi\rangle_{X^*\times X} - \langle f, T(x_0)\varphi\rangle_{X^*\times X}| \\
&\quad \leq |\langle f, T(x)\varphi\rangle_{X^*\times X} - \langle f, T(x)\psi\rangle_{X^*\times X}| + |\langle f, T(x)\psi\rangle_{X^*\times X} - \langle f, T(x_0)\psi\rangle_{X^*\times X}| + \\
&\qquad |\langle f, T(x_0)\varphi\rangle_{X^*\times X} - \langle f, T(x_0)\psi\rangle_{X^*\times X}| \\
&\quad \leq \varepsilon/2 + 2M\|f\|_{X^*}|\varphi - \psi|_p \leq \varepsilon\|f\|_{X^*},
\end{aligned}$$

which implies that for each $\varphi \in (E)_{\mathbf{C}}, x \to \langle f, T(x)\varphi\rangle_{X^*\times X}$ is continuous, hence $T(x) \in C_w(\Omega, \mathcal{L})$. This completes the proof. □

Theorem 5 *Let $V(x) \in G_{lu}^2(\Omega,\mathcal{U})$ satisfy the following two conditions:*

1) *For each $\xi \in E_{\mathbf{C}}, x \to V(x)(\xi)$ belongs to $C_w^n(\Omega)$ ($n \in \mathbf{N}$ or $n = \infty$);*
2) *For each $\alpha, |\alpha| \leq n, D^\alpha V \in G_{lu}^2(\Omega,\mathcal{U})$.*

Then the X-valued generalized functional $S^{-1}V \in C_w^n(\Omega, \mathcal{L})$.

Proof We first carry out the proof for $n = 1$.

Let $D_kT = S^{-1}D_kV$, $k = 1, 2, \ldots, d$. By Theorem 4, $D_kT \in C_w(\Omega, \mathcal{L})$. Thus it remains to verify that for any $x \in \Omega, \varphi \in (E)_{\mathbf{C}}$, $k = 1, 2, \ldots, d$, we have

$$\lim_{h\to 0}\{\frac{1}{h}(\langle f, T(x+he_k)\varphi\rangle_{X^*\times X} - \langle f, T(x)\varphi\rangle_{X^*\times X}) - \langle f, D_kT(x)\varphi\rangle_{X^*\times X}\} = 0. \tag{1}$$

By the given conditions, for $\psi \in \{\mathcal{E}(\xi)|\xi \in E_{\mathbf{C}}\}$ we have (1). With the same argument as that in the proof of Theorem 4, we just prove that there exist $\delta > 0, p > 0$, such that $\{\frac{1}{h}[T(x+he_k) - T(x)] : 0 < |h| < \delta\}$ is a bounded subset of $\mathcal{L}[(E_p)_{\mathbf{C}}, X]$.

Let $x \in \Omega$, $\Omega_{x,k}$ be a neighborhood of x, and $D_kV \in G_u^2(\Omega_{x,k}, \mathcal{U})$. According to Theorem 2, for each $x \in \Omega_{x,k}, x \to \langle f, V(x)\rangle_{X^*\times X}$ is continuously differentiable. We choose δ small enough to satisfy $0 < |h| < \delta$ and $x + he_k \in \Omega_{x,k}$. Then for each $\xi \in E_{\mathbf{C}}$, there exists $x_0 \in \Omega_{x,k}$, such that

$$\begin{aligned}&|\frac{1}{h}(\langle f, V(x+he_k)(\xi)\rangle_{X^*\times X} - \langle f, V(x)(\xi)\rangle_{X^*\times X})| \\ &\quad = |(\langle f, V(x_0)(\xi)\rangle_{X^*\times X})'_k| = |\langle f, D_kV(x_0)(\xi)\rangle_{X^*\times X}| \le \|f\|_{X^*}\|D_kV(x_0)(\xi)\|_X.\end{aligned}$$

Form this, we have

$$\|\frac{1}{h}(V(x+he_k)(\xi) - V(x)(\xi)\|_X \le \|D_kV(x_0)(\xi)\|_X.$$

$D_kV \in G_u^2(\Omega_{x,k}, \mathcal{U})$ implies that D_kV satisfies the uniform condition in $\Omega_{x,k}$. According to Theorem 3, there exists $p \ge 0$, such that $\{\frac{1}{h}[T(x+he_k) - T(x)]\}$ is a bounded subset of $\mathcal{L}[(E_p)_{\mathbf{C}}, X]$.

For $n > 1$, we use the inductive approach. Now we suppose the conclusion is correct for n. Let $V(x) \in G_{lu}^2(\Omega, \mathcal{U})$ and for each $\xi \in E_{\mathbf{C}}, x \to V(x)(\xi)$ belong to $C_w^{n+1}(\Omega)$. Then for each $k = 1, 2, \ldots, d$, and $\xi \in E_{\mathbf{C}}$, $x \to D_kV(x)(\xi)$ belongs to $C_w^n(\Omega)$. Also for each $\alpha, |\alpha| = n$, we have $D^\alpha(D_kV) \in G_{lu}^2(\Omega, \mathcal{U})$. Therefore, by induction we come to $S^{-1}V \in C_w^{n+1}(\Omega, \mathcal{L})$. This completes the proof. □

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