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Boundedness of Convolution-Type Operators on Endpoint Triebel-Lizorkin Spaces

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Abstract This paper focuses on the study of the boundedness of convolution-type Calderón-Zygmund operators on some endpoint Triebel-Lizorkin spaces. Applying wavelets, molecular decomposition and interpolation theory, the author establishes the boundedness on certain endpoint Triebel-Lizorkin spaces $\dot{F}_1^{0,q}$ $(2 < q \leq \infty)$ under a very weak pointwise regularity condition.

Keywords convolution-type Calderón-Zygmund operators; endpoint Triebel-Lizorkin spaces; wavelets; molecular decomposition.

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1. Introduction

Let $\mathscr{D} = \mathscr{D}(\mathbb{R}^n)$ denote the space of Schwartz test functions $\{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp}\varphi$ is compact$\}$ and $\mathscr{D}'$ the space of Schwartz distributions (the dual of $\mathscr{D}$). Suppose that we have a linear continuous mapping $T : \mathscr{D} \to \mathscr{D}'$ associated with a kernel $K(x,y)$ in the sense that

$$\langle Tf, g\rangle = \iint g(x)K(x,y)f(y)\mathrm{d}x\mathrm{d}y \tag{1}$$

for test functions f and g with disjoint support. Assume that $K(x,y)$ is continuous on $\Omega = R^n \times \mathbb{R}^n \backslash \{x = y\}$ and satisfies:

$$|K(x,y)| \leq C_1|x-y|^{-n}. \tag{2}$$

If $2|x-x'| \leq |x-y|$, then

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y',x)| \leq \frac{C_2|x-x'|^\gamma}{|x-y|^{n+\gamma}} \tag{3}$$

for $0 < \gamma \leq 1$. And for $f \in L^2(\mathbb{R}^n)$ with compact support, $Tf(x) = \int K(x,y)f(y)\mathrm{d}y$ holds for almost every $x \in (\operatorname{supp} f)^c$. Assume also that T extends to a bounded operator on $L^2(\mathbb{R}^n)$. Then T is said to be a Calderón-Zygmund (C-Z) operator [1], and written as $T \in \mathrm{CZO}_\gamma$.

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Since C-Z operators were introduced by Coifman and Meyer, there has been significant progress on the study of their boundedness on various function spaces. The prototypical result is the famous T1 theorem of David and Journé [2], which states that under the conditions (2) and (3), T extends to a bounded operator on L^2 iff it satisfies the T1 condition:

$$T1 \in \mathrm{BMO}, \quad T^*1 \in \mathrm{BMO}, \tag{4}$$

and the weak bounded condition:

$$\begin{gathered} |\langle Tf, g\rangle| \leq C_3 R^n(\|f\|_\infty + R\|\nabla f\|_\infty)(\|g\|_\infty + R\|\nabla g\|_\infty), \\ \forall R > 0,\ u \in \mathbb{R}^n,\ f, g \in C_0^1(B(u, R)). \end{gathered} \tag{5}$$

Since then, many authors have been devoted to relaxing the regularity condition (3) (see [3, 4]). But up to now, it is still unknown whether (3) can be replaced by Hörmander condition. To scale the extent of being close to Hörmander condition, Yabuta [4] replaced the smooth condition by the weak pointwise condition. For $R = 1, 2, \ldots$, he introduced the following notation:

$$\omega(R) = \sup_{r>0} \sup_{\substack{|x-x'|\leq r \\ 2^R r \leq |x-y| < 2^{R+1} r}} |x-y|^n \{|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')|\}. \tag{6}$$

It is clear that the condition $\sum_{R\geq 1} \omega(R) < \infty$ approaches Hörmander condition infinitely. On the other hand, Deng et al. [5] proved that Hörmander condition cannot ensure the boundedness at least on certain endpoint function spaces; with the index used to measure the extent close to Hörmander condition, Meyer and Yang achieved the boundedness on $\dot{F}_1^{0,q}$ $(1 \leq q \leq 2)$ under the condition that the index is at least 1. As a result, we try to consider convolution-type operators and Hörmander condition.

For an operator T associated with a kernel $K(x, y)$ in the sense of (1), suppose also that T is a convolution-type operator, then $K(x, y)$ can be written as

$$K(x, y) = K(x - y). \tag{7}$$

In [6], it is proved that Hörmander condition can ensure the boundedness on Besov spaces $\dot{B}_p^{0,q}$ $(1 \leq p, q \leq \infty)$ and on Triebel-Lizorkin spaces $\dot{F}_p^{0,q}$ $(1 < p, q < \infty$ or $1 = p \leq q \leq 2)$, but this idea does not work for endpoint Triebel-Lizorkin spaces $\dot{F}_1^{0,q}$ $(2 < q \leq \infty)$. Also, up to the best knowledge of the author, it is not clear whether Hörmander condition can ensure the boundedness on $\dot{F}_1^{0,q}$ $(2 < q \leq \infty)$. Motivated by Yabuta's notion (6), we introduce the following notion:

$$B(R) = \sup_{j\in\mathbb{Z}} \sup_{|m|\leq 2^{-j-1}} \sup_{2^{R-j-1}\leq |x| < 2^{R-j}} |x|^n |K(x) - K(x+m)|, \tag{8}$$

where $R \in \mathbb{Z}$ and $R \geq 0$. And it is easy to verify that $\{B(R)\}_{R\geq 0}$ is a monotonically decreasing sequence. In this paper, we try to introduce a new idea to get the boundedness on $\dot{F}_1^{0,q}(2 < q \leq \infty)$ under the weak pointwise regularity condition:

$$\sum_{R\geq 0} B(R) < \infty. \tag{9}$$

Definition 1 *We say that $K(x) \in H$ if it satisfies (9) and cancellation condition:*

$$\Big|\int K(x)\psi_R(x)\mathrm{d}x\Big| \le C_0(\|\psi\|_\infty + \|\partial^1\psi\|_\infty + \|\partial^2\psi\|_\infty), \quad \forall \psi(x) \in C_0^2(B(0,1)), \tag{10}$$

where $\psi_R(x) = \psi(\frac{x}{R})$ $(R > 0)$.

The purpose of this paper is to prove

Theorem 1 *Suppose that T is a convolution-type operator associated with a kernel $K(x)$ in the sense of (1) and (7). If $K(x) \in H$, then T is bounded on $\dot{F}_1^{0,q}$ $(2 < q \le \infty)$.*

This paper is organized as follows. Some preliminaries will be introduced in Section 2. Section 3 is devoted to the analysis of the kernel-distribution $K(x)$. The proof of Theorem 1 relies on wavelets and molecular decomposition, which will be given at the end of Section 4.

2. Preliminaries

Firstly, let us introduce some notation. Let $\mathbb{N}$ be the collection of all positive integers. As usual, $\mathbb{Z}$ is the collection of all integers, and $\mathbb{Z}^n$, where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \ldots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. By $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions, and by $\mathcal{S}'$ its topological dual, that is, the space of all tempered distributions. If $\varphi \in \mathcal{S}$, then

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}^\ltimes} e^{-ix\xi}\varphi(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^n$$

denotes the Fourier transform of φ. Here ξx is the scalar product in $\mathbb{R}^n$. Throughout this paper, C denotes a positive constant which is independent of the main parameters involved, but it may vary from line to line.

In this paper, we use tensorial real-valued Meyer wavelets. For $t \in \mathbb{R}$, denote by $\Phi^0(t)$ and $\Phi^1(t)$ the father and mother wavelet, respectively. For $x \in \mathbb{R}^n$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{0,1\}^n$, let $\Phi^\varepsilon(x) = \prod_{i=1}^n \Phi^{\varepsilon_i}(x_i)$. Then there exist two positive numbers C, C' such that

$$\operatorname{supp} \widehat{\Phi^\varepsilon}(\xi) \subset \{\xi : C \le |\xi| \le C'\}, \quad \forall \varepsilon \ne 0. \tag{11}$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let

$$f_{j,k}(x) = 2^{\frac{jn}{2}} f(2^j x - k).$$

Denote

$$E_n = \{0,1\}^n \setminus \{0\} \text{ and } \Lambda_n = \{\lambda = (\varepsilon, j, k), \varepsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

then $\{\Phi^\varepsilon_{j,k}(x)\}_{\lambda \in \Lambda_n}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$.

Let S'/P be the collection of distributions modulo polynomial functions. For any distribution $f(x) \in S'/P$, if we can define $f^\varepsilon_{j,k} = \langle f(x), \Phi^\varepsilon_{j,k}(x)\rangle$ $(\forall (\varepsilon, j, k) \in \Lambda_n)$, then the following equality is true in the sense of distribution

$$f(x) = \sum_{(\varepsilon,j,k)\in\Lambda_n} f^\varepsilon_{j,k}\Phi^\varepsilon_{j,k}(x). \tag{12}$$

For Besov spaces $\dot{B}_p^{0,q}$ $(1 \le p, q \le \infty)$ and Triebel-Lizorkin spaces $\dot{F}_p^{0,q}$ $(1 \le p < \infty, 1 \le q \le \infty)$, which are defined by Triebel in [7], they can be characterized by wavelets [8]. For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let $Q_{j,k} = \{x : 2^j x - k \in [0,1)^n\}$ and let $\chi(2^j x - k)$ be the characteristic function on the cube $Q_{j,k}$.

Lemma 1 *Let $f(x) \in \mathcal{S}'/P$ be represented as (12).*

(i) For $1 \le p,\ q \le \infty$, there exist two constants $C_{p,q}$ and $C'_{p,q}$ such that

$$C_{p,q}\|f(x)\|_{\dot{B}_p^{0,q}} \le \Big(\sum_j 2^{jqn(\frac{1}{2}-\frac{1}{p})}\big(\sum_{\varepsilon,k}|f_{j,k}^{\varepsilon}|^p\big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \le C'_{p,q}\|f(x)\|_{\dot{B}_p^{0,q}}.$$

(ii) Similarly, for $1 \le p < \infty$, $1 \le q \le \infty$, there exist two constants $C_{p,q}$ and $C'_{p,q}$ such that

$$C_{p,q}\|f(x)\|_{\dot{F}_p^{0,q}} \le \Big\|\Big(\sum_{\varepsilon,j,k} 2^{\frac{jqn}{2}}|f_{j,k}^{\varepsilon}|^q \chi(2^j x - k)\Big)^{\frac{1}{q}}\Big\|_{L^p} \le C'_{p,q}\|f(x)\|_{\dot{F}_p^{0,q}}.$$

In addition, we know that $\dot{B}_q^{0,q} = \dot{F}_q^{0,q}$ $(1 \le q \le \infty)$ (see [7]).

Next, we review some facts about the molecular decomposition of Triebel-Lizorkin spaces $\dot{F}_1^{0,q}(1 \le q \le \infty)$ (see [8]). Put $\Theta = \{Q_{j,k}, j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$.

Definition 2 *Given $1 \le q \le \infty$. We call $a(x)$ an $\dot{F}_q^{0,q}$-molecule in $\dot{F}_1^{0,q}$, if there exists a cube $Q_{s,p} \in \Theta$ such that:*

(i) $a(x) = \sum_{\substack{(\varepsilon,j,k)\in\Lambda_n \\ Q_{j,k}\subset Q_{s,p}}} a_{j,k}^{\varepsilon}\Phi_{j,k}^{\varepsilon}(x)$; (2) $\|a(x)\|_{\dot{F}_q^{0,q}} \le |Q_{s,p}|^{\frac{1}{q}-1}$.

Lemma 2 *Given $1 \le q \le \infty$. The following two conditions are equivalent:*

(i) $f(x) \in \dot{F}_1^{0,q}$;

(ii) There exist $\{\lambda_m\}_{m\in\mathbb{N}} \in l^1$ and $\dot{F}_q^{0,q}$-molecules $b_m(x)$ $(m \in \mathbb{N})$ such that $f(x) = \sum_m \lambda_m b_m(x)$.

This lemma will be pivotal in our paper. In fact, to prove that an operator T is bounded on $\dot{F}_1^{0,q}$, it suffices to check that the $\dot{F}_1^{0,q}$-norm of $Ta(x)$ is bounded and the boundedness is independent of $a(x)$, where $a(x)$ is an arbitrary $\dot{F}_q^{0,q}$-molecule. This strategy will be used in Section 4.

In what follows, we recall the well-known interpolation property of Triebel-Lizorkin spaces, which is proved in many real analysis books [8]. For $0 < \theta < 1$, $1 \le p_0,\ p_1 < \infty$ and $1 \le q_0$, $q_1 \le \infty$, let $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Lemma 3 *If $\|T\|_{\dot{F}_{p_i}^{0,q_i} \to \dot{F}_{p_i}^{0,q_i}} \le C_i$ $(i = 0,1)$, then $\|T\|_{\dot{F}_p^{0,q} \to \dot{F}_p^{0,q}} \le C_0^{1-\theta}C_1^{\theta}$.*

Finally, we need also some known results related to the boundedness of non-convolution C-Z operators. Here, we introduce a definition first.

Definition 3 *Given $\gamma \in (0,1]$. For a non-convolution operator T in the sense of (1), we say that $T \in \mathrm{SCZO}_\gamma$ if T satisfies (2), (3), (5) and the following strong $T1$ condition:*

$$T1 = T^*1 = 0. \tag{13}$$

For an operator $T \in \mathrm{SCZO}_\gamma$, the constant C_i $(i = 1,2,3)$ appearing in its definition is called

C-Z constant of T. And the following lemma can be directly deduced from the result of Frazier et al. [9].

Lemma 4 *Given $1 \le q \le \infty$ and $0 < \gamma \le 1$. If $T \in \mathrm{SCZO}_\gamma$, then T is bounded on $\dot{F}_1^{0,q}$.*

For $|\varepsilon||\varepsilon'| \ne 0$, if $m \ge 0$, denote $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) = (\Phi_{m,0}^{\varepsilon'} * \Phi^\varepsilon)(2^{-m}x)$; if $m < 0$, denote $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) = (\Phi_{m,0}^{\varepsilon'} * \Phi^\varepsilon)(x)$. In fact, since $\widehat{\Phi^\varepsilon}(\xi)$ is compactly supported, there exists some constant C such that $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) = 0$, $|m| > C$. In addition, for $|\varepsilon||\varepsilon'| \ne 0$ and $|m| \le C$, we apply the properties of Meyer wavelets to easily verify that $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) \in \mathcal{S}$ and

$$\int x_{i_\varepsilon}^\alpha \widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x)\mathrm{d}x = 0, \quad \forall \alpha \in \mathbb{Z},\ \alpha \ge 0,$$

where i_ε denotes the smallest index i such that $\varepsilon_i \ne 0$. For $|\varepsilon||\varepsilon'| \ne 0$ and $m \in \mathbb{Z}$, let $T_m^{\varepsilon,\varepsilon'}$ be the operator whose kernel is

$$K_m^{\varepsilon,\varepsilon'}(x,y) = \sum_{j,k} 2^{jn}\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(2^jx-k)\Phi^\varepsilon(2^jy-k).$$

Then we have

Lemma 5 *For $|\varepsilon||\varepsilon'| \ne 0$ and $|m| \le C$, there exists $\gamma \in (0,1]$ such that $T_m^{\varepsilon,\varepsilon'} \in \mathrm{SCZO}_\gamma$. And the C-Z constants of $T_m^{\varepsilon,\varepsilon'}$ are independent of ε, ε', m.*

In fact, by the idea used to prove that H^1 defined by a wavelet basis is independent of the wavelet basis chosen (see more details in Section 4 of Chapter 7 in Meyer [10]), we can easily deduce this lemma.

3. Analysis of some distributions

This section is aimed to analyze some kernel-distributions. Assume that M is some positive integer large enough. For any integer $R \ge M+3$, set

$$\tau(R) = B(R-2) + 2^{-R}\big(C_0 + \sum_{R\ge 1} B(R)\big).$$

$\forall(\varepsilon,j,k) \in \Lambda_n$, define $b_{j,k}^\varepsilon = \langle K(x), \Phi_{j,k}^\varepsilon(x)\rangle$, then

Theorem 2 *Given $R \ge M+6$ and $R \in \mathbb{Z}$. There exists a constant C such that*

(i) $|b_{j,k}^\varepsilon| \le \frac{C2^{\frac{jn}{2}}\tau(R)}{(1+|k|)^n}$, $\forall(\varepsilon,j,k) \in \Lambda_n$ *and* $2^{R-1} \le |k| < 2^R$.

(ii) $\sum_{k\in\mathbb{Z}^n} |b_{j,k}^\varepsilon| \le C2^{\frac{jn}{2}}(C_0 + \sum_{r\ge M+3} B(r))$, $\forall \varepsilon \in E_n$ *and* $j \in \mathbb{Z}$.

And C is only dependent on M and n.

To prove it, we will apply the appropriate decomposition of Meyer wavelets which can be deduced from the result in [6]. For $\varepsilon \ne 0$, let i_ε be the smallest index i such that $\varepsilon_i \ne 0$. Denote by $e^\varepsilon \in \mathbb{Z}^n$ the vector whose component is 1 at the i_ε^{th}-position, and is 0 elsewhere.

Lemma 6 *For $\Phi^\varepsilon(x)$ ($\varepsilon \ne 0$) and $N \in \mathbb{N}$, there exist $\Phi^{(\varepsilon,k,N)}(x)$, $\widetilde{\Phi}_{k,N}^\varepsilon(x)(k \in \mathbb{Z}^n)$ such that $\Phi^\varepsilon(x) = \sum_{k\in\mathbb{Z}^n} a_k^{\varepsilon,N}\Phi^{(\varepsilon,k,N)}(x-k)$ and*

(i) $|a_k^{\varepsilon,N}| \le \frac{C_N}{(1+|k|)^N}$;

(ii) $\Phi^{(\varepsilon,k,N)}(x) = \widetilde{\Phi}^{\varepsilon}_{k,N}(x) - \widetilde{\Phi}^{\varepsilon}_{k,N}(x - \frac{1}{2}e^{\varepsilon})$;

(iii) $\operatorname{supp} \widetilde{\Phi}^{\varepsilon}_{k,N}(x) \subset B(0, 2^M)$;

(iv) $\forall 0 \le m \le N$, $\|\widetilde{\Phi}^{\varepsilon}_{k,N}(x)\|_{C^m} \le C_N$;

(v) $\forall \alpha \in \mathbb{Z}$ *and* $0 \le \alpha < N$, $\int x^{\alpha}_{i_\varepsilon} \widetilde{\Phi}^{\varepsilon}_{k,N}(x) \mathrm{d}x = 0$.

Fix $\varepsilon \in E_n$, $m \in \mathbb{Z}^n$ and $N \in \mathbb{N}$. For $k(x) \in K$, set $d^{\varepsilon,m,N}_{j,k} = \langle K(x), \Phi^{(\varepsilon,m,N)}_{j,k}(x)\rangle$.

Lemma 7 *Given* $\varepsilon \in E_n$, $m \in \mathbb{Z}^n$ *and* $N \in \mathbb{N}$, *let* $\Phi^{(\varepsilon,m,N)}(x)$ *be defined as in Lemma 6. Then*

(i) $|d^{\varepsilon,m,N}_{j,k}| \le C_0 2^{\frac{jn}{2}}$, $\forall j \in \mathbb{Z}$ *and* $|k| \le 2^{M+3}$.

(ii) $|d^{\varepsilon,m,N}_{j,k}| \le \frac{C 2^{\frac{jn}{2}} B(R-1)}{(1+|k|)^n}$, $\forall j \in \mathbb{Z}$, $R \ge M+4$, $2^{R-1} \le |k| < 2^R$.

The above constant C *is only dependent on* M, N *and* n.

Proof Given $j \in \mathbb{Z}$ and $|k| \le 2^{M+3}$. By Lemma 6, we know that $\operatorname{supp} \Phi^{(\varepsilon,m,N)}(x-k) \subset B(0, 2^{M+4})$. Further, we have $\operatorname{supp} \Phi^{(\varepsilon,m,N)}(2^{M+4}x - k) \subset B(0,1)$. Together with the condition (10), we have

$$|d^{\varepsilon,m,N}_{j,k}| = 2^{\frac{jn}{2}} \Big| \int K(x) \Phi^{(\varepsilon,m,N)}(\frac{2^{M+4}x}{2^{M+4-j}} - k) \mathrm{d}x \Big| \le C_0 2^{\frac{jn}{2}}.$$

Next, we consider (ii). For $R \ge M+4$ and $2^{R-1} \le |k| < 2^R$, by Lemma 6, we get

$$\begin{aligned} |d^{\varepsilon,m,N}_{j,k}| &= \Big| \int K(x) \Phi^{(\varepsilon,m,N)}_{j,k}(x) \mathrm{d}x \Big| \\ &= 2^{\frac{jn}{2}} \Big| \int K(x) \big(\widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k) - \widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k - \frac{1}{2}e^{\varepsilon})\big) \mathrm{d}x \Big| \\ &= 2^{\frac{jn}{2}} \Big| \int \big(K(x) - K(x + 2^{-j-1}e^{\varepsilon})\big) \widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k) \mathrm{d}x \Big|. \end{aligned} \tag{14}$$

And by $\operatorname{supp} \widetilde{\Phi}^{\varepsilon}_{m,N}(x) \subset B(0, 2^M)$, one knows that, if $x \in \operatorname{supp} \widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k)$, then $2^j |x| \sim (1 + |k|)$ and $2^{-j}(|k| - 2^M) \le |x| \le 2^{-j}(|k| + 2^M)$. Further, one gets that $\operatorname{supp} \widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k) \subset A_j(R)$ where $A_j(R) = \{x : 2^{R-2-j} \le |x| \le 2^{R+1-j}\}$. Since $A_j(R)$ can be decomposed into three parts:

$$\{x : 2^{R-2-j} \le |x| < 2^{R-1-j}\}, \{x : 2^{R-1-j} \le |x| < 2^{R-j}\} \text{ and } \{x : 2^{R-j} \le |x| < 2^{R+1-j}\},$$

together with (7), we have

$$|x|^n |K(x) - K(x + 2^{-j-1}e^{\varepsilon})| \le C B(R-1), \quad \forall x \in A_j(R). \tag{15}$$

Hence, from (14) and (15),

$$|d^{\varepsilon,m,N}_{j,k}| \le \frac{C 2^{\frac{jn}{2}} B(R-1)}{(1+|k|)^n} 2^{jn} \int |\widetilde{\Phi}^{\varepsilon}_{m,N}(2^j x - k)| \mathrm{d}x \le \frac{C 2^{\frac{jn}{2}} B(R-1)}{(1+|k|)^n}. \quad \square$$

Proof of Theorem 2 For any $(\varepsilon, j, k) \in \Lambda_n$, by Lemma 6, there exists an integer $N > n$ large enough such that

$$|b^{\varepsilon}_{j,k}| = \Big| \int K(x) \Phi^{\varepsilon}_{j,k}(x) \mathrm{d}x \Big| = 2^{\frac{jn}{2}} \Big| \int \sum_m a^{\varepsilon,N}_m \Phi^{(\varepsilon,m,N)}(2^j x - k - m) K(x) \mathrm{d}x \Big|.$$

Further,

$$|b^{\varepsilon}_{j,k}| = 2^{\frac{jn}{2}} \Big| \int \sum_t a^{\varepsilon,N}_{t-k} \Phi^{(\varepsilon,t-k,N)}(2^j x - t) K(x) \mathrm{d}x \Big|. \tag{16}$$

Fix $N = 2n+1$. Firstly, we consider (i) in Theorem 2. Given $R \geq M+6$ and $2^{R-1} \leq |k| < 2^R$.

$$\begin{aligned}|b^{\varepsilon}_{j,k}| \leq & 2^{\frac{jn}{2}} \Big| \int \sum_{|t| \leq 2^{M+3}} a^{\varepsilon,N}_{t-k} \Phi^{(\varepsilon,t-k,N)}(2^j x - t) K(x) \mathrm{d}x \Big| + \\ & 2^{\frac{jn}{2}} \Big| \int \sum_{|t| > 2^{M+3}} a^{\varepsilon,N}_{t-k} \Phi^{(\varepsilon,t-k,N)}(2^j x - t) K(x) \mathrm{d}x \Big| \\ := & I^R_{j,k} + II^R_{j,k}. \end{aligned} \tag{17}$$

For the term $I^R_{j,k}$, we apply Lemma 7 to get

$$\begin{aligned} I^R_{j,k} &\leq CC_0 2^{\frac{jn}{2}} \sum_{|t| \leq 2^{M+3}} \frac{1}{(1+|t-k|)^{2n+1}} \\ &\leq CC_0 2^{\frac{jn}{2}} \sum_{|t| \leq 2^{M+3}} \frac{1}{(1+|t-k|)^{2n+1}(1+|t|)^{n+1}} \\ &\leq \frac{CC_0 2^{\frac{jn}{2}}}{(1+|k|)^{n+1}}. \end{aligned} \tag{18}$$

Then, we estimate $II^R_{j,k}$. By Lemma 7, we have

$$\begin{aligned} II^R_{j,k} \leq & C 2^{\frac{jn}{2}} \sum_{r \geq M+4} \sum_{2^{r-1} \leq |t| < 2^r} \frac{B(r-1)}{(1+|t-k|)^{2n+1}(1+|t|)^n} \\ \leq & \frac{C 2^{\frac{jn}{2}}}{(1+|k|)^n} \sum_{r \geq M+4} \sum_{2^{r-1} \leq |t| < 2^r} \frac{B(r-1)}{(1+|t-k|)^{n+1}} \\ = & \frac{C 2^{\frac{jn}{2}}}{(1+|k|)^n} \Big\{ \sum_{r=M+4}^{R-2} \sum_{2^{r-1} \leq |t| < 2^r} \frac{B(r-1)}{(1+|t-k|)^{n+1}} + \\ & \sum_{r \geq R-1} \sum_{2^{r-1} \leq |t| < 2^r} \frac{B(r-1)}{(1+|t-k|)^{n+1}} \Big\} \\ := & \frac{C 2^{\frac{jn}{2}}}{(1+|k|)^n} \{ II^{R,1}_{j,k} + II^{R,2}_{j,k} \}. \end{aligned} \tag{19}$$

As for $II^{R,1}_{j,k}$, by Lemma 7 and $|t-k| \geq |k| - |t| \geq \frac{|k|}{2} \geq |t|$, it follows that

$$II^{R,1}_{j,k} \leq C \sum_{r=M+4}^{R-2} \sum_{2^{r-1} \leq |t| < 2^r} \frac{B(r-1)}{(1+|t|)^n (1+|k|)} \leq \frac{C}{1+|k|} \sum_{r=M+4}^{R-2} B(r-1). \tag{20}$$

As for $II^{R,2}_{j,k}$, noting the monotonically decreasing property of $\{B(R)\}_{R \geq 0}$, we have

$$II^{R,2}_{j,k} \leq \sum_{r \geq R-1} \sum_{2^{r-1} \leq |t| < 2^r} \frac{CB(R-2)}{(1+|t-k|)^{n+1}} \leq CB(R-2). \tag{21}$$

Taking into account (17)–(21), we get

$$|b^{\varepsilon}_{j,k}| \leq \frac{C 2^{\frac{jn}{2}}}{(1+|k|)^n} (C_0 2^{-R} + 2^{-R} \sum_{R \geq 1} B(R) + B(R-2)) = \frac{C 2^{\frac{jn}{2}} \tau(R)}{(1+|k|)^n}.$$

Next, we prove (ii) in Theorem 2. By (16) and Lemma 7, one gets

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}|b^{\varepsilon}_{j,k}|\le&\sum_{k\in\mathbb{Z}^n}\sum_{|t|<2^{M+3}}\frac{CC_0 2^{\frac{jn}{2}}}{(1+|t-k|)^{2n+1}}+\\&\sum_{k\in\mathbb{Z}^n}\sum_{r\ge M+4}\sum_{2^{R-1}\le|t|<2^R}\frac{C2^{\frac{jn}{2}}B(r)}{(1+|t-k|)^{2n+1}(1+|t|)^n}\\\le&CC_0 2^{\frac{jn}{2}}+C2^{\frac{jn}{2}}\sum_{r\ge M+4}B(r)\\\le&C2^{\frac{jn}{2}}\big(C_0+\sum_{r\ge M+3}B(r)\big).\end{aligned}$$

The proof of Theorem 2 is completed. □

4. Proof of Theorem 1

Under the conditions of Theorem 1, if T is bounded on $\dot F_1^{0,\infty}$ and $\dot F_1^{0,1}$, then we can apply the interpolation theory to get the proof of Theorem 1 easily. To obtain the boundedness of T on $\dot F_1^{0,\infty}$, we consider first the action of T on each $\dot F_\infty^{0,\infty}$-molecule.

Lemma 8 *Assume that $a(x)$ is an $\dot F_\infty^{0,\infty}$-molecule on an arbitrary cube $Q_{s,p}\in\Theta$ ($s\in\mathbb{Z}$, $p\in\mathbb{Z}^n$). Under the conditions of Theorem 1, we have*

$$\|Ta(x)\|_{\dot F_1^{0,\infty}}\le C(C_0+\sum_{R\ge1}B(R)),\tag{22}$$

where the constant C is dependent only on M and n.

Proof According to Definition 2, we have

$$Ta(x)=(K*a)(x)=\sum_{\varepsilon,\varepsilon'}\sum_{j',k'}\sum_{Q_{j,k}\subset Q_{s,p}}b^{\varepsilon'}_{j',k'}a^{\varepsilon}_{j,k}(\Phi^{\varepsilon}_{j,k}*\Phi^{\varepsilon'}_{j',k'})(x).$$

Taking the Fourier transform of both sides gives

$$\widehat{Ta}(\xi)=\sum_{(\varepsilon,j,k,\varepsilon',j',k')}2^{-\frac{(j+j')n}{2}}e^{-i\xi(2^{-j}k+2^{-j'}k')}b^{\varepsilon'}_{j',k'}a^{\varepsilon}_{j,k}\widehat{\Phi^{\varepsilon}}(2^{-j}\xi)\widehat{\Phi^{\varepsilon'}}(2^{-j'}\xi).$$

Noting that $\widehat{\Phi^{\varepsilon}}(\xi)$ satisfies (11), we get $|j-j'|\le C$ if $\widehat{\Phi^{\varepsilon}}(2^{-j}\xi)\widehat{\Phi^{\varepsilon'}}(2^{-j'}\xi)\neq0$. Set $m=j'-j$, then

$$Ta(x)=\sum_{\varepsilon,\varepsilon'}\sum_{|m|\le C}\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}}b^{\varepsilon'}_{j+m,k'}a^{\varepsilon}_{j,k}(\Phi^{\varepsilon}_{j,k}*\Phi^{\varepsilon'}_{j+m,k'})(x).$$

Further,

$$\begin{aligned}\|Ta(x)\|_{\dot F_1^{0,\infty}}\le&\sum_{\varepsilon,\varepsilon'}\sum_{|m|\le C}\Big\|\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}}b^{\varepsilon'}_{j+m,k'}a^{\varepsilon}_{j,k}(\Phi^{\varepsilon}_{j,k}*\Phi^{\varepsilon'}_{j+m,k'})(x)\Big\|_{\dot F_1^{0,\infty}}\\:=&\sum_{\varepsilon,\varepsilon'}\sum_{|m|\le C}I(\varepsilon,\varepsilon',m).\end{aligned}\tag{23}$$

Thus, to prove (22), we only need to estimate each $I(\varepsilon,\varepsilon',m)$.

Lemma 9 *Given* $|\varepsilon||\varepsilon'| \neq 0$ *and* $|m| \leq C$, *we have*

$$I(\varepsilon, \varepsilon', m) \leq C\big(C_0 + \sum_{R \geq 1} B(R)\big).$$

Proof (i) Assume that $m = 0$. Put $\widetilde{\Phi}_0^{\varepsilon,\varepsilon'}(x) = (\Phi^{\varepsilon'} * \Phi^{\varepsilon})(x)$, then

$$\begin{aligned} I(\varepsilon, \varepsilon', 0) &= \big\| \sum_{k'} \sum_{Q_{j,k} \subset Q_{s,p}} b_{j,k'}^{\varepsilon'} a_{j,k}^{\varepsilon} (\Phi_{j,k}^{\varepsilon} * \Phi_{j,k'}^{\varepsilon'})(x) \big\|_{\dot{F}_1^{0,\infty}} \\ &= \big\| \sum_{k'} \sum_{Q_{j,k} \subset Q_{s,p}} b_{j,k'}^{\varepsilon'} a_{j,k}^{\varepsilon} \widetilde{\Phi}_0^{\varepsilon,\varepsilon'} (2^j x - (k + k')) \big\|_{\dot{F}_1^{0,\infty}}. \end{aligned}$$

Let $T_0^{\varepsilon,\varepsilon'}$ be the operator whose kernel is $K_0^{\varepsilon,\varepsilon'}(x, y) = \sum_{j,t} 2^{jn} \widetilde{\Phi}_0^{\varepsilon,\varepsilon'}(2^j x - t)\Phi^{\varepsilon}(2^j y - t)$. By Lemmas 4 and 5, we get that $T_0^{\varepsilon,\varepsilon'}$ is bounded on $\dot{F}_1^{0,\infty}$. Hence, by replacing the above t with $k + k'$, one can obtain

$$I(\varepsilon, \varepsilon', 0) \leq C \big\| \sum_{k'} \sum_{Q_{j,k} \subset Q_{s,p}} b_{j,k'}^{\varepsilon'} a_{j,k}^{\varepsilon} \Phi^{\varepsilon}(2^j x - (k + k')) \big\|_{\dot{F}_1^{0,\infty}}.$$

Set $l = k + k'$ and $u_{j,l}^{\varepsilon,\varepsilon'} = \sum_{k: Q_{j,k} \subset Q_{s,p}} b_{j,l-k}^{\varepsilon'} a_{j,k}^{\varepsilon}$, then

$$I(\varepsilon, \varepsilon', 0) \leq C \big\| \sum_{j \geq s} \sum_{l} u_{j,l}^{\varepsilon,\varepsilon'} \Phi^{\varepsilon}(2^j x - l) \big\|_{\dot{F}_1^{0,\infty}} \leq C \int \sup_{j \geq s} \sum_{l} |u_{j,l}^{\varepsilon,\varepsilon'}| \chi(2^j x - l) \mathrm{d}x.$$

For $j \geq s$, let $f_j(x) = \sum_l |u_{j,l}^{\varepsilon,\varepsilon'}| \chi(2^j x - l)$. For $i \in \mathbb{N}$, denote by $2^i Q_{s,p}$ the cube which has the same centre as $Q_{s,p}$, the radius being 2^i times that of $Q_{s,p}$, and the sides parallelling respectively those of $Q_{s,p}$. Put

$$Q_0 = 2Q_{s,p} \text{ and } Q_i = 2^{i+1} Q_{s,p} \backslash 2^i Q_{s,p}, \quad i = 1, 2, \ldots, \tag{24}$$

then we have

$$I(\varepsilon, \varepsilon', 0) \leq C \sum_{i \geq 0} \int_{Q_i} \sup_{j \geq s} f_j(x) \mathrm{d}x. \tag{25}$$

For $0 \leq i < 3M$, by Theorem 2, we can easily get

$$\int_{Q_i} \sup_{j \geq s} f_j(x) \mathrm{d}x \leq C(C_0 + \sum_{R \geq 1} B(R)).$$

On the other hand, for $i \geq 3M$, we estimate $\sup_{j \geq s} f_j(x)$ $(x \in Q_i)$ first.

Given $i \geq 3M$ and $x_0 \in Q_i$. For any $j \geq s$, there exists $l(x_0, j) \in \mathbb{Z}^n$ such that $x_0 \in Q_{j,l(x_0,j)} \subset Q_i$. Together with $Q_{j,k} \subset Q_{s,p}$, we can get $|k - l(x_0, j)| \sim 2^{j-s+i}$. And from $\|a(x)\|_{\dot{F}_\infty^{0,\infty}} \leq |Q_{s,p}|^{-1}$ and Theorem 2, we get

$$f_j(x_0) \leq C \sum_{k: Q_{j,k} \subset Q_{s,p}} |b_{j,l(x_0,j)-k}^{\varepsilon'}||a_{j,k}^{\varepsilon}| \leq C \frac{\tau(j - s + i - M)}{2^{in}|Q_{s,p}|}.$$

Notice that $\{\tau(R)\}_{R \geq M+3}$ is a monotonically decreasing sequence, then

$$f_j(x_0) \leq C \frac{\tau(i - M)}{2^{in}|Q_{s,p}|}.$$

Thus, we obtain $\sup_{j\geq s} f_j(x_0) \leq C\frac{\tau(i-M)}{2^{in}|Q_{s,p}|}$. Further, it follows that

$$\sup_{j\geq s} f_j(x) \leq C\frac{\tau(i-M)}{2^{in}|Q_{s,p}|}, \quad \forall x\in Q_i. \tag{26}$$

Together with (25), one has

$$I(\varepsilon,\varepsilon',0) \leq C(C_0+\sum_{R\geq 1} B(R)) + C\sum_{i\geq 3M}\tau(i-M) \leq C\big(C_0+\sum_{R\geq 1} B(R)\big). \tag{27}$$

(ii) For the case where $0<m\leq C$, denote $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) = (\Phi_{m,0}^{\varepsilon'} * \Phi^{\varepsilon})(2^{-m}x)$. By the analogous idea in (i),

$$\begin{aligned} I(\varepsilon,\varepsilon',m) &= \Big\|\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}} b_{j+m,k'}^{\varepsilon'} a_{j,k}^{\varepsilon}\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(2^{j+m}x-(2^mk+k'))\Big\|_{\dot{F}_1^{0,\infty}} \\ &\leq C\Big\|\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}} b_{j+m,k'}^{\varepsilon'} a_{j,k}^{\varepsilon}\Phi^{\varepsilon}(2^{j+m}x-(2^mk+k'))\Big\|_{\dot{F}_1^{0,\infty}}. \end{aligned}$$

Let $l=2^mk+k'$, $u_{j,l}^{\varepsilon,\varepsilon',m} = \sum_{k:Q_{j,k}\subset Q_{s,p}} b_{j+m,l-2^mk}^{\varepsilon'} a_{j,k}^{\varepsilon}$ and $f_j^m(x) = \sum_l u_{j,l}^{\varepsilon,\varepsilon',m}\Phi^{\varepsilon}(2^{j+m}x-l)$. Then

$$I(\varepsilon,\varepsilon',m) \leq C\Big\|\sum_{j\geq s}\sum_{l} u_{j,l}^{\varepsilon,\varepsilon',m}\Phi^{\varepsilon}(2^{j+m}x-l)\Big\|_{\dot{F}_1^{0,\infty}} \leq C\int \sup_{j\geq s} f_j^m(x)\mathrm{d}x.$$

Similarly to (25), we have

$$I(\varepsilon,\varepsilon',m) \leq C\sum_{i\geq 0}\int_{Q_i}\sup_{j\geq s} f_j^m(x)\mathrm{d}x, \tag{28}$$

where Q_i $(i=0,1,\ldots)$ is defined as (4.3). Then, as we deal with $\sup_{j\geq s} f_j(x)$ $(x\in Q_i)$ in (i), then

$$\int_{Q_i}\sup_{j\geq s} f_j^m(x)\mathrm{d}x \leq C(C_0+\sum_{R\geq 1} B(R)), \quad 0\leq i<3M \tag{29}$$

and

$$\int_{Q_i}\sup_{j\geq s} f_j^m(x) \leq C\tau(i-M), \quad i\geq 3M. \tag{30}$$

Together with (28), we have $I(\varepsilon,\varepsilon',m) \leq C\big(C_0+\sum_{R\geq 1} B(R)\big)$.

(iii) For $-C\leq m<0$, set $\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(x) = (\Phi_{m,0}^{\varepsilon'} * \Phi^{\varepsilon})(x)$. As we do in (i) and (ii), it is not hard to obtain

$$\begin{aligned} I(\varepsilon,\varepsilon',m) &= \Big\|\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}} b_{j+m,k'}^{\varepsilon'} a_{j,k}^{\varepsilon}\widetilde{\Phi}_m^{\varepsilon,\varepsilon'}(2^{j}x-(k+2^{-m}k'))\Big\|_{\dot{F}_1^{0,\infty}} \\ &\leq C\Big\|\sum_{k'}\sum_{Q_{j,k}\subset Q_{s,p}} b_{j+m,k'}^{\varepsilon'} a_{j,k}^{\varepsilon}\Phi^{\varepsilon}(2^{j}x-(k+2^{-m}k'))\Big\|_{\dot{F}_1^{0,\infty}} \\ &\leq C\big(C_0+\sum_{R\geq 1} B(R)\big). \end{aligned}$$

This completes the proof of Lemma 9. □

As for $\dot{F}_1^{0,1}$, we know that $\dot{F}_1^{0,1} = \dot{B}_1^{0,1}$ and Hörmander condition can ensure the boundedness of convolution operators on $\dot{B}_1^{0,1}$ (see Theorem 3.1 in [6]). On the other hand, since the condition

(9) is stronger than Hörmander condition, we have

Lemma 10 *Under the conditions of Theorem 1, T is bounded on $\dot{F}_1^{0,1}$.*

Now, we are in a position to prove Theorem 1. By Lemmas 2 and 8, T is bounded on $\dot{F}_1^{0,\infty}$. Together with Lemma 10, we apply the interpolation property (Lemma 3) to get that T is bounded on $\dot{F}_1^{0,q}(2 < q \leq \infty)$.

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