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Young's Inequality for Positive Operators

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Abstract The classical Young's inequality and its refinements are applied to positive operators on a Hilbert space at first. Based on the classical Poisson integral formula of relevant operators, some new inequalities on unitarily invariant norm of $A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}$ are obtained with effective calculation, where $A,\ B,\ X,\ Y \in \mathcal{B}(\mathcal{H})$ with $A,\ B \geqslant 0$ and $1 < p,\ q < \infty$ with the conjugate exponent $q = p/(p-1)$.

Keywords Young's inequality; positive operator.

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1. Introduction

The classical Young inequality for two nonnegative scalars says that if $a, b \geqslant 0$ and $0 \leqslant v \leqslant 1$, then $a^v b^{1-v} \leqslant va + (1-v)b$ with equality if and only if $a = b$. If $v = \frac{1}{2}$, we obtain the arithmetic-geometric mean inequality $\sqrt{ab} \leqslant \frac{1}{2}(a+b)$. Recently, Kittaneh and Manasrah refined the Young's inequality in [1] and proved that if $a, b \geqslant 0$ and $0 \leqslant v \leqslant 1$, then

$$a^v b^{1-v} + r_0(\sqrt{a} - \sqrt{b}) \leqslant va + (1-v)b,$$

where $r_0 = \min\{v, 1-v\}$.

Young's inequality in operator algebras has been considered in [2] and references therein. Bhatia and Parthasarathy in [3] and Kosaki in [4] proved that if $A,\ B,\ X \in \mathcal{M}_n(\mathbb{C})$ with that A and B are positive semi-definite and if $0 \leqslant v \leqslant 1$, then

$$\|A^v XB^{1-v}\|_2 \leqslant \|vAX + (1-v)XB\|_2. \tag{1}$$

It should be mentioned here that for $v \neq \frac{1}{2}$, the inequality (1) may not hold for other unitary invariant norms. On the other hand, Bhatia and Davis proved in [5] that if $A, B, X \in \mathcal{M}_n(\mathbb{C})$ with A, B being positive semi-definite and if $0 \leqslant v \leqslant 1$, then

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leqslant |||\frac{A^vXB^{1-v} + A^{1-v}XB^v}{2}||| \leqslant |||\frac{AX+XB}{2}|||$$

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is true for any unitary invariant norm $|||\cdot|||$. Moreover, a readable account on $A^{\frac{1}{p}}XB^{\frac{1}{q}} \pm A^{\frac{1}{q}}XB^{\frac{1}{p}}$ ($1 < p < \infty$ and $q = p/(p-1)$) and related inequalities can be found in [6].

The purpose of this article is firstly to improve Young's inequality for positive operator on a Hilbert space $\mathcal{H}$. Secondly, by means of the classical Poisson integral formula, we obtain new estimations for unitary invariant norm of $A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}$ where A, B, X, $Y \in \mathcal{B}(\mathcal{H})$ with A, $B \geqslant 0$ and $1 < p$, $q < \infty$ with the conjugate exponent $q = p/(p-1)$.

2. Young's inequalities for positive operators

In this section we begin with the famous Hölder-McCarthy Inequality.

Lemma 2.1 (Hölder-McCarthy Inequality) *Let $A \in \mathcal{B}(\mathcal{H})$ with $A \geqslant 0$. Then the following properties hold:*

(i) $\langle A^r x, x\rangle \geqslant \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ for $r > 1$ and any $x \in \mathcal{H}$.

(ii) $\langle A^r x, x\rangle \leqslant \langle Ax, x\rangle^r \|x\|^{2(1-r)}$ for $0 \leqslant r \leqslant 1$ and any $x \in \mathcal{H}$.

Theorem 2.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ with A, $B \geqslant 0$ and $0 \leqslant v \leqslant 1$. Then the following statements hold.*

(i) If A is invertible, then $A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}A^{\frac{1}{2}} \leqslant vA + (1-v)B$.

(ii) If B is invertible, then $B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{v}B^{\frac{1}{2}} \leqslant vA + (1-v)B$.

Proof We only prove (i), the proof of (ii) is similar. For any vector $x \in \mathcal{H}$, we have that

$$\begin{aligned}
&\langle (vA^2 + (1-v)B^2)x, x\rangle = v\langle A^2x, x\rangle + (1-v)\langle B^2x, x\rangle \\
&\quad \geqslant \langle A^2x, x\rangle^v \langle B^2x, x\rangle^{(1-v)} \quad \text{(Young's inequalilty)} \\
&\quad = \langle (A^{-1}B^2A^{-1})Ax, Ax\rangle^{1-v}\|Ax\|^{2v} \quad \text{(since A is invertible)} \\
&\quad \geqslant \langle (A^{-1}B^2A^{-1})^v Ax, Ax\rangle = \langle A(A^{-1}B^2A^{-1})^v Ax, x\rangle.
\end{aligned}$$

Hence, $A(A^{-1}B^2A^{-1})^vA \leqslant vA^2 + (1-v)B^2$. Replacing A and B by $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ in above inequality respectively leads to the desired inequality. $\square$

Corollary 2.1 ([7]) *Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and $0 \leqslant v \leqslant 1$. If $AB = BA$, then $A^vB^{1-v} \leqslant vA + (1-v)B$.*

Corollary 2.2 *Let $A, B \in \mathcal{B}(\mathcal{H})$ with A, $B \geqslant 0$ and $0 \leqslant v \leqslant 1$. Then the following statements hold.*

(i) If A is invertible, then $\|A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1-v}{2}}\| \leqslant \|vA + (1-v)B\|^{\frac{1}{2}}$.

(ii) If B is invertible, then $\|B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{\frac{v}{2}}\| \leqslant \|vA + (1-v)B\|^{\frac{1}{2}}$.

Theorem 2.2 *Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and $R(A^{\frac{1}{2}}B^{\frac{1}{2}}) = (A^{\frac{1}{2}}B^{\frac{1}{2}} + (A^{\frac{1}{2}}B^{\frac{1}{2}})^*)/2 \geqslant 0$. Then the following statements hold.*

(i) If A is invertible and $\frac{1}{2} \leqslant v \leqslant 1$, then

$$A^{\frac{1}{2}}(A^{-\frac{1}{2}}(R(A^{\frac{1}{2}}B^{\frac{1}{2}}))A^{-\frac{1}{2}})^{2(1-v)}A^{\frac{1}{2}} + (1-v)(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant vA + (1-v)B. \tag{2}$$

(ii) If B is invertible and $0 \leqslant v \leqslant \frac{1}{2}$, then

$$B^{\frac{1}{2}}(B^{-\frac{1}{2}}(R(A^{\frac{1}{2}}B^{\frac{1}{2}}))B^{-\frac{1}{2}})^{2v}B^{\frac{1}{2}} + v(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant vA + (1-v)B. \tag{3}$$

Proof When $v = \frac{1}{2}$, inequalities (2) and (3) become equalities and there is nothing to prove. In the following, we only prove (i), the proof of (ii) is similar. For any vector $x \in \mathcal{H}$, we have that

$$\begin{aligned}
&\langle (vA^2 + (1-v)B^2 - (1-v)(A-B)^2)x, x\rangle \\
&\quad = v\langle A^2x, x\rangle + (1-v)\langle B^2x, x\rangle - (1-v)(\langle A^2x, x\rangle + \langle B^2x, x\rangle - 2\langle R(AB)x, x\rangle) \\
&\quad = (2v-1)\langle A^2x, x\rangle + 2(1-v)\langle R(AB)x, x\rangle \\
&\quad \geqslant \langle A^2x, x\rangle^{(2v-1)}\langle R(AB)x, x\rangle^{2(1-v)} \quad \text{(Young's inequalilty)} \\
&\quad = \langle ((A^{-1}R(AB)A^{-1})Ax, Ax\rangle^{2(1-v)}\|Ax\|^{2(2v-1)} \quad (A \text{ is invertible}) \\
&\quad \geqslant \langle ((A^{-1}R(AB)A^{-1})^{2(1-v)}Ax, Ax\rangle = \langle A(A^{-1}R(AB)A^{-1})^{2(1-v)}Ax, x\rangle.
\end{aligned}$$

Hence, $A(A^{-1}R(AB)A^{-1})^{2(1-v)}A + (1-v)(A-B)^2 \leqslant vA^2 + (1-v)B^2$ when A is invertible and $\frac{1}{2} \leqslant v \leqslant 1$. Replacing A and B by $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ in above inequality respectively, we get the desired inequality. □

Corollary 2.3 *Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and $AB = BA$. Then*

$$A^vB^{1-v} + r_0(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant vA + (1-v)B, \tag{4}$$

where $r_0 = \min\{v, 1-v\}$.

Proof If $v = \frac{1}{2}$, the inequality (4) becomes an equality.

Firstly, we assume that A, B are invertible positive operators and $v < \frac{1}{2}$. Then, by Theorem 2.2 we have that

$$A^vB^{1-v} + v(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant vA + (1-v)B.$$

To prove the case of the general positive operators, we assume that $A_\epsilon = A + \epsilon I$ and $B_\epsilon = B + \epsilon I$ where ϵ is an arbitrary positive real number. Then A_ϵ and B_ϵ are invertible positive operators. And so by the above special case, we get

$$A_\epsilon^vB_\epsilon^{1-v} + r_0(A_\epsilon^{\frac{1}{2}} - B_\epsilon^{\frac{1}{2}})^2 \leqslant vA_\epsilon + (1-v)B_\epsilon.$$

The desired inequality now follows by letting $\epsilon \to 0$.

If $1 - v < \frac{1}{2}$, then the desired inequality is obtained by similar discussion.

Hence, $A^vB^{1-v} + r_0(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant vA + (1-v)B$ where $r_0 = \min\{v, 1-v\}$. □

As a direct consequence of Corollary 2.3, when $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and $AB = BA$, we get

$$A^vB^{1-v} + A^{1-v}B^v \leqslant A^vB^{1-v} + A^{1-v}B^v + 2r_0(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leqslant A + B$$

and

$$\|A^vB^{1-v} + A^{1-v}B^v\| \leqslant \|A^vB^{1-v} + A^{1-v}B^v + 2r_0(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2\| \leqslant \|A + B\|,$$

where $r_0 = \min\{v, 1-v\}$. However, what about the norm estimation of $A^{\frac{1}{p}}B^{\frac{1}{q}} \pm A^{\frac{1}{q}}B^{\frac{1}{p}}$, where A, $B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and $1 < p$, $q < \infty$ with the conjugate exponent $q = p/(p-1)$? We consider this question in the following section.

3. Related norm inequality for operators

For $0 < \theta < 1$, we set $\mathrm{d}\mu_\theta(t) = a_\theta(t)\mathrm{d}t$ and $\mathrm{d}\nu_\theta(t) = b_\theta(t)\mathrm{d}t$ with

$$a_\theta(t) = \frac{\sin(\pi\theta)}{2(\cos h(\pi t) - \cos(\pi\theta))} \quad \text{and} \quad b_\theta(t) = \frac{\sin(\pi\theta)}{2(\cos h(\pi t) + \cos(\pi\theta))}.$$

For a bounded continuous function $f(z)$ on the strip $\Omega = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Im} z \leqslant 1\}$ which is analytic in the interior, we have the well-known (Poisson) integral formula

$$f(i\theta) = \int_{-\infty}^{+\infty} f(t)\mathrm{d}\mu_\theta(t) + \int_{-\infty}^{+\infty} f(i+t)\mathrm{d}\nu_\theta(t)$$

(see [8] for example), and the total masses of the measures $\mathrm{d}\mu_\theta(t)$, $\mathrm{d}\nu_\theta(t)$ are $1-\theta$, θ, respectively (see [4, Lemma 8] in Appendix B). It should be mentioned that $\mathrm{d}\mu_{\frac{1}{q}} = \mathrm{d}\nu_{\frac{1}{p}}$ and $\mathrm{d}\mu_{\frac{1}{p}} = \mathrm{d}\nu_{\frac{1}{q}}$, where $1 < p < \infty$ is with the conjugate exponent $q = p/(p-1)$. It is plain to see

$$a_{\frac{1}{q}}(t) - b_{\frac{1}{q}}(t) = \frac{\sin(\frac{\pi}{q}) \times 2\cos(\frac{\pi}{q})}{2(\cos h^2(\pi t) - \cos^2(\frac{\pi}{q}))} = \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})}.$$

In [4], Kosaki proved that if $0 < \frac{2}{q} < 1$, then

$$\int_{-\infty}^{+\infty} \mathrm{d}(\mu_{\frac{1}{q}} - \nu_{\frac{1}{q}})(t) = \int_{-\infty}^{+\infty} \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})}\mathrm{d}t = (\frac{2}{p} - 1).$$

Theorem 3.1 *Let A, B, X, $Y \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and let $1 < p$, $q < \infty$ with the conjugate exponent $q = p/(p-1)$. For an arbitrary unitary invariant norm $|||\cdot|||$, we have*

$$|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}||| \leqslant |\frac{2}{p} - 1||||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.$$

Proof There are three conditions to be discussed.

If $0 < \frac{2}{q} < 1$, we consider functions $f_1(t) = A^{1+it}XB^{-it}$ $(t \in \mathbb{R})$ and $g_1(t) = A^{1+it}YB^{-it}$ $(t \in \mathbb{R})$. The two functions extend to bounded continuous (in the strong operator topology) functions on the strip Ω which is analytic in the interior. Thus,

$$\begin{aligned}
A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}} =& f_1(\frac{i}{q}) - g_1(i(1-\frac{1}{q})) \\
=& \int_{-\infty}^{+\infty} A^{it}AXB^{-it}\mathrm{d}\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{it}XBB^{-it}\mathrm{d}\nu_{\frac{1}{q}}(t) - \\
& (\int_{-\infty}^{+\infty} A^{it}AYB^{-it}\mathrm{d}\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{it}YBB^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t)) \\
=& \int_{-\infty}^{+\infty} A^{it}(AX - YB)B^{-it}\mathrm{d}\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{it}(XB - AY)B^{-it}\mathrm{d}\nu_{\frac{1}{q}}(t) \\
=& \int_{-\infty}^{+\infty} A^{it}(AX - YB)B^{-it}\mathrm{d}(\mu_{\frac{1}{q}} - \nu_{\frac{1}{q}})(t) + \\
& \int_{-\infty}^{+\infty} A^{it}(XB - AY + AX - YB)B^{-it}\mathrm{d}\nu_{\frac{1}{q}}(t).
\end{aligned}$$

For vectors ξ, $\eta \in \mathcal{H}$, the function $z \to \langle (f(z) - g(i-z))\xi, \eta\rangle \in \mathbb{C}$ is certainly a bounded continuous function on the strip Ω which is analytic in the interior. Therefore, we have integral expressions such as

$$\begin{aligned}
&\langle (A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}})\xi, \eta\rangle \\
&= \int_{-\infty}^{+\infty} \langle (A^{it}(AX - YB)B^{-it})\xi, \eta\rangle \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})} \mathrm{d}t + \\
&\quad \int_{-\infty}^{+\infty} \langle (A^{it}(XB - AY + AX - YB)B^{-it})\xi, \eta\rangle \frac{\sin(\frac{\pi}{p})}{2(\cos h(\pi t) - \cos(\frac{\pi}{p}))} \mathrm{d}t.
\end{aligned}$$

Let

$$\begin{aligned}
Y_n = &\int_{-n}^{n} \langle (A^{it}(AX - YB)B^{-it})\xi, \eta\rangle \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})} \mathrm{d}t + \\
&\int_{-n}^{n} \langle (A^{it}(XB - AY + AX - YB)B^{-it})\xi, \eta\rangle \frac{\sin(\frac{\pi}{p})}{2(\cos h(\pi t) - \cos(\frac{\pi}{p}))} \mathrm{d}t.
\end{aligned}$$

Obviously, $\{Y_n\}$ converges to $A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}$ in the weak operator topology as $n \to \infty$. Since $|||\cdot|||$ is lower semi-continuous relative to the weak operator topology, we have that $|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}||| \leqslant \liminf_{n\to\infty} |||Y_n|||$. Moreover,

$$\begin{aligned}
\liminf_{n\to\infty} |||Y_n||| \leqslant &\int_{-\infty}^{\infty} \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})} \mathrm{d}t |||AX - YB||| + \\
&\int_{-\infty}^{\infty} \frac{\sin(\frac{\pi}{p})}{2(\cos h(\pi t) - \cos(\frac{\pi}{p}))} \mathrm{d}t |||XB - AY + AX - YB||| \\
\leqslant &(\frac{2}{p} - 1)|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.
\end{aligned}$$

Hence, $|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}||| \leqslant (\frac{2}{p} - 1)|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||$.

If $\frac{2}{q} > 1$, we consider functions $f_2(t) = A^{-it}XB^{1+it}$ $(t \in \mathbb{R})$ and $g_2(t) = A^{-it}YB^{1+it}$ $(t \in \mathbb{R})$. The two functions extend to bounded continuous functions in the strong operator topology on the strip Ω which is analytic in the interior. Thus,

$$\begin{aligned}
&A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}} = f_2(\frac{i}{p}) - g_2(i(1 - \frac{1}{p})) \\
&= \int_{-\infty}^{+\infty} A^{-it}AXB^{1+it} \mathrm{d}\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{1-it}XB^{it} \mathrm{d}\nu_{\frac{1}{p}}(t) - \\
&\quad (\int_{-\infty}^{+\infty} A^{-it}AYB^{1+it} \mathrm{d}\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{1-it}YBB^{it} \mathrm{d}\nu_{\frac{1}{q}}(t)) \\
&= \int_{-\infty}^{+\infty} A^{-it}(XB - AY)B^{it} \mathrm{d}\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it}(AX - YB)B^{it} \mathrm{d}\nu_{\frac{1}{p}}(t) \\
&= \int_{-\infty}^{+\infty} A^{-it}(AX - YB)B^{-it} \mathrm{d}(\nu_{\frac{1}{p}} - \mu_{\frac{1}{p}})(t) + \int_{-\infty}^{+\infty} A^{-it}(XB - AY + AX - YB)B^{it} \mathrm{d}\mu_{\frac{1}{p}}(t) \\
&= \int_{-\infty}^{+\infty} A^{-it}(YB - AX)B^{-it} \mathrm{d}(\mu_{\frac{1}{p}} - \nu_{\frac{1}{p}})(t) + \int_{-\infty}^{+\infty} A^{-it}(XB - AY + AX - YB)B^{it} \mathrm{d}\mu_{\frac{1}{p}}(t).
\end{aligned}$$

Taking similar discussion when $0 < \frac{2}{q} < 1$, we get that

$$|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}||| \leqslant (1 - \frac{2}{p})|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.$$

When $\frac{q}{2} = 1$, it follows that $q = 2$ and

$$\begin{aligned} A^{\frac{1}{2}}XB^{\frac{1}{2}} - A^{\frac{1}{2}}YB^{\frac{1}{2}} &= f_1(\frac{i}{2}) - g_1(\frac{i}{2}) \\ &= \int_{-\infty}^{+\infty} A^{it}(AX - YB)B^{-it}\mathrm{d}\mu_{\frac{1}{2}}(t) + \int_{-\infty}^{+\infty} A^{it}(XB - AY)B^{-it}\mathrm{d}\nu_{\frac{1}{2}}(t) \\ &= \int_{-\infty}^{+\infty} A^{it}(AX - YB + XB - AY)B^{-it}\mathrm{d}\nu_{\frac{1}{2}}(t). \end{aligned}$$

By similar discussion of the case $0 < \frac{2}{q} < 1$, we get that

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}} - A^{\frac{1}{2}}YB^{\frac{1}{2}}||| \leqslant \frac{1}{2}|||XB - AY + AX - YB|||.$$

Therefore, $|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{q}}YB^{\frac{1}{p}}||| \leqslant |\frac{2}{p} - 1||||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||$. □

The above argument was motivated by [4, 8, 9] where quadratic Sakai Radon-Nikodym derivatives in the operator algebra theory were studied. Note that Theorem 1 in [9] and some related inequalities in [9] are the direct results of our Theorem 3.1.

Corollary 3.1 *Let A, $B \in \mathcal{B}(\mathcal{H})$ with $A, B \geqslant 0$ and let $1 < p$, $q < \infty$ with the conjugate exponent $q = p/(p-1)$. Then*

$$\|A^{\frac{1}{p}}B^{\frac{1}{q}} - A^{\frac{1}{q}}B^{\frac{1}{p}}\| \leqslant |\frac{2}{p} - 1|\|A - B\| \ \ \textit{and} \ \ \|A^{\frac{1}{p}}B^{\frac{1}{q}} + A^{\frac{1}{q}}B^{\frac{1}{p}}\| \leqslant (|\frac{2}{p} - 1| + \frac{2}{q})\|A + B\|.$$

The following lemma is an elementary result by direct computation.

Lemma 3.1 *Let a, b be positive real numbers and $1 < p$, $q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f(t) = at^p + bt^{-q}$ $(t \in (0, \infty))$, then $f_{\min} = f((\frac{bq}{ap})^{\frac{1}{pq}}) = ap(\frac{bq}{ap})^{\frac{1}{q}}$.*

A new estimate of unitarily invariant norm of $A^{\frac{1}{p}}XB^{\frac{1}{q}}$ is given in Corollary 3.2.

Corollary 3.2 *Let A, B, $X \in \mathcal{B}$ $(\mathcal{H})$ with $A, B \geqslant 0$ and let $1 < p$, $q < \infty$ with the conjugate exponent $q = p/(p-1)$. For an arbitrary unitary invariant norm $|||\cdot|||$, the inequality*

$$|||A^{\frac{1}{p}}XB^{\frac{1}{q}}||| \leqslant (|2 - p| + p - 1)^{\frac{1}{p}}|||AX|||^{\frac{1}{p}}|||XB|||^{\frac{1}{q}}$$

holds.

Proof Let $Y = 0$ in Theorem 3.1. We get that

$$\begin{aligned} |||A^{\frac{1}{p}}XB^{\frac{1}{q}}||| &\leqslant |\frac{2}{p} - 1||||AX||| + \frac{1}{q}|||XB + AX||| \\ &\leqslant (|\frac{2}{p} - 1| + \frac{1}{q})|||AX||| + \frac{1}{q}|||XB|||. \end{aligned}$$

By changing A, B to t^pA, $t^{-q}B$ with $t > 0$, we have that

$$|||A^{\frac{1}{p}}XB^{\frac{1}{q}}||| \leqslant (|\frac{2}{p} - 1| + \frac{1}{q})t^p|||AX||| + \frac{1}{q}t^{-q}|||XB|||.$$

The minimum of the right side is $(|2-p|+p-1)^{\frac{1}{p}}|||AX|||^{\frac{1}{p}}|||XB|||^{\frac{1}{q}}$ as a function of t by Lemma 3.1, and so the corollary is proved. □

In fact, by repeating the similar argument for the functions $f(z)=A^{-iz}XB^{-iz}$ and $g(z)=A^{-iz}YB^{-iz}$ we get that

$$A^{\frac{1}{p}}XB^{\frac{1}{p}}=f(\frac{i}{p})=\int_{-\infty}^{+\infty}A^{it}XB^{-it}\mathrm{d}\mu_{\frac{1}{p}}(t)+\int_{-\infty}^{+\infty}A^{it}AXBB^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t)$$

and

$$A^{\frac{1}{q}}YB^{\frac{1}{q}}=g(\frac{i}{q})=\int_{-\infty}^{+\infty}A^{it}YB^{-it}\mathrm{d}\mu_{\frac{1}{q}}(t)+\int_{-\infty}^{+\infty}A^{it}AYBB^{-it}\mathrm{d}\nu_{\frac{1}{q}}(t).$$

Hence,

$$A^{\frac{1}{p}}XB^{\frac{1}{p}}+A^{\frac{1}{q}}YB^{\frac{1}{q}}=\int_{-\infty}^{+\infty}A^{it}(X+AYB)B^{-it}\mathrm{d}\mu_{\frac{1}{p}}(t)+\int_{-\infty}^{+\infty}A^{it}(AXB+Y)B^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t)$$

by the fact that $d\mu_{\frac{1}{q}}=\mathrm{d}\nu_{\frac{1}{p}}$ and $\mathrm{d}\mu_{\frac{1}{p}}=\mathrm{d}\nu_{\frac{1}{q}}$. Clearly, when $\frac{2}{q}>1$, we have that

$$\begin{aligned}A^{\frac{1}{p}}XB^{\frac{1}{p}}+A^{\frac{1}{q}}YB^{\frac{1}{q}}=&\int_{-\infty}^{+\infty}A^{it}(X+AYB)B^{-it}\mathrm{d}(\mu_{\frac{1}{p}}-\nu_{\frac{1}{p}})(t)+\\&\int_{-\infty}^{+\infty}A^{it}(AXB+Y+X+AYB)B^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t);\end{aligned}$$

when $0<\frac{2}{q}\leqslant 1$, we get that

$$\begin{aligned}&A^{\frac{1}{p}}(-X)B^{\frac{1}{p}}+A^{\frac{1}{q}}(-Y)B^{\frac{1}{q}}\\&=\int_{-\infty}^{+\infty}A^{it}(-(X+AYB))B^{-it}\mathrm{d}\mu_{\frac{1}{p}}(t)+\int_{-\infty}^{+\infty}A^{it}(-(AXB+Y))B^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t)\\&=\int_{-\infty}^{+\infty}A^{it}(X+AYB)B^{-it}\mathrm{d}(\nu_{\frac{1}{p}}-\mu_{\frac{1}{p}})(t)+\int_{-\infty}^{+\infty}A^{it}(-(AXB+Y+X+AYB))B^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t)\\&=\int_{-\infty}^{+\infty}A^{it}(X+AYB)B^{-it}\mathrm{d}(\mu_{\frac{1}{q}}-\nu_{\frac{1}{q}})(t)+\int_{-\infty}^{+\infty}A^{it}(-(AXB+Y+X+AYB))B^{-it}\mathrm{d}\nu_{\frac{1}{p}}(t).\end{aligned}$$

The above expressions obviously show Theorem 3.2 is true.

Theorem 3.2 *Let A, B, X, $Y\in\mathcal{B}(\mathcal{H})$ with $A,B\geqslant 0$ and let $1<p$, $q<\infty$ with the conjugate exponent $q=p/(p-1)$. For an arbitrary unitarily invariant norm $|||\cdot|||$, the inequality*

$$|||A^{\frac{1}{p}}XB^{\frac{1}{p}}+A^{\frac{1}{q}}YB^{\frac{1}{q}}|||\leqslant|\frac{2}{p}-1||||X+AYB|||+\frac{1}{p}|||X+Y+A(X+Y)B|||$$

holds.

Finally we would like to point out that there are some special cases of our results in Theorems 3.1 and 3.2. Moreover, our estimate can be improved a little bit by the standard interpolation argument for a particular unitarily invariant norm.

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