Young’s Inequality for Positive Operators

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Abstract The classical Young’s inequality and its refinements are applied to positive operators
on a Hilbert space at first. Based on the classical Poisson integral formula of relevant operators,
some new inequalities on unitarily invariant norm of $A^p X B^q - A^q Y B^p$ are obtained with
effective calculation, where $A, B, X, Y \in B(H)$ with $A, B \geq 0$ and $1 < p, q < \infty$ with the
conjugate exponent $q = p/(p - 1)$.

Keywords Young’s inequality; positive operator.

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1. Introduction

The classical Young inequality for two nonnegative scalars says that if $a, b \geq 0$ and $0 \leq v \leq 1$,
then $a^v b^{1-v} \leq va + (1 - v)b$ with equality if and only if $a = b$. If $v = \frac{1}{2}$, we obtain the arithmetic-
geometric mean inequality $\sqrt{ab} \leq \frac{1}{2}(a + b)$. Recently, Kittaneh and Manasrah refined the Young’s
inequality in [1] and proved that if $a, b \geq 0$ and $0 \leq v \leq 1$, then
$$a^v b^{1-v} + r_0(\sqrt{a} - \sqrt{b}) \leq va + (1 - v)b,$$
where $r_0 = \min\{v, 1 - v\}$.

Young’s inequality in operator algebras has been considered in [2] and references therein.
Bhatia and Parthasarathy in [3] and Kosaki in [4] proved that if $A, B, X \in M_n(\mathbb{C})$ with that $A$
and $B$ are positive semi-definite and if $0 \leq v \leq 1$, then
$$\|A^v X B^{1-v}\|_2 \leq \|vAX + (1 - v)XB\|_2.$$ (1)

It should be mentioned here that for $v \neq \frac{1}{2}$, the inequality (1) may not hold for other unitary
invariant norms. On the other hand, Bhatia and Davis proved in [5] that if $A, B, X \in M_n(\mathbb{C})$
with $A, B$ being positive semi-definite and if $0 \leq v \leq 1$, then
$$\|A^v X B^q\| \leq \|\frac{A^v X B^{1-v} + A^{1-v} X B^v}{2}\| \leq \|\frac{AX + XB}{2}\|.$$
is true for any unitary invariant norm $\| \cdot \|$. Moreover, a readable account on $A^{\frac{1}{2}} X B^{\frac{1}{2}} \pm A^{\frac{1}{2}} X B^{\frac{1}{2}}$ (1 < p < $\infty$ and $q = p/(p - 1)$) and related inequalities can be found in [6].

The purpose of this article is firstly to improve Young’s inequality for positive operator on a Hilbert space $\mathcal{H}$. Secondly, by means of the classical Poisson integral formula, we obtain new estimations for unitary invariant norm of $A^{\frac{1}{2}} X B^{\frac{1}{2}} - A^{\frac{1}{2}} Y B^{\frac{1}{2}}$ where $A, B, X, Y \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $1 < p, q < \infty$ with the conjugate exponent $q = p/(p - 1)$.

2. Young’s inequalities for positive operators

In this section we begin with the famous Hölder-McCarthy Inequality.

**Lemma 2.1** (Hölder-McCarthy Inequality) Let $A \in \mathcal{B}(\mathcal{H})$ with $A \geq 0$. Then the following properties hold:

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ for $r > 1$ and any $x \in \mathcal{H}$.

(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \|x\|^{2(1-r)}$ for $0 \leq r \leq 1$ and any $x \in \mathcal{H}$.

**Theorem 2.1** Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $0 \leq v \leq 1$. Then the following statements hold.

(i) If $A$ is invertible, then $A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}A^{\frac{1}{2}} \leq vA + (1-v)B$.

(ii) If $B$ is invertible, then $B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{v}B^{\frac{1}{2}} \leq vA + (1-v)B$.

**Proof** We only prove (i), the proof of (ii) is similar. For any vector $x \in \mathcal{H}$, we have that

\[
\langle (vA^2 + (1-v)B^2)x, x \rangle = v\langle A^2 x, x \rangle + (1-v)\langle B^2 x, x \rangle \\
\geq \langle A^2 x, x \rangle^v \langle B^2 x, x \rangle^{1-v} \quad \text{(Young’s inequality)} \\
= \langle (A^{-1}B^2A^{-1})Ax, Ax \rangle^{1-v}\|Ax\|^{2v} \quad \text{(since $A$ is invertible)} \\
\geq \langle (A^{-1}B^2A^{-1})^v Ax, Ax \rangle = \langle A^{-1}B^2A^{-1})^v Ax, x \rangle.
\]

Hence, $A(A^{-1}B^2A^{-1})^v A \leq vA^2 + (1-v)B^2$. Replacing $A$ and $B$ by $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ in above inequality respectively leads to the desired inequality. □

**Corollary 2.1** ([7]) Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $0 \leq v \leq 1$. If $AB = BA$, then $A^{v}B^{1-v} \leq vA + (1-v)B$.

**Corollary 2.2** Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $0 \leq v \leq 1$. Then the following statements hold.

(i) If $A$ is invertible, then $\| A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v} \| \leq \| vA + (1-v)B \|^{\frac{1}{2}}$.

(ii) If $B$ is invertible, then $\| B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{v} \| \leq \| vA + (1-v)B \|^{\frac{1}{2}}$.

**Theorem 2.2** Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $R(A^{\frac{1}{2}}B^{\frac{1}{2}}) = (A^{\frac{1}{2}}B^{\frac{1}{2}} + (A^{\frac{1}{2}}B^{\frac{1}{2}})^*)/2 \geq 0$. Then the following statements hold.

(i) If $A$ is invertible and $\frac{1}{2} \leq v \leq 1$, then

\[
A^{\frac{1}{2}}(R(A^{\frac{1}{2}}B^{\frac{1}{2}}))A^{\frac{1}{2}}(1-v)(A^{\frac{1}{2}} - B^{\frac{1}{2}})^2 \leq vA + (1-v)B. \tag{2}
\]
If $B$ is invertible and $0 \leq v \leq \frac{1}{2}$, then
\[
B^{\frac{1}{2}}(B^{-\frac{1}{2}}(R(A^\frac{1}{2}B^\frac{1}{2}))B^{-\frac{1}{2}})^{2v}B^{\frac{1}{2}} + v(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq vA + (1 - v)B.
\] (3)

**Proof** When $v = \frac{1}{2}$, inequalities (2) and (3) become equalities and there is nothing to prove. In the following, we only prove (i), the proof of (ii) is similar. For any vector $v \in \mathcal{H}$, we have that
\[
\langle (v^2A^2 + (1 - v)B^2 - (1 - v)(A - B)^2)x, x \rangle = v\langle A^2x, x \rangle + (1 - v)\langle B^2x, x \rangle - (1 - v)(\langle A^2x, x \rangle + \langle B^2x, x \rangle - 2\langle R(AB)x, x \rangle)
\]
\[
= (2v - 1)\langle A^2x, x \rangle + 2(1 - v)\langle R(AB)x, x \rangle
\]
\[
\geq (A^2x, x)^{(2v - 1)}(R(AB)x, x)^{2(1 - v)} \text{ (Young's inequality)}
\]
\[
= ((A^{-1}R(AB)A^{-1})Ax, Ax)^{2(1 - v)}||Ax||^{2(2v - 1)} \text{ (A is invertible)}
\]
\[
\geq ((A^{-1}R(AB)A^{-1})^{2(1 - v)}Ax, Ax) = \langle A(A^{-1}R(AB)A^{-1})^{2(1 - v)}Ax, x \rangle.
\]

Hence, $A(A^{-1}R(AB)A^{-1})^{2(1 - v)}A + (1 - v)(A - B)^2 \leq vA^2 + (1 - v)B^2$ when $A$ is invertible and $\frac{1}{2} \leq v \leq 1$. Replacing $A$ and $B$ by $A^\frac{1}{2}$ and $B^\frac{1}{2}$ in above inequality respectively, we get the desired inequality. 

**Corollary 2.3** Let $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $AB = BA$. Then
\[
A^vB^{1-v} + r_0(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq vA + (1 - v)B,
\] (4)
where $r_0 = \min\{v, 1 - v\}$.

**Proof** If $v = \frac{1}{2}$, the inequality (4) becomes an equality.

Firstly, we assume that $A, B$ are invertible positive operators and $v < \frac{1}{2}$. Then, by Theorem 2.2 we have that
\[
A^vB^{1-v} + v(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq vA + (1 - v)B.
\]

To prove the case of the general positive operators, we assume that $A_\epsilon = A + \epsilon I$ and $B_\epsilon = B + \epsilon I$ where $\epsilon$ is an arbitrary positive real number. Then $A_\epsilon$ and $B_\epsilon$ are invertible positive operators. And so by the above special case, we get
\[
A_\epsilon^vB_\epsilon^{1-v} + r_0(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq vA_\epsilon + (1 - v)B_\epsilon.
\]
The desired inequality now follows by letting $\epsilon \to 0$.

If $1 - v < \frac{1}{2}$, then the desired inequality is obtained by similar discussion.

Hence, $A^vB^{1-v} + r_0(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq vA + (1 - v)B$ where $r_0 = \min\{v, 1 - v\}$. 

As a direct consequence of Corollary 2.3, when $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $AB = BA$, we get
\[
A^vB^{1-v} + A^{1-v}B^v \leq A^vB^{1-v} + A^{1-v}B^v + 2r_0(A^\frac{1}{2} - B^\frac{1}{2})^2 \leq A + B
\]
and
\[
\|A^vB^{1-v} + A^{1-v}B^v\| \leq \|A^vB^{1-v} + A^{1-v}B^v + 2r_0(A^\frac{1}{2} - B^\frac{1}{2})^2\| \leq \|A + B\|,
\]
where $r_0 = \min\{v, 1 - v\}$. However, what about the norm estimation of $A^\frac{1}{2}B^\frac{1}{2} \pm A^\frac{1}{2}B^\frac{1}{2}$, where $A, B \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and $1 < p, q < \infty$ with the conjugate exponent $q = p/(p - 1)$? We
consider this question in the following section.

3. Related norm inequality for operators

For $0 < \theta < 1$, we set $d\mu_\theta(t) = a_\theta(t)dt$ and $d\nu_\theta(t) = b_\theta(t)dt$ with

$$a_\theta(t) = \frac{\sin(\pi \theta)}{2\cos(h(\pi t) - \cos(\pi \theta))} \quad \text{and} \quad b_\theta(t) = \frac{\sin(\pi \theta)}{2\cos(h(\pi t) + \cos(\pi \theta))}.$$ 

For a bounded continuous function $f(z)$ on the strip $\Omega = \{z \in \mathbb{C} : 0 \leq \Im z \leq 1\}$ which is analytic in the interior, we have the well-known (Poisson) integral formula

$$f(i\theta) = \int_{-\infty}^{+\infty} f(t)d\mu_\theta(t) + \int_{-\infty}^{+\infty} f(i + t)d\nu_\theta(t)$$

(see [8] for example), and the total masses of the measures $d\mu_\theta(t)$, $d\nu_\theta(t)$ are $1 - \theta$, $\theta$, respectively (see [4, Lemma 8] in Appendix B). It should be mentioned that $d\mu_{\frac{1}{q}} = d\nu_{\frac{1}{p}}$ and $d\mu_{\frac{1}{p}} = d\nu_{\frac{1}{q}}$, where $1 < p < \infty$ is with the conjugate exponent $q = p/(p - 1)$. It is plain to see

$$a_{\frac{1}{q}}(t) - b_{\frac{1}{q}}(t) = \frac{\sin(\frac{\pi}{q}) \times 2\cos(\frac{\pi}{q})}{2\cos(h(2\pi t) - \cos(\frac{2\pi}{q}))} = \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})}.$$ 

In [4], Kosaki proved that if $0 < \frac{2}{q} < 1$, then

$$\int_{-\infty}^{+\infty} d(\mu_{\frac{1}{q}} - \nu_{\frac{1}{q}})(t) = \int_{-\infty}^{+\infty} \frac{\sin(\frac{2\pi}{q})}{\cos h(2\pi t) - \cos(\frac{2\pi}{q})}dt = \left(\frac{2}{p} - 1\right).$$

**Theorem 3.1** Let $A, B, X, Y \in \mathcal{B}(\mathcal{H})$ with $A, B \geq 0$ and let $1 < p, q < \infty$ with the conjugate exponent $q = p/(p - 1)$. For an arbitrary unitary invariant norm $||| \cdot |||$, we have

$$|||A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{p}}YB^{\frac{1}{q}}||| \leq \frac{2}{p} - 1|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.$$ 

**Proof** There are three conditions to be discussed.

If $0 < \frac{2}{q} < 1$, we consider functions $f_1(t) = A^{1+it}XB^{-it}$ ($t \in \mathbb{R}$) and $g_1(t) = A^{1+it}YB^{-it}$ ($t \in \mathbb{R}$). The two functions extend to bounded continuous (in the strong operator topology) functions on the strip $\Omega$ which is analytic in the interior. Thus,

$$A^{\frac{1}{p}}XB^{\frac{1}{q}} - A^{\frac{1}{p}}YB^{\frac{1}{q}} = f_1(\frac{i}{q}) - g_1(i(1 - \frac{1}{q}))$$

$$= \int_{-\infty}^{+\infty} A^{it}AXB^{-it}d\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{it}XBB^{-it}d\nu_{\frac{1}{q}}(t) -$$

$$\left(\int_{-\infty}^{+\infty} A^{it}AYB^{-it}d\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{it}YBB^{-it}d\nu_{\frac{1}{q}}(t)\right)$$

$$= \int_{-\infty}^{+\infty} A^{it}(AX - YB)B^{-it}d\mu_{\frac{1}{q}}(t) + \int_{-\infty}^{+\infty} A^{it}(XB - AY)B^{-it}d\nu_{\frac{1}{q}}(t)$$

$$\int_{-\infty}^{+\infty} A^{it}(AX - YB)B^{-it}d(\mu_{\frac{1}{q}} - \nu_{\frac{1}{q}})(t) +$$

$$\int_{-\infty}^{+\infty} A^{it}(XB - AY + AX - YB)B^{-it}d\nu_{\frac{1}{q}}(t).$$
For vectors $\xi, \eta \in \mathcal{H}$, the function $z \rightarrow \langle (f(z) - g(i-z))\xi, \eta \rangle \in \mathbb{C}$ is certainly a bounded continuous function on the strip $\Omega$ which is analytic in the interior. Therefore, we have integral expressions such as

$$\langle (A^+ X B^+ - A^+ Y B^+)\xi, \eta \rangle = \int_{-\infty}^{+\infty} (A^t(AX - YB)B^{-it})\xi, \eta \rangle \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})} dt + \int_{-\infty}^{+\infty} (A^t(XB - AY + AX - YB)B^{-it})\xi, \eta \rangle \frac{\sin(\frac{\pi}{p})}{2(\cosh(\pi t) - \cos(\frac{\pi}{p}))} dt.$$

Let

$$Y_n = \int_{-\infty}^{+\infty} (A^t(AX - YB)B^{-it})\xi, \eta \rangle \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})} dt + \int_{-\infty}^{+\infty} (A^t(XB - AY + AX - YB)B^{-it})\xi, \eta \rangle \frac{\sin(\frac{\pi}{p})}{2(\cosh(\pi t) - \cos(\frac{\pi}{p}))} dt.$$

Obviously, $\{Y_n\}$ converges to $A^+ X B^+ - A^+ Y B^+$ in the weak operator topology as $n \to \infty$. Since $|||\cdot|||$ is lower semi-continuous relative to the weak operator topology, we have that $|||A^+ X B^+ - A^+ Y B^+||| \leq \liminf_{n \to \infty} |||Y_n|||$. Moreover,

$$\liminf_{n \to \infty} |||Y_n||| \leq \int_{-\infty}^{+\infty} \frac{\sin(\frac{2\pi}{q})}{\cosh(2\pi t) - \cos(\frac{2\pi}{q})} dt |||AX - YB||| + \int_{-\infty}^{+\infty} \frac{\sin(\frac{\pi}{p})}{2(\cosh(\pi t) - \cos(\frac{\pi}{p}))} dt |||XB - AY + AX - YB|||$$

$$\leq \left(\frac{2}{q} - 1\right)|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.$$

Hence, $|||A^+ X B^+ - A^+ Y B^+||| \leq \left(\frac{2}{q} - 1\right)|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||$. If $\frac{2}{q} > 1$, we consider functions $f_2(t) = A^{-it}X^1 B^1 + it$ and $g_2(t) = A^{-it}Y^1 B^1 + it$. The two functions extend to bounded continuous functions in the strong operator topology on the strip $\Omega$ which is analytic in the interior. Thus,

$$A^+ X B^+ - A^+ Y B^+ = f_2\left(\frac{i}{p}\right) - g_2\left(i(1 - \frac{1}{p})\right)$$

$$= \int_{-\infty}^{+\infty} A^{-it} AX B^1 + it d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it} X B^1 + it d\nu_{\frac{1}{p}}(t) -$$

$$\left( \int_{-\infty}^{+\infty} A^{-it} AY B^1 + it d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it} Y B^1 + it d\nu_{\frac{1}{p}}(t) \right)$$

$$= \int_{-\infty}^{+\infty} A^{-it} (XB - AY) B^1 + it d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it} (AX - YB) B^1 + it d\nu_{\frac{1}{p}}(t)$$

$$= \int_{-\infty}^{+\infty} A^{-it} (AX - YB) B^{-it} d\nu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it} (XB - AY + AX - YB) B^1 + it d\nu_{\frac{1}{p}}(t)$$

$$= \int_{-\infty}^{+\infty} A^{-it} (YB - AX) B^{-it} d\mu_{\frac{1}{p}}(t) + \int_{-\infty}^{+\infty} A^{-it} (XB - AY + AX - YB) B^1 + it d\nu_{\frac{1}{p}}(t).$$
Taking similar discussion when $0 < \frac{2}{q} < 1$, we get that
\[
|||A^\frac{1}{p}XB^\frac{1}{q} - A^\frac{1}{p}YB^\frac{1}{q}||| \leq (1 - \frac{2}{p})|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||.
\]

When $\frac{q}{2} = 1$, it follows that $q = 2$ and
\[
A^\frac{1}{p}XB^\frac{1}{q} - A^\frac{1}{p}YB^\frac{1}{q} = f_1(t) - g_1(t)
\]
\[
= \int_{-\infty}^{+\infty} A^t(AX - YB)B^{-it}d\mu_2(t) + \int_{-\infty}^{+\infty} A^t(XB - AY)B^{-it}d\nu_2(t)
\]
\[
= \int_{-\infty}^{+\infty} A^t(AX - YB + XB - AY)B^{-it}d\nu_2(t).
\]

By similar discussion of the case $0 < \frac{2}{q} < 1$, we get that
\[
|||A^\frac{1}{p}XB^\frac{1}{q} - A^\frac{1}{p}YB^\frac{1}{q}||| \leq \frac{1}{2}|||XB - AY + AX - YB|||.
\]

Therefore, $|||A^\frac{1}{p}XB^\frac{1}{q} - A^\frac{1}{p}YB^\frac{1}{q}||| \leq |\frac{2}{p} - 1|||AX - YB||| + \frac{1}{q}|||XB - AY + AX - YB|||$. \hfill \Box

The above argument was motivated by [4, 8, 9] where quadratic Sakai Radon-Nikodym derivatives in the operator algebra theory were studied. Note that Theorem 1 in [9] and some related inequalities in [9] are the direct results of our Theorem 3.1.

**Corollary 3.1** Let $A, B \in B(H)$ with $A, B \geq 0$ and let $1 < p, q < \infty$ with the conjugate exponent $q = \frac{p}{p-1}$. Then
\[
\|A^\frac{1}{p}B^\frac{1}{q} - A^\frac{1}{p}B^\frac{1}{q}\| \leq \frac{2}{p} - 1\|A - B\| \quad \text{and} \quad \|A^\frac{1}{p}B^\frac{1}{q} + A^\frac{1}{p}B^\frac{1}{q}\| \leq (\frac{2}{p} - 1 + \frac{2}{q})\|A + B\|.
\]

The following lemma is an elementary result by direct computation.

**Lemma 3.1** Let $a, b$ be positive real numbers and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f(t) = at^p + bt^{-q}$ ($t \in (0, \infty)$), then $f_{\min} = f((\frac{b}{ap})^{\frac{1}{p-q}}) = ap(\frac{b}{ap})^{\frac{1}{q}}$.

A new estimate of unitarily invariant norm of $A^\frac{1}{p}XB^\frac{1}{q}$ is given in Corollary 3.2.

**Corollary 3.2** Let $A, B, X \in B(H)$ with $A, B \geq 0$ and let $1 < p, q < \infty$ with the conjugate exponent $q = \frac{p}{p-1}$. For an arbitrary unitary invariant norm $|||\cdot|||$, the inequality
\[
|||A^\frac{1}{p}XB^\frac{1}{q}||| \leq (2 - p + p - 1)^\frac{1}{p}|||AX|||^\frac{1}{p}|||XB|||^\frac{1}{q}
\]
holds.

**Proof** Let $Y = 0$ in Theorem 3.1. We get that
\[
|||A^\frac{1}{p}XB^\frac{1}{q}||| \leq \frac{2}{p} - 1|||AX||| + \frac{1}{q}|||XB + AX||| \leq (\frac{2}{p} - 1 + \frac{1}{q})|||AX||| + \frac{1}{q}|||XB|||.
\]

By changing $A, B$ to $t^pA, t^{-q}B$ with $t > 0$, we have that
\[
|||A^\frac{1}{p}XB^\frac{1}{q}||| \leq (\frac{2}{p} - 1 + \frac{1}{q})t^p|||AX||| + \frac{1}{q}t^{-q}|||XB|||.
\]
The minimum of the right side is $\langle 2 - p \rangle^\frac{p}{p-1} |||AX||| |||XB|||^\frac{p}{p}$ as a function of $t$ by Lemma 3.1, and so the corollary is proved. □

In fact, by repeating the similar argument for the functions $f(z) = A^{-iz}XB^{-iz}$ and $g(z) = A^{-iz}YB^{-iz}$ we get that

$$A^\frac{1}{q} XB^\frac{1}{p} = f\left(\frac{i}{p}\right) = \int_{-\infty}^{+\infty} A^{it} XB^{-it} d\mu_A^\frac{1}{p}(t) + \int_{-\infty}^{+\infty} A^{it} XB^{-it} d\mu_B^\frac{1}{p}(t)$$

and

$$A^\frac{1}{q} YB^\frac{1}{p} = g\left(\frac{i}{q}\right) = \int_{-\infty}^{+\infty} A^{it} YB^{-it} d\mu_A^\frac{1}{q}(t) + \int_{-\infty}^{+\infty} A^{it} YB^{-it} d\mu_B^\frac{1}{q}(t).$$

Hence,

$$A^\frac{1}{q} XB^\frac{1}{p} + A^\frac{1}{q} YB^\frac{1}{p} = \int_{-\infty}^{+\infty} A^{it}(X + AYB)B^{-it} d\mu_A^\frac{1}{p}(t) + \int_{-\infty}^{+\infty} A^{it}(AXB + Y)B^{-it} d\mu_B^\frac{1}{p}(t)$$

by the fact that $d\mu_A^\frac{1}{q} = d\mu_A^\frac{1}{p}$ and $d\mu_B^\frac{1}{q} = d\mu_B^\frac{1}{p}$. Clearly, when $\frac{2}{q} > 1$, we have that

$$A^\frac{1}{q} XB^\frac{1}{p} + A^\frac{1}{q} YB^\frac{1}{p} = \int_{-\infty}^{+\infty} A^{it}(X + AYB)B^{-it} d(\mu_A^\frac{1}{p} - \mu_B^\frac{1}{p})(t) + \int_{-\infty}^{+\infty} A^{it}(AXB + Y + X + AYB)B^{-it} d\nu_B^\frac{1}{p}(t);$$

when $0 < \frac{2}{q} < 1$, we get that

$$A^\frac{1}{q} (-X)B^\frac{1}{p} + A^\frac{1}{q} (-Y)B^\frac{1}{p}$$

$$= \int_{-\infty}^{+\infty} A^{it}(-(-X + AYB))B^{-it} d\mu_A^\frac{1}{p}(t) + \int_{-\infty}^{+\infty} A^{it}(-(-AXB + Y))B^{-it} d\mu_B^\frac{1}{p}(t)$$

$$= \int_{-\infty}^{+\infty} A^{it}(X + AYB)B^{-it} d(\nu_A^\frac{1}{p} - \mu_B^\frac{1}{p})(t) + \int_{-\infty}^{+\infty} A^{it}(-(-AXB + Y + X + AYB))B^{-it} d\nu_B^\frac{1}{p}(t)$$

The above expressions obviously show Theorem 3.2 is true.

**Theorem 3.2** Let $A, B, X, Y \in B(H)$ with $A, B \succeq 0$ and let $1 < p, q < \infty$ with the conjugate exponent $q = p/(p - 1)$. For an arbitrary unitarily invariant norm $||| \cdot |||$, the inequality

$$|||A^\frac{1}{q} XB^\frac{1}{p} + A^\frac{1}{q} YB^\frac{1}{p}||| \leq \|\frac{2}{p} - 1\| |||X + AYB||| + \|\frac{1}{p}\| |||X + Y + A(X + Y)B|||$$

holds.

Finally, we would like to point out that there are some special cases of our results in Theorems 3.1 and 3.2. Moreover, our estimate can be improved a little bit by the standard interpolation argument for a particular unitarily invariant norm.
References


