Journal of Mathematical Research & Exposition
Sept., 2011, Vol. 31, No. 5, pp. 937–945
DOI:10.3770/j.issn:1000-341X.2011.05.021
Http://jmre.dlut.edu.cn

Center Conditions and Bifurcation of Limit Cycles at Nilpotent Critical Point in a Quintic Lyapunov System

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Abstract In this paper, center conditions and bifurcation of limit cycles at the nilpotent critical point in a class of quintic polynomial differential system are investigated. With the help of computer algebra system MATHEMATICA, the first 8 quasi Lyapunov constants are deduced. As a result, the necessary and sufficient conditions to have a center are obtained. The fact that there exist 8 small amplitude limit cycles created from the three-order nilpotent critical point is also proved. Henceforth we give a lower bound of cyclicity of three-order nilpotent critical point for quintic Lyapunov systems.

Keywords three-order nilpotent critical point; center-focus problem; bifurcation of limit cycles; quasi-Lyapunov constant.

Document code A
MR(2010) Subject Classification 34C05; 34C07
Chinese Library Classification O175.12; O175.13

1. Introduction

Consider an autonomous planar ordinary differential equation having a three-order nilpotent critical point with the form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=y-3x^2y+a_{12}xy^2+a_{03}y^3+x^3y+a_{22}x^2y^2+a_{13}xy^3+a_{04}y^4+a_{41}x^4y,\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=-2x^3+y^2+b_{21}x^2y+b_{12}xy^2+a_{22}y^3+b_{50}x^5.\end{aligned}\tag{1}$$

The main goal of this paper is to use the integral factor method theory to distinguish center-focus and generate limit cycles from the origin of the above system.

Let $DX(p)$ denote the differential matrix of X at the critical point p. When the matrix $DX(p)$ has its two eigenvalues equal to zero, but the matrix is not identically null, p is said to be a nilpotent critical point. In a suitable coordinate system the Lyapunov system with the origin as a nilpotent critical point can be written as

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=y+\sum_{i+j=2}^{\infty}a_{ij}x^iy^j=X(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t}&=\sum_{i+j=2}^{\infty}b_{ij}x^iy^j=Y(x,y).\end{aligned}\tag{2}$$

Received May 9, 2010; Accepted November 20, 2010
Supported by the Natural Science Foundation of Shandong Province (Grant No. Y2007A17).
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Suppose that the function $y = y(x)$ satisfies $X(x, y) = 0, y(0) = 0$. Lyapunov proved in [3] that the origin of system (2) is a monodromic critical point (i.e., a center or a focus) if and only if

$$\begin{aligned} &Y(x, y(x)) = \alpha x^{2n+1} + o(x^{2n+1}),\ \alpha < 0, \\ &[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}]_{y=y(x)} = \beta x^n + o(x^n), \\ &\beta^2 + 4(n+1)\alpha < 0, \end{aligned} \tag{3}$$

where n is a positive integer. The monodromy problem in this case was solved in [4] and the center problem in [12]. Nevertheless, in practice, given an analytic system with a nilpotent monodromic critical point it is not an easy task to know if it is a center or a focus. As far as we know, there are essentially three differential ways of obtaining the Lyapunov constant: by using normal form theory [9], by computing the Poincaré return map [6] or by using Lyapunov functions [13]. The three tools explained above have been also used to study the center-focus problem of nilpotent critical points, see, for instance, [1, 8, 12], respectively. Takens proved in [15] that system (2) can be formally transformed into a generalized Liénard system. Recently Stróżyna and Żołądek proved in [14] that indeed this normal form can be achieved through an analytic change of variables. The authors of [2] proved that using a reparametrization of the time can simplify the system (2) even more.

There are very few results known for concrete families of differential systems with monodromic nilpotent critical points. Gasull and Torregrosa [10] have generalized the scheme of computation of the Lyapunov constants for systems of the form

$$\begin{aligned} \dot{x} &= y + \sum_{k \geq n+1} F_k(x, y), \\ \dot{y} &= -x^{2n-1} + \sum_{k \geq 2n} G_k(x, y), \end{aligned} \tag{4}$$

where F_k and G_k are $(1, n)$-quasi-homogeneous functions of degree k. Using their technique, one can obtain the center conditions for some concrete examples, for instance, the family studied in [7] and [10].

For a given family of polynomial differential equations, the number of Lyapunov constants needed to solve the center-focus problem is also related with the so-called cyclicity of the point (i.e., the number of limit cycles generated by small perturbations of the coefficients of the given differential equation inside the family considered). The three tools of obtaining the Lyapunov constants mentioned above have been used to generate limit cycles from nilpotent critical points, see for instance [1, 2, 5], respectively. Let $N(n)$ be the maximum possible number of limit cycles bifurcating from nilpotent critical points for analytic vector fields of degree n. [5] got $N(3) \geq 2, N(5) \geq 5, N(7) \geq 9$; [2] got $N(3) \geq 3, N(5) \geq 5$; For a family of Kukles system with 6 parameters, [1] got $N(3) \geq 3$. Hence in this paper, employing the integral factor method introduced in [11], we will prove $N(5) \geq 8$. To the best of our knowledge, our results on the lower bounds of cyclicity of three-order nilpotent critical points for quintic systems are new.

We will organize this paper as follows. In Section 2, we state some preliminary knowledge given in [11] which is useful throughout the paper. In Section 3, using the linear recursive formulae in [11] to do direct computation, we obtain with relative ease the first 8 quasi-Lyapunov constants and the necessary and sufficient conditions of center. This paper is ended with Section 4 in which the 8-order weak focus conditions and the fact that there exist 8 limit cycles in the neighborhood of the three-order nilpotent critical point are proved.

2. Preliminary knowledge

The ideas of this section come from [11], where the center-focus problem of three-order nilpotent critical points in the planar dynamical systems was studied. We first recall the related notions and results. For more details, we refer to [11].

The origin of system (2) is a three-order monodromic critical point if and only if the system can be written as the following real autonomous planar system

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i y^j = X(x,y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^i y^j = Y(x,y).\end{aligned} \tag{5}$$

It is differential from the center-focus problem for the elementary critical points. We give the following key results, which define the quasi-Lyapunov constants and provide a way of computing them.

Definition 2.1 *If there exists a natural number s and a formal series $M(x,y) = x^4 + y^2 + o(r^4)$, such that*

$$\frac{\partial}{\partial x}\Big(\frac{X}{M^{s+1}}\Big) + \frac{\partial}{\partial y}\Big(\frac{Y}{M^{s+1}}\Big) = \frac{1}{M^{s+2}}\sum_{m=1}^{\infty}(2m-4s-1)\lambda_m[x^{2m+4} + o(r^{2m+4})] \tag{6}$$

holds, then, λ_m is called the m-th quasi-Lyapunov constant of the origin of system (5).

Theorem 2.1 *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}, \quad \beta \geq 3, \tag{7}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{8}$$

such that

$$\Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\Big)M - (s+1)\Big(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\Big) = \sum_{m=3}^{\infty}\omega_m(s,\mu)x^m, \tag{9}$$

where for all k, $M_k(x,y)$ is a k-homogeneous polynomial of x, y and $s\mu = 0$.

It is easy to see that (9) is linear with respect to the function M, so that we can easily find the following recursive formulae for the calculation of $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Theorem 2.2 *For $\alpha \geq 1$, $\alpha + \beta \geq 3$ in (8) and (9), $c_{\alpha\beta}$ can be uniquely determined by the* *recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{10}$$

For $m \geq 1$, $\omega_m(s,\mu)$ can be uniquely determined by the recursive formula

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{11}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k-(s+1)(\alpha-k+1)]a_{kj}c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j-(s+1)(\beta-j+1)]b_{kj}c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{12}$$

Notice that in (12), we set

$$\begin{aligned} &c_{00} = c_{10} = c_{01} = 0, \\ &c_{20} = c_{11} = 0, c_{02} = 1, \\ &c_{\alpha\beta} = 0, \text{ if } \alpha < 0 \text{ or } \beta < 0. \end{aligned} \tag{13}$$

We see from Theorem 2.2 that if the origin of system (5) is s-class or ∞-class, then, by choosing $\{c_{\alpha\beta}\}$, such that

$$\omega_{2k+1}(s,\mu) = 0, \quad k = 1, 2, \dots, \tag{14}$$

we can obtain a solution group of $\{c_{\alpha\beta}\}$ of (14). Thus, we have

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m-4s-1}. \tag{15}$$

Consider the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \delta x + y + \sum_{k+j=2}^{\infty} a_{kj}(\gamma)x^k y^j, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 2\delta y + \sum_{k+j=2}^{\infty} b_{kj}(\gamma)x^k y^j, \end{aligned} \tag{16}$$

where $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{m-1}\}$ is $(m-1)$-dimensional parameter vector. Let $\gamma_0 = \{\gamma_1^{(0)}, \gamma_2^{(0)}, \dots, \gamma_{m-1}^{(0)}\}$ be a point at the parameter space. Suppose that for $\|\gamma - \gamma_0\| \ll 1$, the functions of the right hand of system (16) are power series of x, y with a non-zero convergence radius and have continuous partial derivatives with respect to γ. In addition,

$$a_{20}(\gamma) \equiv \mu, b_{20}(\gamma) \equiv 0, b_{11}(\gamma) \equiv 2\mu, b_{30}(\gamma) \equiv -2. \tag{17}$$

For an integer k, let $\nu_{2k}(-2\pi,\gamma)$ be the k-order focal value of the origin of system $(16)_{\delta=0}$.

Theorem 2.3 *If for $\gamma = \gamma_0$, the origin of system $(16)_{\delta=0}$ is an m-order weak focus, and the Jacobin*

$$\left.\frac{\partial(\nu_2, \nu_4, \dots, \nu_{2m-2})}{\partial(\gamma_1, \gamma_2, \dots, \gamma_{m-1})}\right|_{\gamma=\gamma_0} \neq 0, \tag{18}$$

then, there exist two positive numbers δ^* and γ^*, such that for $0 < |\delta| < \delta^*, 0 < \|\gamma - \gamma_0\| < \gamma^*$, in a neighborhood of the origin, system (16) has at most m limit cycles which enclose the origin (an elementary node) $O(0,0)$. In addition, under the above conditions, there exist $\tilde{\gamma}$, $\tilde{\delta}$, such that when $\gamma = \tilde{\gamma}$, $\delta = \tilde{\delta}$, there exist exact m limit cycles of (16) in a small neighborhood of the origin.

Clearly, the recursive formula by Theorem 2.2 is linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to realize the computations of quasi-Lyapunov constants by using computer algebraic system like MATHEMATICA.

3. Quasi-Lyapunov constants and center conditions

According to Theorem 2.1, for system (1), we can find a positive integer s and a formal series $M(x, y) = x^4 + y^2 + o(r^4)$, such that (9) holds. Applying the recursive formulae presented in Theorem 2.2 to carry out calculations in MATHEMATICA, we have

$$
\begin{aligned}
&\omega_3 = \omega_4 = \omega_5 = 0,\\
&\omega_6 = -\frac{1}{3}b_{12}(-1 + 4s),\\
&\omega_7 = 3(s - 1)c_{03},\\
&\omega_8 = -\frac{2}{5}(a_{12} + 3a_{22})(-3 + 4s),\\
&\omega_9 = -\frac{16a_{22}}{3}(-1 + s).
\end{aligned}
\tag{19}
$$

From (15) and (19), we obtain the first two quasi-Lyapunov constants of system (1):

$$
\begin{aligned}
&\lambda_1 = \frac{\omega_6}{1 - 4s} = \frac{1}{3}b_{12},\\
&\lambda_2 = \frac{\omega_8}{3 - 4s} = \frac{2}{5}(a_{12} + 3a_{22}).
\end{aligned}
\tag{20}
$$

We see from $\omega_7 = \omega_9 = 0$ that

$$
c_{03} = 0, \quad s = 1. \tag{21}
$$

Furthermore, taking $s = 1$, we obtain the following conclusion.

Proposition 3.1 *For system (1), one can determine successively the terms of the formal series $M(x, y) = x^4 + y^2 + o(r^4)$, such that*

$$
\Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\Big)M - 2\Big(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\Big) = \sum_{m=1}^{11} \lambda_m[(2m - 5)x^{2m+4} + o(r^{26})], \tag{22}
$$

where λ_m is the m-th quasi-Lyapunov constant at the origin of system (1), $m = 1, 2, \ldots, 11$.

After careful computation with the help of MATHEMATICS 7.0, it is easy to get

Theorem 3.1 *For system (1), the first 8 quasi-Lyapunov constants at the origin are given by*

$$\begin{aligned}
\lambda_1 &= \frac{b_{21}}{3},\\
\lambda_2 &= \frac{2(a_{12}+3a_{22})}{5},\\
\lambda_3 &= \frac{4a_{22}(-2+5b_{12})}{35},\\
\lambda_4 &= -\frac{4(589050a_{04}-945751a_{22})}{4417875},\\
\lambda_5 &= -\frac{2a_{22}(-8417334479+7549657500a_{03}-19635000a_{13})}{8504409375},\\
\lambda_6 &= \frac{a_{22}(-41308398226396071+18506492430387500a_{13}+234715831812000000a_{22}^2)}{36132896721796875},\\
\lambda_7 &= -\frac{a_{22}f}{306280730949080866940786228701171875 0},\\
\lambda_8 &= -\frac{a_{22}g}{1217083054608910095005949276301281738281 25},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
f = &-56050486027489156821497600884250855 23-\\
&520231104268154209895045104676730000 00a_{22}^2+\\
&192764494558328970793835088000000000000 00a_{22}^4,\\
g = &-176192177659808858837860030433270976220 5407-\\
&176625095271752914300840786071551507163750 00a_{22}^2+\\
&577370095349981319122458035310920000000000 00000a_{22}^4.
\end{aligned}$$

In the above expressions of λ_k, we have already let $\lambda_1=\lambda_2=\cdots=\lambda_{k-1}=0$, $k=2,\ldots,8$.

From Theorem 3.1, we obtain the following assertion.

Proposition 3.2 *The first 8 quasi-Lyapunov constants at the origin of system (1) are zero if and only if the following conditions are satisfied:*

$$a_{12}=a_{22}=b_{21}=a_{04}=0.$$

The Proposition 3.2 implies the following

Proposition 3.3 *The origin of system (1) is a center when conditions of Proposition 3.2 hold.*

Proof When conditions of Proposition 3.2 hold, system (1) can be brought to

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= y-3x^2y+a_{03}y^3+x^3y+a_{13}xy^3+a_{41}x^4y,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -2x^3+y^2+b_{12}xy^2+b_{50}x^5
\end{aligned} \tag{24}$$

whose vector field is symmetric with respect to the x-axis. So the origin of (24) is a center. □

We see from Propositions 3.2 and 3.3 that

Theorem 3.2 *The origin of system (1) is a center if and only if the first 8 quasi-Lyapunov* *constants are zero, that is, the conditions in Proposition 3.2 are satisfied.*

4. Multiple bifurcation of limit cycles

This section is devoted to proving that when the three-order nilpotent critical point $O(0,0)$ is a 8-order weak focus, the perturbed system of (1) can generate 8 limit cycles enclosing an elementary node at the origin of perturbation system (1).

Using the fact $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0,\ \lambda_8 \neq 0$, we obtain

Theorem 4.1 *The origin of system (1) is a 8-order weak focus if and only if*

$$\begin{aligned}
&b_{21} = 0, a_{12} = -3a_{22},\\
&b_{12} = \frac{2}{5}, a_{04} = -\frac{945751a_{22}}{589050},\\
&a_{03} = -\frac{-8417334479 - 19635000a_{13}}{7549657500},\\
&a_{13} = -\frac{-41308398226396071 + 234715831812000000a_{22}^2}{18506492430387500},
\end{aligned} \tag{25}$$

where a_{22} are the real roots of $f = 0$.

Proof By letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$, we obtain

$$\begin{gathered}
b_{21} = 0,\ a_{12} = -3a_{22},\\
b_{12} = \frac{2}{5},\ a_{04} = -\frac{945751a_{22}}{589050},\\
a_{03} = -\frac{-8417334479 - 19635000a_{13}}{7549657500},\\
a_{13} = -\frac{-41308398226396071 + 234715831812000000a_{22}^2}{18506492430387500}.
\end{gathered}$$

Solving the equation $f = 0$, we could get four real solutions

$$A_1 \approx -0.593627, \quad A_2 \approx 0.593627,$$

$$A_3 \approx -0.28725177555672643i, \quad A_4 \approx 0.28725177555672643i$$

and when $a_{22} = A_1 or A_2$, we have

$$\text{Resultant}[f, g] \neq 0.$$

So $\lambda_8 \neq 0$, and the origin of system (1) is a 8-order weak focus.

Now we study the perturbed system of (1) as follows:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \delta x + y + \mu x^2 - 3x^3y + a_{12}xy^2 + a_{03}y^3 + x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \delta y - 2x^3 + \mu xy + y^2 + b_{21}xry + a_{22}x^3 + b_{40}x^4.
\end{aligned} \tag{26}$$

Theorem 4.2 *If the origin of system (1) is a 8-order weak focus, for $0 < \delta \ll 1$, making a small perturbation to the coefficients of system (1), then, for system (26), in a small neighborhood of* *the origin, there exist exactly 8 small amplitude limit cycles enclosing the origin $O(0,0)$, which is an elementary node.*

Proof When conditions of (25) hold, we see that $a_{22} = A_i$ are the simple zeros of $f = 0$. Hence, when $a_{22} \approx -0.593627$,

$$\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, a_{12}, b_{12}, a_{04}, a_{03}, a_{13}, a_{22})} = 0.149119;$$

when $a_{22} \approx 0.593627$,

$$\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{\partial(b_{21}, a_{12}, b_{12}, a_{04}, a_{03}, a_{13}, a_{22})} = -0.149119.$$

So Theorem 4.2 holds according to Theorem 2.3. □

Appendix A

Detailed recursive MATHEMATICA code to compute the quasi-Lyapunov constants at the origin of system (1):

$c[0,0] = 0, c[1,0] = 0, c[0,1] = 0, c[2,0] = 0, c[1,1] = 0, c[0,2] = 1;$

$c[0,j] := d[j]/; (j > 2)$

$c[k,j] := 0/; (k < 0 || j < 0)$

$$\begin{aligned}
c[k,j] = &-\frac{1}{3927k(1+s)}(7854c[-6+k,2+j] + 3927jc[-6+k,2+j] + 7854sc[-6+k,2+j]+\\
&3927jsc[-6+k,2+j] - 344c[-4+k,j] + 43kc[-4+k,j] - 172sc[-4+k,j]-\\
&15708c[-4+k,2+j] - 7854jc[-4+k,2+j] - 15708sc[-4+k,2+j]-\\
&23562c[-3+k,j] + 3927kc[-3+k,j] - 11781sc[-3+k,j] + 3927ksc[-3+k,j]+\\
&3927b_{21}jc[-3+k,1+j] + 3927b_{21}sc[-3+k,1+j] + 3927b_{21}jsc[-3+k,1+j]-\\
&15708a_{22}c[-2+k,-1+j] + 3927a_{22}kc[-2+k,-1+j] - 7854a_{22}sc[-2+k,-1+j]+\\
&3927a_{22}ksc[-2+k,-1+j] + 47124c[-2+k,j] - 7854b_{12}c[-2+k,j]-\\
&11781kc[-2+k,j] + 23562sc[-2+k,j] + 3927b_{12}jsc[-2+k,j] - 11781ksc[-2+k,j]-\\
&7854a_{13}c[-1+k,-2+j] + 3927a_{13}kc[-1+k,-2+j] - 3927a_{13}sc[-1+k,-2+j]+\\
&3927a_{13}ksc[-1+k,-2+j] - 7854a_{12}c[-1+k,-1+j] - 15708a_{22}c[-1+k,-1+j]+\\
&3927a_{22}jc[-1+k,-1+j] + 3927a_{12}kc[-1+k,-1+j] - 3927a_{12}sc[-1+k,-1+j]-\\
&3927a_{22}sc[-1+k,-1+j] + 3927a_{22}jsc[-1+k,-1+j] + 3927a_{12}ksc[-1+k,-1+j]-\\
&7854c[-1+k,j] + 3927jc[-1+k,j] + 3927jsc[-1+k,j] + 3927a_{04}kc[k,-3+j]+\\
&3927a_{04}ksc[k,-3+j] + 3927a_{03}kc[k,-2+j] + 3927a_{03}ksc[k,-2+j]+\\
&43ksc[-4+k,j] - 7854jsc[-4+k,2+j] + 3927b_{12}jc[-2+k,j]),
\end{aligned}$$

$$\begin{aligned}
\omega_m = &\frac{1}{3927}(-3927c[-5+m,1] - 3927sc[-5+m,1] + 301c[-3+m,-1] - 43mc[-3+m,-1]+\\
&129sc[-3+m,-1] - 43msc[-3+m,-1] + 7854c[-3+m,1] + 7854sc[-3+m,1]-
\end{aligned}$$

$$
\begin{aligned}
&3927mc[-2+m,-1]+7854sc[-2+m,-1]-3927msc[-2+m,-1]+3927b_{21}c[-2+m,0]+\\
&11781a_{22}c[-1+m,-2]-3927a_{22}mc[-1+m,-2]+3927a_{22}sc[-1+m,-2]-\\
&3927a_{22}msc[-1+m,-2]-35343c[-1+m,-1]+11781b_{12}c[-1+m,-1]-\\
&3927a_{13}mc[m,-3]-3927a_{13}msc[m,-3]+3927a_{12}c[m,-2]+19635a_{22}c[m,-2]-\\
&3927a_{12}mc[m,-2]+7854a_{22}sc[m,-2]-3927a_{12}msc[m,-2]+11781c[m,-1]+\\
&3927sc[m,-1]-3927a_{04}c[1+m,-4]-3927a_{04}mc[1+m,-4]-3927a_{04}sc[1+m,-4]-\\
&3927a_{04}msc[1+m,-4]-3927a_{03}c[1+m,-3]-3927a_{03}mc[1+m,-3]-\\
&3927a_{03}msc[1+m,-3]-3927c[1+m,-1]-3927mc[1+m,-1]-3927sc[1+m,-1]+\\
&19635c[-2+m,-1]+11781mc[-1+m,-1]-3927msc[1+m,-1]-\\
&3927a_{03}sc[1+m,-3]),
\end{aligned}
$$

$$\lambda_m = \frac{\omega_{2m+4}}{2m-4s-1}.$$

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