Journal of Mathematical Research & Exposition
Nov., 2011, Vol. 31, No. 6, pp. 951–964
DOI:10.3770/j.issn:1000-341X.2011.06.001
Http://jmre.dlut.edu.cn

Minimal Rank Preserving Additive Mappings on Upper Triangular Matrices

Yu GUO[1,2,3,*], **Jin Chuan HOU**[2,3]

1. *Department of Mathematics, Shanxi Datong University, Shanxi* 037009*, P. R. China;*
2. *Department of Mathematics, Shanxi University, Shanxi* 030006*, P. R. China;*
3. *Department of Mathematics, Taiyuan University of Technology, Shanxi* 030024*, P. R. China*

Abstract The additive mappings that preserve the minimal rank on the algebra of all $n \times n$ upper triangular matrices over a field of characteristic 0 are characterized.

Keywords rank; minimal rank; upper triangular matrices; additive mappings.

Document code A
MR(2010) Subject Classification 15A03; 15A04
Chinese Library Classification O151

1. Introduction

Let $\mathbb{F}$ be a field, $\mathcal{M}_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over $\mathbb{F}$. By $\mathcal{T}_n(\mathbb{F})$ we denote the algebra of all $n \times n$ upper triangular matrices over $\mathbb{F}$. For $A \in \mathcal{M}_n(\mathbb{F})$, define $\mathrm{mr}(A)$ to be the $\min\{\mathrm{rank}(A - \lambda I) : \lambda \in \mathbb{F}\}$, which is called the minimal rank of A (see [1]). Let $\Gamma_k = \{A : \mathrm{mr}(A) = k\}$, $0 \le k \le n$. A mapping $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is called a minimal rank preserving mapping if $\phi(\Gamma_k) \subset \Gamma_k$ holds for all $k = 0, 1, 2, \ldots, n$.

The minimal rank has been studied intensively because of its many applications in architecture, engineering and control theory, etc. For example, the minimal rank method can be used as a method of structural damage detection in architecture and engineering [2–4], and it also has important applications in the eigenstructure assignment and the dynamical order assignment for singular systems [5].

As showed in [1], if $\mathbb{F}$ is an algebraically closed field of characteristic 0, then a linear mapping $\phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is minimal rank preserving if and only if there exist an invertible matrix $S \in \mathcal{M}_n(\mathbb{F})$, a linear mapping $h : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ and a nonzero element $\alpha \in \mathbb{F}$ such that $\phi(A) = \alpha SAS^{-1} + h(A)I$ for all $A \in \mathcal{M}_n(\mathbb{F})$, or $\phi(A) = \alpha SA^{\mathrm{T}}S^{-1} + h(A)I$ for all $A \in \mathcal{M}_n(\mathbb{F})$, where A^{T} is the transpose of A. This result was generalized to additive mappings in [6]. It is interesting to notice that the question of characterizing minimal rank preserving mappings is connected with the question of characterizing the mappings preserving the number of nontrivial (or nonconstant)

Received June 9, 2010; Accepted October 3, 2010
Supported by the National Natural Science Foundation of China (Grant Nos. 10771157; 10871111) and Research Grant to Returned Scholars of Shanxi Province (Grant No. 2007-38).
* Corresponding author
E-mail address: guoyu3@yahoo.com.cn (Y. GUO)

invariant polynomials (i.e., invariant factors [7]) of matrices [1, 8]. For $A \in \mathcal{M}_n(\mathbb{F})$, denote by $\mathrm{i}(A)$ the number of nontrivial invariant polynomials of A. By an observation of Oliveira et al. [8], we have that $\mathrm{mr}(A) + \mathrm{i}(A) = n$ whenever $\mathbb{F}$ is an algebraically closed field of characteristic 0 (also see [1]). And the authors in [9] showed that, if $\mathbb{F}$ is an arbitrary number field, then $\mathrm{mr}(X) + \mathrm{i}(X) - \mathrm{k}(X) = n$, where $\mathrm{k}(X)$ denotes the number of nontrivial invariant polynomials which have no roots in $\mathbb{F}$. For upper triangular matrix case, it is clear that $\mathrm{mr}(A)+\mathrm{i}(A) = n$ holds for all $n \times n$ upper triangular matrix A over a field of characteristic 0. Thus every minimal rank preserving mapping on the algebra of upper triangular matrices over any field of characteristic 0 is a mapping preserving the number of nontrivial invariant polynomials.

In this note, we are interested in the question of characterizing additive mappings on the upper triangular matrix algebra $\mathcal{T}_n(\mathbb{F})$ that preserve the minimal rank. We mention here that the question of characterizing linear or additive mappings on upper triangular matrices preserving rank or rank-one have been studied by several authors [10–12]. Note that, unlike the case for $\mathcal{M}_n(\mathbb{F})$, the situation for $\mathcal{T}_n(\mathbb{F})$ is more difficult and the structure of rank-one preserving additive mappings on $\mathcal{T}_n(\mathbb{F})$ is quite complicated (see, for example, [10, 11]). However, as what we will show, the structure of minimal rank preserving additive mappings is nice.

2. Notation and preliminaries

Let φ be a homomorphism of $\mathbb{F}$. Assume that $\mathcal{U}$ and $\mathcal{V}$ are vector spaces over $\mathbb{F}$, an additive mapping $L : \mathcal{U} \to \mathcal{V}$ is called φ-quasilinear if $L(\lambda u) = \varphi(\lambda)Lu$ for all $\lambda \in \mathbb{F}$ and $u \in \mathcal{U}$. If $A = [a_{ij}]$ is a matrix, A_φ (some times, $\varphi(A)$) will stand for the matrix $[\varphi(a_{ij})]$. Clearly, the mapping $A \mapsto A_\varphi$ is additive and multiplicative. The flip mapping $A \mapsto A^f$ is defined by $A^f = JA^{\mathrm{T}}J$, where $J = \sum_{i=1}^n E_{i,n+1-i}$ and E_{ij} is the matrix with (i, j)-entry 1 and others 0. It is clear that every additive mapping from $\mathcal{T}_n(\mathbb{F})$ into itself of the form $A \mapsto \alpha TA_\varphi T^{-1} + h(A)I$ or $A \mapsto \alpha T(A_\varphi)^f T^{-1} + h(A)I$ is an additive mapping preserving minimal rank of matrices, where α is a nonzero scalar, $T \in \mathcal{T}_n(\mathbb{F})$ is nonsingular, φ is a nonzero homomorphism of $\mathbb{F}$ and $h : \mathcal{T}_n(\mathbb{F}) \to \mathbb{F}$ is an additive mapping. However, there are additive mappings of other forms that preserve minimal rank as well. Our purpose is to give a complete classification of all additive mappings preserving minimal rank on $\mathcal{T}_n(\mathbb{F})$.

Throughout this paper, $\{e_i\}_{i=1}^n$ stands for the standard basis of $\mathbb{F}^n$, that is, $e_1=(1, 0, 0, \ldots, 0, 0)^{\mathrm{T}}$, $e_2=(0, 1, 0, \ldots, 0, 0)^{\mathrm{T}}$, $\ldots$, $e_n=(0, 0, 0, \ldots, 0, 1)^{\mathrm{T}}$. For vectors $x=(x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $f = (f_1, f_2, \ldots, f_n)^{\mathrm{T}} \in \mathbb{F}^n$, we denote by $x \otimes f$ the rank-one matrix $xf^{\mathrm{T}} = [x_i f_j]$. Thus, $E_{ij} = e_i \otimes e_j$. For any mapping $\psi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$, $\psi^f : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ is the mapping defined by $\psi^f(A) = \psi(A)^f$, $A \in \mathcal{T}_n(\mathbb{F})$.

The following properties of the minimal rank, which are needed to prove our results, follow immediately from its definition. Assume that $A \in \mathcal{T}_n(\mathbb{F})$, $n \geq 2$ and $\lambda \in \mathbb{F}$, where $\mathbb{F}$ is an arbitrary field.

(a) $\mathrm{mr}(A + \lambda I) = \mathrm{mr}(A)$;

(b) $\mathrm{mr}(TAT^{-1}) = \mathrm{mr}(A)$ for any invertible matrix $T \in \mathcal{T}_n(\mathbb{F})$;

(c) $\mathrm{mr}(\lambda A) = \mathrm{mr}(A)$ if $\lambda \neq 0$;

(d) $\mathrm{mr}(A^f) = \mathrm{mr}(A)$;

(e) $0 \leq \mathrm{mr}(A) \leq n-1$ and $\mathrm{mr}(A) \leq \mathrm{rank}(A)$;

(f) $\mathrm{mr}(A) = 0$ if and only if $A = \alpha I$ for some $\alpha \in \mathbb{F}$;

(g) If $\mathrm{rank}(A) = 1$, then $\mathrm{mr}(A) = 1$;

(h) $\mathrm{mr}(A_\varphi) = \mathrm{mr}(A)$ for any nonzero homomorphism φ of $\mathbb{F}$.

3. The main result and its proof

The following is our main result.

Theorem 1 *Let $\mathbb{F}$ be a field of characteristic 0, $n \geq 3$, and $\phi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ be an additive injective mapping. Then ϕ preserves minimal rank if and only if there exists an invertible matrix $T \in \mathcal{T}_n(\mathbb{F})$, a nonzero scalar $\alpha \in \mathbb{F}$, a nonzero homomorphism φ of $\mathbb{F}$, an additive function $f : \mathbb{F} \to \mathbb{F}$ and an additive mapping $h : \mathcal{T}_n(\mathbb{F}) \to \mathbb{F}$ such that either*

$$\phi(A) = \alpha T A_\varphi T^{-1} + f(a_{11} - a_{nn})E_{1n} + h(A)I \quad \text{for all} \quad A = [a_{ij}] \in \mathcal{T}_n(\mathbb{F})$$

or

$$\phi(A) = \alpha T A_\varphi^f T^{-1} + f(a_{11} - a_{nn})E_{1n} + h(A)I \quad \text{for all} \quad A = [a_{ij}] \in \mathcal{T}_n(\mathbb{F}).$$

In order to prove Theorem 1, some lemmas are needed.

Lemma 2 *If $A \in \mathcal{T}_3(\mathbb{F})$, then $\mathrm{rank}(A) = 2$ and $\mathrm{mr}(A) = 1$ imply that A is similar to $\mathrm{Diag}(a, a, 0)$ for some nonzero $a \in \mathbb{F}$.*

Proof If $A \in \mathcal{T}_3(\mathbb{F})$ satisfies $\mathrm{rank}(A) = 2$ and $\mathrm{mr}(A) = 1$, then there exist $x, f \in \mathbb{F}^3$ and $a \in \mathbb{F}$ such that $A = x \otimes f + aI$. Since $\mathrm{rank}(A) = 2$, $a \neq 0$ and A is not invertible, we see that $x \otimes f$ is not nilpotent. It follows that $\sigma(x \otimes f) = \{\langle x, f\rangle, 0\}$ and $a = -\langle x, f\rangle \neq 0$, here $\langle x, f\rangle = x^{\mathrm{T}} f$ and $\sigma(x \otimes f)$ denotes the set of all eigenvalues of $x \otimes f$. Thus $x \otimes f$ is similar to $\mathrm{Diag}(-a, 0, 0)$ which implies that A is similar to $\mathrm{Diag}(a, a, 0)$ for some nonzero $a \in \mathbb{F}$. □

Lemma 3 *Let ϕ be an additive mapping on $\mathcal{T}_n(\mathbb{F})$, $n \geq 3$, such that $\mathrm{mr}(\phi(E)) = 1$ whenever $\mathrm{rank}(E) = 1$. Then $\mathrm{rank}(A + B) = \mathrm{rank}(A - B) = \mathrm{rank}(\phi(A)) = \mathrm{rank}(\phi(B)) = 1$ implies $\mathrm{rank}(\phi(A + B)) = 1$.*

Proof If $\mathrm{rank}(A + B) = 1$, then, by the hypotheses, $\mathrm{mr}(\phi(A + B)) = 1$. There are two cases to be considered.

Case 1 $n \geq 4$. Note that $\mathrm{mr}(\phi(A + B)) = 1$, thus $\phi(A + B) = E + \lambda I$ for some E with $\mathrm{rank}(E) = 1$. As ϕ is additive, we get $\phi(A) + \phi(B) - E = \lambda I$. If $\lambda \neq 0$, then $4 \leq n = \mathrm{rank}(\lambda I) = \mathrm{rank}(\phi(A) + \phi(B) - E) \leq \mathrm{rank}(\phi(A)) + \mathrm{rank}(\phi(B)) + \mathrm{rank}(E) = 3$, a contradiction. It follows that $\lambda = 0$, and so $\mathrm{rank}(\phi(A + B)) = \mathrm{rank}(E) = 1$.

Case 2 $n = 3$. $\mathrm{mr}(\phi(A+B)) = 1$ implies that $1 \leq \mathrm{rank}(\phi(A+B)) \leq \mathrm{rank}(\phi(A)) + \mathrm{rank}(\phi(B)) =$

2. If $\mathrm{rank}(\phi(A+B)) = 2$, then by Lemma 2, $\phi(A)+\phi(B)$ is similar to $\mathrm{Diag}(a,a,0)$ for some nonzero $a \in \mathbb{F}$. Together with the assumption $\mathrm{rank}(\phi(A)) = \mathrm{rank}(\phi(B)) = 1$, we deduce that $\sigma(\phi(A)-\phi(B)) = \{a, -a, 0\}$, which leads to $\mathrm{mr}(\phi(A)-\phi(B)) = 2$. However, by the hypotheses, $\mathrm{mr}(\phi(A)-\phi(B)) = 1$ since $\mathrm{rank}(A-B) = 1$, a contradiction. $\Box$

Lemma 4 *Let ϕ be an additive mapping on $\mathcal{T}_n(\mathbb{F})$, $n \geq 3$, with the properties:*

(i) $\mathrm{rank}(E) = 1$ *implies* $\mathrm{mr}(\phi(E)) = 1$, *and*

(ii) $\mathrm{rank}(\phi(\lambda E_{ij})) = 1$ *for any nonzero $\lambda \in \mathbb{F}$ and any i, j with $1 \leq i \leq j \leq n$.*

Then, for any $A \in \mathcal{T}_n(\mathbb{F})$, that A is of rank one implies that $\phi(A)$ is of rank one.

Proof For $z \in \mathbb{F}^n$, denote $\mathcal{S}(z) = \{i : z_i \neq 0,\ z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}\}$. For any rank-one matrix $E = x \otimes y$, let $\mathcal{K}(E) = \#\mathcal{S}(x) + \#\mathcal{S}(y)$, where $\#\mathcal{S}(x)$ denotes the number of elements in $\mathcal{S}(x)$. We will prove Lemma 4 by induction on $\mathcal{K}(E)$. It is clear that $2 \leq \mathcal{K}(E) \leq n+1$ since $E \in \mathcal{T}_n(\mathbb{F})$. If $\mathcal{K}(E) = 2$, then there exist some nonzero $\mu \in \mathbb{F}$ and i, j with $1 \leq i \leq j \leq n$ such that $E = \mu E_{ij}$. By the property (ii) we obtain that $\mathrm{rank}(\phi(E)) = 1$. Now assume that $\mathrm{rank}(\phi(E)) = 1$ holds for all rank-one upper triangular matrices E with $\mathcal{K}(E) \leq k$, $2 \leq k \leq n$. For any rank-one matrix $E = x \otimes y \in \mathcal{T}_n(\mathbb{F})$ with $\mathcal{K}(E) = k+1$, we have to show that $\mathrm{rank}(\phi(E)) = 1$. Obviously, either $\#\mathcal{S}(x) \geq 2$ or $\#\mathcal{S}(y) \geq 2$.

Case 1 $\#\mathcal{S}(y) \geq 2$. In this case, decompose y as $y = y' + y''$ with $\#\mathcal{S}(y') < \#\mathcal{S}(y)$, $\#\mathcal{S}(y'') < \#\mathcal{S}(y)$, $\mathcal{S}(y') \subseteq \mathcal{S}(y)$, and $\mathcal{S}(y'') \subseteq \mathcal{S}(y)$. Thus, $x \otimes y'$, $x \otimes y'' \in \mathcal{T}_n(\mathbb{F})$. and $\mathcal{K}(x \otimes y') \leq k$, $\mathcal{K}(x \otimes y'') \leq k$. So, by the induction assumption, we have $\mathrm{rank}(\phi(x \otimes y')) = 1 = \mathrm{rank}(\phi(x \otimes y''))$. Also note that $\mathrm{rank}(x \otimes y' + x \otimes y'') = 1 = \mathrm{rank}(x \otimes y' - x \otimes y'')$. Applying Lemma 3, we obtain that $\mathrm{rank}(\phi(E)) = \mathrm{rank}(\phi(x \otimes y') + \phi(x \otimes y'')) = 1$.

Case 2 $\#\mathcal{S}(x) \geq 2$. The proof is similar to that of Case 1. $\Box$

The next lemma comes from [11], which gives a characterization of rank-one preserving additive mappings on upper triangular matrices.

Before stating Lemma 5, let us recall some more notations from [11]. As usual, by $\mathcal{T}_n^1$ we denote the set of all rank-one matrices in $\mathcal{T}_n(\mathbb{F})$. For any integers $1 \leq s$, $t \leq n$, we denote by $\mathcal{T}_{s,t}$ the subspace of $\mathcal{T}_n(\mathbb{F})$ consisting of all matrices $[a_{ij}]$ in which $a_{ij} = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq s-1$, and $a_{ij} = 0$ for all $t < i \leq n$ and $1 \leq j \leq n$. Particularly, $\mathcal{T}_{1,n} = \mathcal{T}_n(\mathbb{F})$, $\mathcal{T}_{1,1} = \{[a_{ij}] : a_{ij} = 0 \text{ whenever } i \neq 1\}$ and $\mathcal{T}_{n,n} = \{[a_{ij}] : a_{ij} = 0 \text{ whenever } j \neq n\}$. For the sake of convenience, we denote $\mathcal{T}_{1,0} = \mathcal{T}_{n+1,n} = \{0\}$. Let $\mathcal{S}$ be a nonempty subspace of $\mathcal{M}_{n1}$, k be a positive integer such that $k \leq \min\{n, \dim\mathcal{S}\}$. A matrix P is said to be k-regular with respect to $(\varphi, \mathcal{S})$ if $P(x_1)_\varphi, \ldots, P(x_k)_\varphi$ are linearly independent whenever $x_1, \ldots, x_k$ are linearly independent vectors in $\mathcal{S}$. In particular, P is one-regular with respect to $(\varphi, \mathcal{S})$, if $Px_\varphi \neq 0$ for all nonzero vectors $x \in \mathcal{S}$, and thus, PA_φ is of rank one whenever A is of rank one, $A \in \mathcal{T}_n(\mathbb{F})$. We use $\langle u_1, u_2, \ldots, u_r\rangle$ to denote the subspace spanned by the vectors $u_1, u_2, \ldots, u_r$. With an upper triangular matrix algebra $\mathcal{T}_n(\mathbb{F})$, we associate two chains of subspaces

$$\{0\} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \mathbb{F}^n,$$

and

$$\{0\} = \mathcal{V}_{n+1} \subset \mathcal{V}_n \subset \cdots \subset \mathcal{V}_2 \subset \mathcal{V}_1 = \mathbb{F}^n,$$

where $\mathcal{U}_i = \langle e_j : 1 \leq j \leq i\rangle$, $\mathcal{V}_i = \langle e_j : i \leq j \leq n\rangle$ for all $1 \leq i \leq n$, $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{F}^n$. The flip map of a vector $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^{\mathrm{T}}$ is defined by $\nu^f = (\nu_n, \ldots, \nu_2, \nu_1)^{\mathrm{T}}$.

Lemma 5 ([11, Corollary 3.12]) *Let $\psi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$, $n \geq 2$, be an additive mapping. Then ψ preserves rank-one matrices if and only if ψ or ψ^f takes one of the following forms:*

(i) There exist nonzero vectors $u \in \mathcal{U}_s$, $v \in \mathcal{V}_t$ for some integers $1 \leq s \leq t \leq n$ such that for each integer $1 \leq i \leq n$, either

(a) $\psi(A) = u \otimes F_i(A)$ for all $A \in \mathcal{T}_{i,i}$, or

(b) $\psi(A) = G_i(A) \otimes v$ for all $A \in \mathcal{T}_{i,i}$,

where $F_i : \mathcal{T}_{i,i} \to \mathcal{V}_s$, $G_i : \mathcal{T}_{i,i} \to \mathcal{U}_t$ are additive with $F_i|_{\mathcal{T}_n^1}$, $G_i|_{\mathcal{T}_n^1}$ injective; or

(ii) There exist integers $1 \leq s \leq t$ and $1 \leq i \leq j \leq n$, and a nonzero field homomorphism $\varphi : \mathbb{F} \to \mathbb{F}$ such that

(c) $\psi(A) = u \otimes F(A)$ for all $A \in \mathcal{T}_{1,s-1}$,

(d) $\psi(A) = TA_\varphi S$ for all $A \in \mathcal{T}_{s,t}$, and

(e) $\psi(A) = G(A) \otimes v$ for all $A \in \mathcal{T}_{t+1,n}$, where T, $S^f \in \mathcal{M}_n(\mathbb{F})$ are of rank ≥ 2 one-regular matrices with respect to $(\varphi, \mathcal{U}_t)$ and $(\varphi, \mathcal{V}_s^f)$ respectively satisfying $TE_{kl}S \in \mathcal{T}_n(\mathbb{F})$ for all $1 \leq k \leq l \leq n$, $u \in \mathcal{U}_i$, $v \in \mathcal{V}_j$ are nonzero vectors, and $F : \mathcal{T}_{1,s-1} \to \mathcal{V}_i$, $G : \mathcal{T}_{t+1,n} \to \mathcal{U}_j$ are additive mappings with $F|_{\mathcal{T}_n^1}$, $G|_{\mathcal{T}_n^1}$ injective such that $Tx_\varphi = \alpha(x)u$ and $F(x \otimes y) = \alpha(x)S^{\mathrm{T}}y_\varphi$ for all $x \otimes y \in \mathcal{T}_{s,s-1}$, and $S^{\mathrm{T}}y_\varphi = \lambda(y)v$ and $G(x \otimes y) = \lambda(y)Tx_\varphi$ for all $x \otimes y \in \mathcal{T}_{t+1,t}$, with $\alpha : \mathcal{U}_{s-1} \to \mathbb{F}$, $\lambda : \mathcal{V}_{t+1} \to \mathbb{F}$ injective φ-quasilinear.

Now we are in a position to give our proof of the main result, Theorem 1.

Proof of Theorem 1 We only need to check the "only if" part. Assume that $\phi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ is an additive injective mapping preserving the minimal rank.

Claim 1 There exists an additive mapping $\psi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ which preserves minimal rank of matrices as well rank-one matrices and an additive functional $h : \mathcal{T}_n(\mathbb{F}) \to \mathbb{F}$ such that

$$\phi(A) = \psi(A) + h(A)I \quad \text{for all} \quad A \in \mathcal{T}_n(\mathbb{F}).$$

If E is a matrix of rank one, then by property (g) we have $\mathrm{mr}(E) = 1$, and so $\mathrm{mr}(\phi(E)) = 1$. Therefore $\phi(E) = F + \delta I$ for some rank-one matrix $F \in \mathcal{T}_n(\mathbb{F})$ and $\delta \in \mathbb{F}$. In particular $\phi(E_{ij}) = F_{ij} + \delta_{ij}I$ and for any nonzero $\lambda \in \mathbb{F}$ we have $\phi(\lambda E_{ij}) = F_{ij}(\lambda) + \delta_{ij}(\lambda)I$ for some rank-one matrices F_{ij}, $F_{ij}(\lambda) \in \mathcal{T}_n(\mathbb{F})$ and scalars $\delta_{ij}, \delta_{ij}(\lambda) \in \mathbb{F}$, $1 \leq i \leq j \leq n$. Similarly, $\phi((1+\lambda)E_{ij}) = F_{ij}(1+\lambda) + \delta_{ij}(1+\lambda)I = F_{ij} + F_{ij}(\lambda) + (\delta_{ij} + \delta_{ij}(\lambda))I$. Write $F_{ij} = x_{ij} \otimes f_{ij}$ and $F_{ij}(\lambda) = y_{ij}(\lambda) \otimes g_{ij}(\lambda)$. Thus $F_{ij}(1+\lambda) = F_{ij} + F_{ij}(\lambda) = x_{ij} \otimes f_{ij} + y_{ij}(\lambda) \otimes g_{ij}(\lambda)$ is a rank-one matrix whenever $\lambda \neq -1$. It follows that either x_{ij} and $y_{ij}(\lambda)$ are linearly dependent, or f_{ij} and $g_{ij}(\lambda)$ are linearly dependent.

For any fixed pair of (i, j), without loss of generality, we assume that, there exists $\lambda \neq 0$ such that x_{ij} and $y_{ij}(\lambda)$ are linearly dependent. The case that f_{ij} and $g_{ij}(\lambda)$ are linearly dependent can be dealt with similarly. Thus we can assume that $y_{ij}(\lambda) = x_{ij}$. There are two cases that we have to consider.

Case 1 f_{ij} and $g_{ij}(\lambda)$ are linearly independent.

For any $\lambda_1 \in \mathbb{F}$, $\phi((1+\lambda+\lambda_1)E_{ij}) = F_{ij}(1+\lambda+\lambda_1) + \delta_{ij}(1+\lambda+\lambda_1)I$. On the other hand, writing $\delta_{ij} + \delta_{ij}(\lambda) + \delta_{ij}(\lambda_1) = \delta$, we have $\phi((1+\lambda+\lambda_1)E_{ij}) = \phi(E_{ij}) + \phi(\lambda E_{ij}) + \phi(\lambda_1 E_{ij}) = x_{ij} \otimes f_{ij} + x_{ij} \otimes g_{ij}(\lambda) + y_{ij}(\lambda_1) \otimes g_{ij}(\lambda_1) + \delta I = x_{ij} \otimes (f_{ij} + g_{ij}(\lambda)) + y_{ij}(\lambda_1) \otimes g_{ij}(\lambda_1) + \delta I$. If $\lambda_1 \neq -1$ and $1+\lambda+\lambda_1 \neq 0$, considering $\phi((1+\lambda_1)E_{ij}) = x_{ij} \otimes f_{ij} + y_{ij}(\lambda_1) \otimes g_{ij}(\lambda_1) + \delta_{ij}(1+\lambda_1)I$, if x_{ij} and $y_{ij}(\lambda_1)$ are linearly independent, then $g_{ij}(\lambda_1)$ and f_{ij} are linearly dependent, which implies that $\operatorname{rank}(F_{ij}(1+\lambda+\lambda_1)) = 2$, a contradiction. If $\lambda_1 = -1$ or $1+\lambda+\lambda_1 = 0$, it is clear that x_{ij} and $y_{ij}(\lambda_1)$ are linearly dependent. Thus, for any $\lambda_1 \in \mathbb{F}$, x_{ij} and $y_{ij}(\lambda_1)$ are also linearly dependent.

Case 2 f_{ij} and $g_{ij}(\lambda)$ are linearly dependent.

In this case, it is clear that, for any $\lambda \in \mathbb{F}$, $F_{ij}(\lambda) = \alpha(\lambda) x_{ij} \otimes f_{ij} \in \langle F_{ij} \rangle$, where $\alpha(\lambda) \in \mathbb{F}$.

So in both cases, we can assume that $F_{ij}(\lambda) = x_{ij} \otimes g_{ij}(\lambda)$ holds for all $\lambda \in \mathbb{F}$, and therefore, $\phi(\lambda E_{ij}) = x_{ij} \otimes g_{ij}(\lambda) + \delta_{ij}(\lambda)I$ for all λ. Thus we obtain that, for any $\lambda_1, \lambda_2 \in \mathbb{F}$, $\phi((\lambda_1 + \lambda_2)E_{ij}) = \phi(\lambda_1 E_{ij}) + \phi(\lambda_2 E_{ij})$, that is $x_{ij} \otimes g_{ij}(\lambda_1+\lambda_2) + \delta_{ij}(\lambda_1+\lambda_2)I = x_{ij} \otimes g_{ij}(\lambda_1) + x_{ij} \otimes g_{ij}(\lambda_2) + \delta_{ij}(\lambda_1)I + \delta_{ij}(\lambda_2)I = x_{ij} \otimes (g_{ij}(\lambda_1) + g_{ij}(\lambda_2)) + (\delta_{ij}(\lambda_1) + \delta_{ij}(\lambda_2))I$. As $n \geq 3$, it follows that $g_{ij}(\lambda_1+\lambda_2) = g_{ij}(\lambda_1) + g_{ij}(\lambda_2)$, $\delta_{ij}(\lambda_1+\lambda_2) = \delta_{ij}(\lambda_1) + \delta_{ij}(\lambda_2)$. Hence $\delta_{ij} : \mathbb{F} \to \mathbb{F}$ and $g_{ij} : \mathbb{F} \to \mathbb{F}^n$ are additive.

So far we have shown that, for any pair (i,j) with $1 \leq i \leq j \leq n$, there is an additive function $\delta_{ij} : \mathbb{F} \to \mathbb{F}$ and an additive map $F_{ij} : \mathbb{F} \to \mathcal{T}_n^1$ such that $\phi(\lambda E_{ij}) = F_{ij}(\lambda) + \delta_{ij}(\lambda)I$ for all $\lambda \in \mathbb{F}$.

Now define $h : \mathcal{T}_n(\mathbb{F}) \to \mathbb{F}$ by $h(A) = \sum_{i \leq j} \delta_{ij}(a_{ij})$ for any $A = [a_{ij}] = \sum_{i \leq j} a_{ij} E_{ij} \in \mathcal{T}_n(\mathbb{F})$. If $A = [a_{ij}], B = [b_{ij}] \in \mathcal{T}_n(\mathbb{F})$, then $h(A+B) = h(\sum_{i \leq j}(a_{ij}+b_{ij})E_{ij}) = \sum_{i \leq j} \delta_{ij}(a_{ij}) + \sum_{i \leq j} \delta_{ij}(b_{ij}) = h(A) + h(B)$. Thus h is an additive functional on $\mathcal{T}_n(\mathbb{F})$.

Define $\psi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ by $\psi(A) = \phi(A) - h(A)I$. Then it is easily seen that ψ is a minimal rank preserving additive mapping, and for any $\lambda \in \mathbb{F}$, $\psi(\lambda E_{ij}) = F_{ij}(\lambda)$ is of rank one. Hence, by Lemma 4, we see that ψ preserves rank-one matrices, too. So Claim 1 is true.

Claim 2 $\psi(I) = \beta I$ for some scalar $\beta \neq 0$.

Since ϕ is a minimal rank preserving mapping, there exists some scalar $\gamma \in \mathbb{F}$ such that $\phi(I) = \gamma I$. It is obvious that $\gamma \neq 0$ as ϕ is an additive injection and $\phi(0) = 0$. Thus we may assume that $\phi(I) = I$. Denote $\phi(E_{ii}) = F_{ii} + \lambda_{ii} I$, $i = 1, 2, \ldots, n$, where $F_{ii} \in \mathcal{T}_n^1$. Then $\phi(I) = \phi(\sum_{i=1}^n E_{ii}) = \sum_{i=1}^n F_{ii} + \sum_{i=1}^n \lambda_{ii} I = I$. It follows that $\psi(I) = \beta I$ for some $\beta \in \mathbb{F}$ with $\beta + \sum_{i=1}^n \lambda_{ii} = 1$. We have to show that $\beta \neq 0$. Otherwise suppose, to reach a contradiction, that $\beta = 0$, that is,

$$\psi(I) = \sum_{i=1}^n F_{ii} = x_{11} \otimes f_{11} + x_{22} \otimes f_{22} + \cdots + x_{nn} \otimes f_{nn} = 0$$

with $n \geq 3$. Without loss of generality we may assume that $\{x_{ii}\}_{i=1}^k$ is the maximal linearly independent subset of $\{x_{ii}\}_{i=1}^n$, where $1 \leq k < n$. Consider $\psi(E_{11} + 2E_{22} + \cdots + nE_{nn})$. It is clear that $\mathrm{mr}(\psi(E_{11}+2E_{22}+\cdots+nE_{nn}))=n-1$ as $\mathrm{mr}(E_{11}+2E_{22}+\cdots+nE_{nn})=n-1$. On the other hand,

$$\begin{aligned}\psi(E_{11}+2E_{22}+\cdots+nE_{nn})&=x_{11}\otimes f_{11}+2x_{22}\otimes f_{22}+\cdots+nx_{nn}\otimes f_{nn}\\ &=x_{11}\otimes f_{11}+2x_{22}\otimes f_{22}+\cdots+kx_{kk}\otimes f_{kk}+\\ &\quad(k+1)(\sum_{i=1}^{k}\alpha_{k+1,i}x_{ii})\otimes f_{k+1,k+1}+\cdots+n(\sum_{i=1}^{k}\alpha_{ni}x_{ii})\otimes f_{nn}\\ &=\sum_{i=1}^{k}x_{ii}\otimes g_{ii}\end{aligned}$$

for some $g_{ii}\in\mathbb{F}^n$ and $\alpha_{ji}\in\mathbb{F}$, $1\le i\le k$, $k+1\le j\le n$. Then, it is clear that $\mathrm{mr}(\psi(E_{11}+2E_{22}+\cdots+nE_{nn}))<n-1$ if $k<n-1$, a contradiction. If $k=n-1$, then $x_{nn}=\sum_{i=1}^{n-1}\alpha_{ni}x_{ii}$. It follows that

$$\begin{aligned}0=&x_{11}\otimes f_{11}+x_{22}\otimes f_{22}+\cdots+x_{n-1,n-1}\otimes f_{n-1,n-1}+(\sum_{i=1}^{n-1}\alpha_{ni}x_{ii})\otimes f_{nn}\\ =&x_{11}\otimes(f_{11}+\alpha_{n1}f_{nn})+x_{22}\otimes(f_{22}+\alpha_{n2}f_{nn})+\\ &\cdots+x_{n-1,n-1}\otimes(f_{n-1,n-1}+\alpha_{n,n-1}f_{nn}),\end{aligned}$$

which forces that $f_{ii}+\alpha_{ni}f_{nn}=0$, $1\le i\le n-1$. Thus, $x_{11}\otimes f_{11}+2x_{22}\otimes f_{22}+\cdots+nx_{nn}\otimes f_{nn}=y\otimes f_{nn}$ for some $y\in\mathbb{F}^n$. But this implies that $\mathrm{mr}(\psi(E_{11}+2E_{22}+\cdots+nE_{nn}))=1$, a contradiction, too. Hence we must have $\beta\neq 0$.

Claim 3 There exist an invertible matrix $T\in\mathcal{T}_n(\mathbb{F})$, an additive function $f:\mathbb{F}\to\mathbb{F}$ and a nonzero homomorphism $\varphi:\mathbb{F}\to\mathbb{F}$ such that

$$\psi(A)=\beta TA_{\varphi}T^{-1}+f(a_{11}-a_{nn})E_{1n}\quad\text{for all}\quad A\in\mathcal{T}_n(\mathbb{F})$$

or

$$\psi(A)=\beta TA_{\varphi}^{f}T^{-1}+f(a_{11}-a_{nn})E_{1n}\quad\text{for all}\quad A\in\mathcal{T}_n(\mathbb{F}).$$

By Claim 2 and replacing ψ by $\beta^{-1}\psi$ we may assume that $\psi(I)=I$.

Since ψ is a rank-one preserving additive mapping, by Lemma 5, we obtain that ψ or ψ^f takes one of the forms of (i) and (ii) in Lemma 5. Since ψ also preserves the minimal rank, it is clear that the form (i) in the lemma cannot occur. So, ψ or ψ^f, say in the sequel, ψ takes the form (ii) in Lemma 5. Obviously, (ii)-(c) only holds for the case of $s=2$ so that $\mathcal{T}_{1,s-1}=\mathcal{T}_{1,1}$ and (ii)-(e) only holds for the case $t=n-1$ so that $\mathcal{T}_{t+1,n}=\mathcal{T}_{n,n}$. Thus we get $s=2$ and $t=n-1$. Going further, we may assert that $E_{11}TASE_{11}=E_{nn}TASE_{nn}=0$ for all $A\in\mathcal{T}_{2,n-1}$. In fact, assume, to reach a contradiction, that $E_{11}TASE_{11}\neq 0$ for some $A\in\mathcal{T}_{2,n-1}$, then there exists a rank-one matrix $A'\in\mathcal{T}_{2,n-1}$ such that $E_{11}TA'_{\varphi}SE_{11}\neq 0$ as well. It follows that

$$\psi(A'+A_0)=TA'_{\varphi}S+u\otimes F(A_0)\in\mathcal{T}_n^1$$

for all $A_0\in\mathcal{T}_{1,1}$, which is impossible since ψ preserves the minimal rank. Similarly, we can get that $E_{nn}TASE_{nn}=0$ holds for all $A\in\mathcal{T}_{2,n-1}$. Since $TAS\in\mathcal{T}_n(\mathbb{F})$ for every $A\in\mathcal{T}_{2,n-1}$, T,

$S \in \mathcal{M}_n(\mathbb{F})$ can be chosen with t_{nk}, t_{kn}, s_{1k}, s_{k1} arbitrary, $k = 1, 2, \ldots, n$. We firstly choose $t_{nk} = t_{kn} = s_{1k} = s_{k1} = 0$.

We assert further that, for our case, $i = 1$ and $j = n$ in Lemma 5 (ii). Assume, on the contrary, that $i > 1$. As $j \geq i > 1$, $u \in \mathcal{U}_i$, $F(e_1 \otimes e_1) \in \mathcal{V}_i$, $v \in \mathcal{V}_j$, $G(e_n \otimes e_n) \in \mathcal{U}_j$, $e_2 \otimes e_2 + \cdots + e_{n-1} \otimes e_{n-1} \in \mathcal{T}_{2,n-1}$, and ψ is additive, we see that

$$I = \psi(I) = u \otimes F(e_1 \otimes e_1) + T(e_2 \otimes e_2 + \cdots + e_{n-1} \otimes e_{n-1})S + G(e_n \otimes e_n) \otimes v.$$

It follows from $i > 1$ that $E_{11}(u \otimes F(e_1 \otimes e_1))E_{11} = 0$ and hence $E_{11}\psi(I)E_{11} = 0$, which contradicts $\psi(I) = I$. So, we must have $i = 1$ and $u \in \langle e_1 \rangle$. Similarly, $j < n$ leads to a contradiction that $E_{nn} = E_{nn}\psi(I)E_{nn} = 0$. Hence $j = n$ and $v \in \langle e_n \rangle$. Thus we may assume that $u = e_1$ and $v = e_n$. Furthermore, by Lemma 5, F and G are φ-quasilinear respectively on $\{e_1 \otimes y : y \in \langle e_2, \ldots, e_n \rangle\}$ and $\{x \otimes e_n : x \in \langle e_1, e_2, \ldots, e_{n-1} \rangle\}$. It follows that ψ is φ-quasilinear on $\mathcal{T}_{2,n-1} = \{A = [a_{ij}] \in \mathcal{T}_n(\mathbb{F}) : a_{11} = a_{nn} = 0\}$.

Thus, for any $A \in \mathcal{T}_n(\mathbb{F})$, because $A' = A - a_{11}E_{11} - a_{nn}E_{nn} \in \mathcal{T}_{2,n-1}$ and ψ is additive, we have

$$\psi(A) = \psi(a_{11}E_{11} + A' + a_{nn}E_{nn}) = e_1 \otimes F(a_{11}E_{11}) + TA'_{\varphi}S + G(a_{nn}E_{nn}) \otimes e_n. \tag{3.1}$$

To see the behavior of F and G on $\langle e_1 \otimes e_1 \rangle$ and $\langle e_n \otimes e_n \rangle$, respectively, we apply the fact $\psi(\mathbb{F}I) \subseteq \mathbb{F}I$. For any $\alpha \in \mathbb{F}$, By Eq.(3.1), $\psi(\alpha I) = e_1 \otimes F(\alpha e_1 \otimes e_1) + \varphi(\alpha)T(e_2 \otimes e_2 + \cdots + e_{n-1} \otimes e_{n-1})S + G(\alpha e_n \otimes e_n) \otimes e_n = \tau(\alpha)I$ for some $\tau(\alpha) \in \mathbb{F}$. It follows from Eq.(3.1) and $\psi(I) = I$ that $\tau(\alpha)e_2 \otimes e_2 = (e_2 \otimes e_2)\psi(\alpha I)(e_2 \otimes e_2) = \varphi(\alpha)(e_2 \otimes e_2)$. Hence $\tau(\alpha) = \varphi(\alpha)$ and

$$\begin{aligned}
&\varphi(\alpha)e_1 \otimes e_1 = e_1 \otimes e_1\psi(\alpha I) \\
&\quad = e_1 \otimes F(\alpha e_1 \otimes e_1) + \varphi(\alpha)(e_1 \otimes e_1)T(\sum_{i=2}^{n-1} e_i \otimes e_i)S(\sum_{j=2}^{n} e_j \otimes e_j) + \\
&\qquad \langle G(\alpha e_n \otimes e_n), e_1 \rangle e_1 \otimes e_n \\
&\quad = e_1 \otimes F(\alpha e_1 \otimes e_1) + \varphi(\alpha)e_1 \otimes (\sum_{j=2}^{n}\sum_{i=2}^{n-1} \langle Se_j, e_i \rangle \langle Te_i, e_1 \rangle e_j) + \\
&\qquad \langle G(\alpha e_n \otimes e_n), e_1 \rangle e_1 \otimes e_n.
\end{aligned}$$

Thus we have

$$F(\alpha e_1 \otimes e_1) = \varphi(\alpha)[e_1 - \sum_{j=2}^{n}(\sum_{i=2}^{n-1} \langle Se_j, e_i \rangle \langle Te_i, e_1 \rangle)e_j] - \langle G(\alpha e_n \otimes e_n), e_1 \rangle e_n. \tag{3.2}$$

Similarly, by considering $\psi(\alpha I)e_n \otimes e_n$, one gets

$$G(\alpha e_n \otimes e_n) = \varphi(\alpha)[e_n - \sum_{j=1}^{n-1}(\sum_{i=2}^{n-1} \langle Se_n, e_i \rangle \langle Te_i, e_j \rangle)e_j] - \langle e_n, F(\alpha e_1 \otimes e_1) \rangle e_1. \tag{3.3}$$

Note that we have chosen T and S so that

$$T = \begin{bmatrix} t_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{22} & S_{23} \\ 0 & S_{32} & s_{nn} \end{bmatrix}.$$

Since ψ preserves rank-one matrices, $\psi(e_1\otimes(e_1+e_j)) = e_1\otimes F(e_1\otimes e_1)+T(e_1\otimes e_j)S$ and Eq.(3.2) together imply that $t_{j1}=0$ for each $j=2,\ldots,n-2$. Hence $T_{21}=0$. Similarly, considering $\psi((e_j+e_n)\otimes e_n)$ and applying Eq.(3.3) yields $S_{32}=0$. It follows that $T_{22}A_{22}S_{22}\in\mathcal{T}_{n-2}(\mathbb{F})$ for all $A_{22}\in\mathcal{T}_{n-2}(\mathbb{F})$ and $T_{22}S_{22}=I_{n-2}$. Therefore, both T_{22} and S_{22} are upper triangular matrices and $S_{22}=T_{22}^{-1}$. Consequently, $T,S\in\mathcal{T}_n(\mathbb{F})$. Let $U=\begin{bmatrix} t & 0 & 0\\ 0 & T_{22} & 0\\ 0 & 0 & s\end{bmatrix}$, where $t=1$ if $t_{11}=0$; $t=t_{11}$ if $t_{11}\neq 0$; $s=1$ if $s_{nn}=0$; $s=s_{nn}^{-1}$ if $s_{nn}\neq 0$. Then $U\in\mathcal{T}_n(\mathbb{F})$ is invertible. Replacing ψ by $U^{-1}\psi U$ if necessary, we may assume that

$$T=\begin{bmatrix} 1 & T_{12} & 0\\ 0 & I_{n-2} & 0\\ 0 & 0 & 1\end{bmatrix} \quad\text{and}\quad S=\begin{bmatrix} 1 & 0 & 0\\ 0 & I_{n-2} & S_{23}\\ 0 & 0 & 1\end{bmatrix}.$$

Without effecting the value of ψ, we may replace above T and S by

$$T_1=\begin{bmatrix} 1 & T_{12} & -T_{12}S_{23}\\ 0 & I_{n-2} & -S_{23}\\ 0 & 0 & 1\end{bmatrix} \quad\text{and}\quad S_1=\begin{bmatrix} 1 & -T_{12} & 0\\ 0 & I_{n-2} & S_{23}\\ 0 & 0 & 1\end{bmatrix}.$$

As $S_1=T_1^{-1}$, replacing ψ by $T_1^{-1}\psi T_1$ if necessary, we may assume that

$$\psi(A)=e_1\otimes F_1(a_{11}E_{11})+(A-a_{11}E_{11}-a_{nn}E_{nn})_\varphi+G_1(a_{nn}E_{nn})\otimes e_n$$

holds for every $A=[a_{ij}]\in\mathcal{T}_n(\mathbb{F})$. Again, $\psi(\alpha I)=\varphi(\alpha)I$ and thus F_1 and G_1 have the same representation as in Eqs.(3.2) and (3.3), respectively. It follows that there exist scalars $f_2,\ldots,f_{n-1}$; $g_2,\ldots,g_{n-1}\in\mathbb{F}$ and additive functions f_n, g_1 from $\mathbb{F}$ into $\mathbb{F}$ such that

$$F_1(\alpha E_{11})=\varphi(\alpha)(e_1+f_2e_2+\cdots+f_{n-1}e_{n-1})+f_n(\alpha)e_n$$

and

$$G_1(\alpha E_{nn})=g_1(\alpha)e_1+\varphi(\alpha)(g_1e_2+\cdots+g_{n-1}e_{n-1}+e_n).$$

Therefore, for every $A=[a_{ij}]\in\mathcal{T}_n(\mathbb{F})$, we have

$$\begin{aligned}\psi(A)=&e_1\otimes(\varphi(a_{11})(e_1+f_2e_2+\cdots+f_{n-1}e_{n-1})+f_n(a_{11})e_n)+\\ &(A_\varphi-\varphi(a_{11})e_1\otimes e_1-\varphi(a_{nn})e_n\otimes e_n)\\ &+(g_1(a_{nn})e_1+\varphi(a_{nn})(g_1e_2+\cdots+g_{n-1}e_{n-1}+e_n))\otimes e_n\\ =&e_1\otimes(\varphi(a_{11})(f_2e_2+\cdots+f_{n-1}e_{n-1})+f_n(a_{11})e_n)+A_\varphi+\\ &(g_1(a_{nn})e_1+\varphi(a_{nn})(g_1e_2+\cdots+g_{n-1}e_{n-1}))\otimes e_n.\end{aligned}$$

Applying the fact $\psi(\alpha I)=\varphi(\alpha)I$, we get

$$f_2=\cdots=f_{n-1}=g_2=\cdots=g_{n-1}=0 \;\text{ and }\; f_n(\alpha)+g_1(\alpha)=0$$

for all $\alpha\in\mathbb{F}$. Let $f=f_n=-g_1$. It follows that

$$\psi(A)=A_\varphi+f(a_{11}-a_{nn})e_1\otimes e_n,$$

this completes the proof of Claim 3 and thus the proof of Theorem 1. □

A closer look at the proof of Theorem 1 reveals the following corollary.

Corollary 6 *Let $\mathbb{F}$ be a field, $n \geq 3$, $\phi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ be an additive injection. If ϕ satisfies $A \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_{n-1} \Rightarrow \mathrm{mr}(\phi(A)) = \mathrm{mr}(A)$, then ϕ has the same form as that in Theorem 1.*

It is well known that every nonzero homomorphism on $\mathbb{R}$ is the identity. Thus the following corollary is immediate.

Corollary 7 *Let $n \geq 3$, $\phi : \mathcal{T}_n(\mathbb{R}) \to \mathcal{T}_n(\mathbb{R})$ be an additive injection satisfying $\mathrm{mr}(\phi(A)) = \mathrm{mr}(A)$ for any $A \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_{n-1}$. Then there exist a nonsingular matrix $T \in \mathcal{T}_n(\mathbb{R})$, a nonzero real number α, an additive function $f : \mathbb{R} \to \mathbb{R}$ and an additive function $h : \mathcal{T}_n(\mathbb{R}) \to \mathbb{R}$ such that either*

$$\phi(A) = \alpha TAT^{-1} + f(a_{11} - a_{nn})E_{1n} + h(A)I \quad \text{for all} \quad A \in \mathcal{T}_n(\mathbb{R})$$

or

$$\phi(A) = \alpha TA^f T^{-1} + f(a_{11} - a_{nn})E_{1n} + h(A)I \quad \text{for all} \quad A \in \mathcal{T}_n(\mathbb{R}).$$

4. Related results and unsolved problem

It is easy to check that the summand $f(a_{11} - a_{nn})E_{1n}$ does not occur and the functional h is linear provided that ϕ is linear in Theorem 1. In fact, we have little more.

Theorem 8 *Let $\mathbb{F}$ be a field of characteristic 0, τ be a nonzero homomorphism of $\mathbb{F}$, $n \geq 3$, and $\phi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ be an injective τ-quasilinear mapping. Then ϕ preserves minimal rank if and only if there exists an invertible matrix $T \in \mathcal{T}_n(\mathbb{F})$, a nonzero scalar $\alpha \in \mathbb{F}$, a τ-quasilinear mapping $h : \mathcal{T}_n(\mathbb{F}) \to \mathbb{F}$ such that either*

$$\phi(A) = \alpha TA_\tau T^{-1} + h(A)I \quad \text{for all} \quad A = [a_{ij}] \in \mathcal{T}_n(\mathbb{F})$$

or

$$\phi(A) = \alpha TA_\tau^f T^{-1} + h(A)I \quad \text{for all} \quad A = [a_{ij}] \in \mathcal{T}_n(\mathbb{F}).$$

Proof If ϕ is injective τ-quasilinear and preserves minimal rank, then ϕ takes the form stated in Theorem 1 with $\varphi = \tau$ and f, h being τ-quasilinear. Thus there is a scalar $c \in \mathbb{F}$ such that $f(\lambda) = c\tau(\lambda)$. It follows that either

$$\phi(A) = \alpha TA_\tau T^{-1} + c(\tau(a_{11}) - \tau(a_{nn}))E_{1n} + h(A)I$$

or

$$\phi(A) = \alpha TA_\tau^f T^{-1} + c(\tau(a_{11}) - \tau(a_{nn}))E_{1n} + h(A)I.$$

But $(c(\tau(a_{11}) - \tau(a_{nn}))E_{1n})^f = c(\tau(a_{11}) - \tau(a_{nn}))E_{1n}$ and

$$A_\tau + c(\tau(a_{11}) - \tau(a_{nn}))E_{1n} = SA_\tau S^{-1}$$

for all $A \in \mathcal{T}_n(\mathbb{F})$ with $S = I - cE_{1n}$. Hence ϕ has the form as desired. □

We also have a direct proof of Theorem 8 without using Theorem 1. In fact, by a similar argument as that in the proof of Theorem 1, we get, for some τ-quasilinear functional h, $\psi(A) = \phi(A) - h(A)I$ preserves rank-one matrices as well as the minimal rank of matrices. Then applying the following Lemma 9 completes the proof immediately.

The lemma below may also be regarded as a generalization of [10, Theorem 3.1]. Let $\mathcal{M}_{mn}(\mathbb{F})$ be the vector space of all $m \times n$ matrices over $\mathbb{F}$.

Lemma 9 *Let $\mathcal{L}$ be a subspace of $\mathcal{M}_{mn}(\mathbb{F})$ with $\mathbb{F}$ being an arbitrary field and $\tau : \mathbb{F} \to \mathbb{F}$ being a nonzero homomorphism. Assume that $\mathcal{L}$ satisfies the following conditions:*

(i) $\mathcal{L}$ contains $x_0 \otimes \mathbb{F}^n$ for some $x_0 \in \mathbb{F}^m$;

(ii) $\mathcal{L}$ contains $\mathbb{F}^m \otimes y_0$ for some $y_0 \in \mathbb{F}^n$;

(iii) $\mathcal{L}$ is spanned by its rank-one matrices.

Let $\psi : \mathcal{L} \to \mathcal{M}_{kl}(\mathbb{F})$ be a τ-quasilinear mapping preserving rank-one matrices. Then either

(a) $m \leq k$, $n \leq l$, and there exist a $k \times m$ matrix T of rank m and an $n \times l$ matrix S of rank n such that

$$\psi(A) = TA_\tau S \quad \textit{for every} \quad A \in \mathcal{L};$$

or

(b) $m \leq l$, $n \leq k$, and there exist a $k \times n$ matrix T of rank n and an $m \times l$ matrix S of rank m such that

$$\psi(A) = TA_\tau^{\mathrm{T}} S \quad \textit{for every} \quad A \in \mathcal{L};$$

or

(c) $\psi(\mathcal{L})$ is contained in a subspace consisting of some rank-one matrices.

Proof The range of ψ on $x_0 \otimes \mathbb{F}^n$ is a $\tau(\mathbb{F})$-vector space of rank-one matrices. So $\psi(x_0 \otimes \mathbb{F}^n) = u_0 \otimes W$ or $V \otimes v_0$ for some $\tau(\mathbb{F})$-subspace W of $\mathbb{F}^l$ and some vector $u_0 \in \mathbb{F}^k$ or for some $\tau(\mathbb{F})$-subspace V of $\mathbb{F}^k$ and some vector $v_0 \in \mathbb{F}^l$. Replacing ψ by the mapping $\psi_1(A) = \psi(A)^{\mathrm{T}}$ if necessary, we may assume without loss of generality that $\psi(x_0 \otimes \mathbb{F}^n) = u_0 \otimes W$. Because the kernel of ψ contains no matrices of rank one, we see that $\dim W = n$. Consequently, $l \geq n$ and $\psi(x_0 \otimes y) = u_0 \otimes g(y)$ for some injective τ-quasilinear transformation $g : \mathbb{F}^n \to \mathbb{F}^l$, i.e., $\psi(x_0 \otimes y) = u_0 \otimes S^{\mathrm{T}} y_\tau$ for an $n \times l$ matrix S of rank n.

Similarly, $\psi(\mathbb{F}^m \otimes y_0)$ is a $\tau(\mathbb{F})$-vector space of rank-one matrices and hence takes one of the two forms $\psi(\mathbb{F}^m \otimes y_0) \subseteq u_1 \otimes \mathbb{F}^l$ and $\psi(\mathbb{F}^m \otimes y_0) \subseteq \mathbb{F}^k \otimes v_1$. We consider these two cases, respectively.

Case 1 $\psi(\mathbb{F}^m \otimes y_0) \subseteq u_1 \otimes \mathbb{F}^l$. In this case, there exists an injective τ-quasilinear transformation $h : \mathbb{F}^m \to \mathbb{F}^l$ such that $\psi(x \otimes y_0) = u_1 \otimes h(x)$. As $\psi(x_0 \otimes y_0) = u_0 \otimes w = u_1 \otimes v$ for some nonzero vectors w and v, then u_0 and u_1 are linearly dependent. Hence we may assume that $u_1 = u_0$. Assume that there exist nonzero vectors x, y, u, v such that $\psi(x \otimes y) = u \otimes v$ and u is linearly independent of u_0. Let $A_1 = x \otimes y$, $A_2 = (x + x_0) \otimes y$, $A_3 = x \otimes (y + y_0)$ and $A_4 = (x + x_0) \otimes (y + y_0)$, and let $B_j = \psi(A_j)$, $1 \leq j \leq 4$. Then, as ψ is additive, $B_1 = u \otimes v$, $B_2 = u \otimes v + u_0 \otimes g(y)$, $B_3 = u \otimes v + u_0 \otimes h(x)$ and $B_4 = u \otimes v + u_0 \otimes (h(x) + g(y) + g(y_0))$. Since u and u_0 are linearly independent and B_j is of rank one, $1 \leq j \leq 4$, we conclude that $v = \alpha g(y) = \beta h(x) = \gamma g(y_0)$ for some nonzero scalars α, β, γ. Particularly, we get $\alpha S^{\mathrm{T}} y_\tau = \gamma S^{\mathrm{T}}(y_0)_\tau$. As S^{T} is injective as a transformation, it follows that $\alpha y_\tau = \gamma (y_0)_\tau$ and hence y is linearly dependent on y_0 as τ is injective. However, this implies that $u \otimes v = \psi(x \otimes y) = \psi(\delta x \otimes y_0) = \tau(\delta) u_0 \otimes h(x)$, contradicting the assumption that u and u_0 are linearly independent. Hence we must have $\psi(x \otimes y) \in u_0 \otimes \mathbb{F}^l$ holds for every rank-one matrix $x \otimes y \in \mathcal{L}$. Since $\mathcal{L}$ is spanned by its rank-one elements, we see that $\psi(\mathcal{L}) \subseteq u_0 \otimes \mathbb{F}^l$ and ψ has the form (c).

Case 2 $\psi(\mathbb{F}^m \otimes y_0) \subseteq \mathbb{F}^k \otimes v_1$. As before, we have that $\psi(x \otimes y_0) = Tx_\tau \otimes v_0$, for a $k \times m$ matrix T of rank m, i.e., an injective linear transformation T from $\mathbb{F}^m$ into $\mathbb{F}^k$. Note that $u_0 \otimes S^{\mathrm{T}}(y_0)_\tau = \psi(x_0 \otimes y_0) = T(x_0)_\tau \otimes v_0$. After absorbing a constant in u_0 and v_0 if necessary, we may assume that $T(x_0)_\tau = u_0$ and $S^{\mathrm{T}}(y_0)_\tau = v_0$. Now consider an arbitrary rank-one matrix $x \otimes y \in \mathcal{L}$ and let $\psi(x \otimes y) = u \otimes v$. Let A_j and $B_j = \psi(A_j)$ be rank-one matrices as in Case 1, $j = 1, 2, 3, 4$. Then $B_1 = u \otimes v$, $B_2 = u \otimes v + u_0 \otimes S^{\mathrm{T}} y_\tau$, $B_3 = u \otimes v + Tx_\tau \otimes v_0$ and $B_4 = u \otimes v + u_0 \otimes S^{\mathrm{T}} y_\tau + Tx_\tau \otimes v_0 + u_0 \otimes v_0$. If u_0, Tx_τ are linearly independent and $v_0, S^{\mathrm{T}} y_\tau$ are linearly independent, then it is easily checked that $\psi(x \otimes y) = Tx_\tau \otimes S^{\mathrm{T}} y_\tau$ (also, [10, Lemma 3.1]). If $Tx_\tau = cu_0 = cT(x_0)_\tau$ for a scalar c, then $x = \alpha x_0$ for some scalar α with $\tau(\alpha) = c$. In this case we also have $\psi(x \otimes y) = \psi(\alpha x_0 \otimes y) = cu_0 \otimes S^{\mathrm{T}} y_\tau = Tx_\tau \otimes S^{\mathrm{T}} y_\tau$. A similar argument proves the same conclusion when $S^{\mathrm{T}} y_\tau$ and v_0 are linearly dependent. Therefore, $\psi(x \otimes y) = T(x \otimes y)_\tau S$ for every $x \otimes y \in \mathcal{L}$. By the assumption (iii) we conclude that $\psi(A) = TA_\tau S$ for every $A \in \mathcal{L}$. □

The situation for $n = 2$ is quite different. We give an example which shows that a minimal rank preserving additive mapping $\phi : \mathcal{T}_2(\mathbb{F}) \to \mathcal{T}_2(\mathbb{F})$ may have the form not as that stated in Theorem 1.

Example 10 Let $\mathbb{F} = \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be an injective additive mapping and be not of a scalar multiple of any homomorphism of $\mathbb{C}$ (i.e., there exist no constant $a \in \mathbb{C}$ and no homomorphism τ of $\mathbb{C}$ so that $g = a\tau$). Define $\phi : \mathcal{T}_2(\mathbb{C}) \to \mathcal{T}_2(\mathbb{C})$ by

$$\phi\left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & g(a_{12}) \\ 0 & a_{22} \end{bmatrix}.$$

It is clear that ϕ is additive, injective and ϕ preserves the minimal rank. However, ϕ is not of the form stated in Theorem 1. Actually, if ϕ is of the form as in Theorem 1, we may assume, without loss of generality, that $\phi(A) = \alpha T A_\varphi T^{-1} + f(a_{11} - a_{22})E_{12} + h(A)I$ for all $A = [a_{ij}] \in \mathcal{T}_2(\mathbb{C})$ (Here, α, T, φ, f and h are defined as in Theorem 1). Write

$$T = \begin{bmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{bmatrix}.$$

A simple computation shows that $g(a_{12}) = \frac{\alpha t_{11}}{t_{22}} \varphi(a_{12})$ for every $a_{12} \in \mathbb{C}$. As $t = \frac{\alpha t_{11}}{t_{22}}$ is a constant, we see that $g = t\varphi$ is a scalar multiple of the homomorphism φ, a contradiction.

There exist numerous injective additive mappings $g : \mathbb{C} \to \mathbb{C}$ that are not of the form $t\varphi$, i.e., a scalar multiple of a homomorphism of $\mathbb{C}$. To see this, we regard $\mathbb{C}$ as an infinite dimensional linear space over the rational number field $\mathbb{Q}$. Take a Hamel basis $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of $\mathbb{C}$. Then for any $z\in\mathbb{C}$, $z=\sum_{i=1}^n \xi_i\alpha_i$ for some $\xi_i\in\mathbb{Q}$ and $\alpha_i\in\{\alpha_\lambda\}_{\lambda\in\Lambda}$. Let $\Omega_1=\{\omega_\lambda:\lambda\in\Lambda\}$ and $\Omega_2=\{\omega'_\lambda:\lambda\in\Lambda\}$ be arbitrary two Hamel bases of the $\mathbb{Q}$-linear space $\mathbb{C}$ and $g:\mathbb{C}\to\mathbb{C}$ be a $\mathbb{Q}$-linear transformation defined by

$$g(\sum_{i=1}^n \xi_i\omega_i)=\sum_{i=1}^n \xi_i\omega'_i,\ z=\sum_{i=1}^n \xi_i\omega_i\in\mathbb{C}\ (\xi_i\in\mathbb{Q}).$$

Then g is an additive injective mapping on the field $\mathbb{C}$ but in general is not a scalar multiple of a homomorphism. For instance, let $\omega_1=e$ (here, as usual, $e=\exp(1)=\sum_{n=0}^\infty\frac{1}{n!}$), $\omega_2=e^2$, $\omega_3=e^3$, $\omega_4=\pi$, $\omega_5=\pi^2$, $\omega_6=\pi^3$, $\omega'_1=\omega_1$, $\omega'_2=\omega_2$, $\omega'_3=\omega_6$, $\omega'_4=\omega_4$, $\omega'_5=\omega_5$, $\omega'_6=\omega_3$. Take two Hamel bases Ω_1 and Ω_2 such that $\omega_i\in\Omega_1$ and $\omega'_i\in\Omega_2$. Let g be any bijective mapping from Ω_1 onto Ω_2 satisfying $g(\omega_i)=\omega'_i$, $i=1,2,\dots,6$. g determines a bijective additive mapping from $\mathbb{C}$ onto itself. Then $g(\omega_1\omega_2)=g(\omega_3)=\pi^3$ and $g(\omega_4\omega_5)=g(\omega_6)=e^3$. If $g=a\tau$ for some nonzero scalar a and homomorphism τ, it follows that $g(\omega_1\omega_2)=a\tau(\omega_1\omega_2)=a\tau(\omega_1)\tau(\omega_2)=\frac{1}{a}a\tau(\omega_1)a\tau(\omega_2)=\frac{1}{a}g(\omega_1)g(\omega_2)=\frac{1}{a}\omega_1\omega_2=\frac{e^3}{a}$, and similarly, $g(\omega_4\omega_5)=\frac{1}{a}\omega_4\omega_5=\frac{\pi^3}{a}$. This leads to $\pi^3=\frac{e^3}{a}$ and $e^3=\frac{\pi^3}{a}$, a contradiction.

In fact, for $n=2$, we have

Theorem 11 *Let $\mathbb{F}$ be a field of characteristic 0 and $\phi:\mathcal{T}_2(\mathbb{F})\to\mathcal{T}_2(\mathbb{F})$ be an additive bijective mapping. Then ϕ preserves the minimal rank if and only if $\phi(\mathbb{F}I)=\mathbb{F}I$.*

Proof Assume that $\phi(\mathbb{F}I)=\mathbb{F}I$. If $A\in\mathcal{T}_2(\mathbb{F})$ and $\mathrm{mr}(A)=0$, then there exists a $\lambda\in\mathbb{F}$ such that $A=\lambda I$. Thus $\phi(A)=\phi(\lambda I)=\delta I$ for some $\delta\in\mathbb{F}$, which implies that $\mathrm{mr}(\phi(A))=0$. For any $A\in\mathcal{T}_2(\mathbb{F})$ with $\mathrm{mr}(A)\neq 0$, we have $\mathrm{mr}(A)=1$. If $\mathrm{mr}(\phi(A))=0$, then $\phi(A)=\delta I$ for some scalar δ. As $\phi(\mathbb{F}I)=\mathbb{F}I$, there exists a λ such that $\phi(\lambda I)=\delta I=\phi(A)$, which contradicts the injectivity of ϕ. So we must have $\mathrm{mr}(\phi(A))=1$ and hence ϕ preserves the minimal rank. Conversely, if ϕ preserves the minimal rank, then ϕ preserves the minimal rank zero, which implies that $\phi(\mathbb{F}I)\subseteq\mathbb{F}I$. If the equality does not hold, then there is a δ such that $\delta I\notin\phi(\mathbb{F}I)$. As ϕ is surjective, $\delta I=\phi(A)$ for some $A\notin\mathbb{F}I$, this leads to a contradiction that $1=\mathrm{mr}(A)=\mathrm{mr}(\phi(A))=0$. So, $\phi(\mathbb{F}I)=\mathbb{F}I$. □

Remark 12 Let $\mathbb{F}$ be a field of characteristic 0 and $n\geq 2$. It is obvious that there is no additive mapping $\phi:\mathcal{T}_n(\mathbb{F})\to\mathcal{T}_n(\mathbb{F})$ satisfying $\mathrm{mr}(A)=\mathrm{rank}(\phi(A))$ for all $A\in\mathcal{T}_n(\mathbb{F})$. Indeed, if there exists an additive mapping ϕ satisfying $\mathrm{mr}(A)=\mathrm{rank}(\phi(A))$ for all $A\in\mathcal{T}_n(\mathbb{F})$, then for any rank-one matrix A we have $\phi(A)\neq 0$, or else, $0=\mathrm{rank}(\phi(A))=\mathrm{mr}(A)$, which leads to a contradiction that $A=\alpha I$ for some scalar α. Thus $\mathrm{rank}(\phi(A))\geq 1$. On the other hand, $\mathrm{rank}(\phi(A))=\mathrm{mr}(A)\leq\mathrm{rank}(A)=1$. So ϕ preserves rank-one matrices. By Lemma 6 we know that $0=\mathrm{mr}(I)=\mathrm{rank}(\phi(I))\neq 0$, a contradiction. It is also obvious that there is no additive mapping $\phi:\mathcal{T}_n(\mathbb{F})\to\mathcal{T}_n(\mathbb{F})$ satisfying $\mathrm{rank}(A)=\mathrm{mr}(\phi(A))$ for all $A\in\mathcal{T}_n(\mathbb{F})$. In fact, if such ϕ exists, then we get $n=\mathrm{rank}(I)=\mathrm{mr}(\phi(I))\leq n-1$, a contradiction.

Before drawing conclusions, we raise a question that is still open. Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. The following problem appeared in [1]: How to characterize the linear mappings from $\mathcal{M}_n(\mathbb{F})$ into itself preserving the relationship of having same minimal rank (i.e., $\mathrm{mr}(A) = \mathrm{mr}(B) \Rightarrow \mathrm{mr}(\phi(A)) = \mathrm{mr}(\phi(B))$)? This problem is equivalent to the problem of characterizing the linear mappings ψ satisfying that, for each $0 \le i \le n-1$, there is a $0 \le j \le n-1$ such that $\phi(\Gamma_i) \subset \Gamma_j$. Concerning upper triangular matrices, the following problem is also natural and interesting.

Problem 13 Let $\mathbb{F}$ be a field of characteristic 0 and $n \ge 2$. How to characterize the additive (or linear) mappings $\phi : \mathcal{T}_n(\mathbb{F}) \to \mathcal{T}_n(\mathbb{F})$ which satisfy that, for any $A, B \in \mathcal{T}_n(\mathbb{F})$, $\mathrm{mr}(\phi(A)) = \mathrm{mr}(\phi(B))$ whenever $\mathrm{mr}(A) = \mathrm{mr}(B)$?

Acknowledgement The authors would like to thank the referee's helpful comments which improved this paper.

References

[1] SO W. *Linear operators preserving the minimal rank* [J]. Linear Algebra Appl., 1999, **302/303**: 461–468.

[2] YANG Qiuwei, LIU Jike. *Structural damage detection with minimal update method* [J]. Journal of Vbtation and Shock (Shanghai), 2008, **27**(4): 7–9.

[3] ZHU Zi, CAO Shen. *Damage identification of truss structures using a minimum rank method* [J]. Journal of Hebei Institute of Architectural Science and Technology, 2006, **23**(3): 7–10.

[4] ZHU Zi, DONG Cong. *Structural damage identification based on minimal rank method* [J]. Journal of Architecture and Civil Engineering (Xi'an), 2006, **23**(3): 37–40.

[5] ZHANG Guoshan, ZHANG Qingling, ZHAO Zhiwu. *The minimum rank of the matrix pencil $A+BKC$ and its applications* [J]. Control and Decision (Shenyang), 1998, **13**(S): 508–512.

[6] ZHANG Xiuling, CUI Hongyan, HOU Jinchuan. *Jordan semi-triple mappings preserving rank and minimal rank* [C]. Proceedings of The Third International Workshop on Matrix Analysis, 2009, **3**: 273–276.

[7] PEASE M C. *Methods of Matrix Algebra* [M]. New York: Academic Press, 1965, 211–212.

[8] DE OLIVEIRA G N, MARQUES DE SÀ E, DIAS DA SILVA J A. *On the eigenvalues of the matrix $A + XBX^{-1}$* [J]. Linear and Multilinear Algebra, 1977/78, **2**: 119–128.

[9] ZHANG Zhixu, WU Haiyan, ZHANG Ling. *The minimal rank of matrices and it's applications* [J]. Journal of Jiamusi University(Natural Science Edition), 2006, **24**(1): 122–124.

[10] BELL J, SOUROUR A R. *Additive rank-one preserving mappings on triangular matrix algebras* [J]. Linear Algebra Appl., 2000, **312**(1-3): 13–33.

[11] CHOOI W L, LIM M H. *Additive rank-one preservers on block triangular matrix spaces* [J]. Linear Algebra Appl., 2006, **416**(2-3): 588–607.

[12] CUI Jianlian, HOU Jinchuan, LI Bingren. *Linear preservers on upper triangular operator matrix algebras* [J]. Linear Algebra Appl., 2001, **336**: 29–50.