Journal of Mathematical Research & Exposition
Nov., 2011, Vol. 31, No. 6, pp. 965–976
DOI:10.3770/j.issn:1000-341X.2011.06.002
Http://jmre.dlut.edu.cn

Biderivations of the Algebra of Strictly Upper Triangular Matrices over a Commutative Ring

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Abstract Let $N_n(R)$ be the algebra consisting of all strictly upper triangular $n \times n$ matrices over a commutative ring R with the identity. An R-bilinear map $\phi : N_n(R) \times N_n(R) \longrightarrow N_n(R)$ is called a biderivation if it is a derivation with respect to both arguments. In this paper, we define the notions of central biderivation and extremal biderivation of $N_n(R)$, and prove that any biderivation of $N_n(R)$ can be decomposed as a sum of an inner biderivation, central biderivation and extremal biderivation for $n \geq 5$.

Keywords biderivation; strictly upper triangular matrix; algebra.

Document code A
MR(2010) Subject Classification 17B30; 17B40
Chinese Library Classification O177.1

1. Introduction

Let R be a commutative ring with the identity. Denote by $N_n(R)$, where n is a positive integer greater than 1, the R-algebra of strictly upper triangular $n \times n$ matrices over R. In recent years, many significant researches have been done in automorphisms, Lie automorphisms, derivations and Lie derivations of $N_n(R)$. Cao [4] studied the R-algebra automorphisms of $N_n(R)$. Cao [5, 6] investigated the Lie automorphisms group of $N_n(R)$. Ou et al. [8] and Ji [7] determined the Lie derivations of $N_n(R)$. Wang and Li [9] studied the structure of all triple derivations of $N_n(R)$. Someone described the derivations of $N_n(R)$.

Let T be an R-algebra and M a T-bimodule. Recall that an R-linear map $d : T \longrightarrow M$ is called a derivation if $d(ab) = d(a)b + ad(b)$ for all a, b in T. For example, take $x \in M$, and set $D_x(a) = ax - xa$ for all a in T. In the following, we denote by $[x, y]$ the commutator (the Lie product) of the elements $x, y \in T$. A bilinear map $\phi : T \times T \longrightarrow T$ is called a biderivation if it is a derivation with respect to both components, meaning that,

$$\phi(xy, z) = x\phi(y, z) + \phi(x, z)y \quad \text{and} \quad \phi(x, yz) = \phi(x, y)z + y\phi(x, z)$$

for all $x, y, z \in T$. If T is a noncommutative algebra, then the map $\phi(x, y) = \lambda[x, y]$ for all $x, y \in T$, where $\lambda \in Z(T)$, the center of T, is a basic example of biderivation, and we call it an inner biderivation.

Received April 8, 2010; Accepted May 28, 2010
Supported by the National Natural Science Foundation of China (Grant No. 10971117).
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Biderivations are a subject of research in various areas. Bresar et al. [3] proved that all biderivations on noncommutative prime rings are inner. Zhang et al. [10] showed that biderivations of nest algebras are usally inner. There are special cases of nest algebras with non-inner biderivations. Zhao et al. used the results in [10] to prove that every biderivation of an upper triangular matrix algebra is a sum of an inner biderivation and a special biderivation. Benkovic [1] generalized this result to triangular algebras. He proved that every biderivation of some certain triangular algebras is a sum of an inner biderivation and an extremal biderivation. The aim of this paper is to describe the biderivations of $N_n(R)$ for $n \geq 5$. In the cases $2 \leq n \leq 4$, we have already given the structure of the biderivations of $N_n(R)$.

This paper is organized as follows. In Section 2, we construct certain special biderivation of $N_n(R)$ so as to build every biderivation of $N_n(R)$. In Section 3, we decompose any biderivation of $N_n(R)$ into a sum of those known ones.

Now we introduce some preliminary notations and results about matrix algebras.

Let $e_{i,j}$ be the standard matrix units for $1 \leq i, j \leq n$. It is well known that the matrix set $\{e_{i,j} : 1 \leq i < j \leq n\}$ forms a basis of $N_n(R)$ and any x in $N_n(R)$ can be uniquely written as $x = \sum_{1 \leq i < j \leq n} x_{i,j} e_{i,j}$ with $x_{i,j} \in R$. Set

$$M_r = \{ \sum_{j-i \geq r} x_{i,j} e_{i,j} \in N_n(R) : x_{i,j} \in R\}, \quad r = 1, 2, \ldots, n-1,$$

and $M_r = 0$ for $r \geq n$. It is clear that each M_r is an ideal of $N_n(R)$ and $x_1 x_2 \cdots x_r \in M_r$ for any $x_1, x_2, \ldots, x_r \in N_n(R)$. It follows that $M_r M_s \subseteq M_{r+s}$. It is not difficult to know that the center of the R-algebra $N_n(R)$ is $M_{n-1} = Re_{1,n}$.

To prove the main result of this paper, we need the following result, which might not be written down explicitly to our knowledge.

Lemma 1.1 *Let R be an arbitrary commutative ring with the identity and D a derivation of $N_n(R)$ for $n \geq 4$. Then there is an $n \times n$ diagonal matrix d, an element $y \in N_n(R)$ and $c = (c_2, \ldots, c_{n-2}) \in R^{n-3}$ such that $D(x) = [d, x] + [y, x] + \sum_{i=2}^{n-2} x_{i,i+1} c_i e_{1,n}$ for all $x = \sum_{1 \leq i < j \leq n} x_{i,j} e_{i,j} \in N_n(R)$.*

2. Certain standard biderivations of $N_n(R)$

In this section, we will give three types of standard derivations of $N_n(R)$, which will be used to describe arbitrary biderivations of $N_n(R)$.

(A) Inner biderivations: Let $d \in R$. Then the map $\phi : N_n(R) \times N_n(R) \longrightarrow N_n(R)$, $(x, y) \longmapsto d[x, y]$, is a biderivation of $N_n(R)$. We call it the inner biderivation of $N_n(R)$ induced by d.

Note that the inner biderivation defined here is different from the one defined in the papers [1–3, 10, 11]. Since the center of $N_n(R)$ is $Re_{1,n}$, and for every $x, y \in N_n(R)$, we have that $e_{1,n}[x, y] = 0$. Hence the inner biderivation of $N_n(R)$ defined in [1–3, 10, 11] is the zero map.

(B) Central biderivations: Let A be an $(n-1) \times (n-1)$ matrix over R. We define an R-bilinear map $\phi_A : N_n(R) \times N_n(R) \longrightarrow N_n(R)$ by $\phi_A(y, x) = \sum_{i,j=1}^{n-1} y_{i,i+1} a_{i,j} x_{j,j+1} e_{1,n}$ for all

$x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j}$, $y = \sum_{1\le i<j\le n} y_{i,j}e_{i,j} \in N_n(R)$. Then it is easy to check that ϕ_A is a biderivation of $N_n(R)$. We call it the central biderivation of $N_n(R)$ induced by A.

(C) Extremal biderivations: Let $a \in R$. The R-bilinear map $\phi_a : N_n(R) \times N_n(R) \longrightarrow N_n(R)$, which is defined by $\phi_a(y,x) = a(y_{2,3}x_{1,3}e_{1,n} + y_{2,3}x_{2,3}e_{2,n} + y_{1,3}x_{2,3}e_{1,n})$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j}$, $y = \sum_{1\le i<j\le n} y_{i,j}e_{i,j} \in N_n(R)$, is a biderivation on $N_n(R)$.

In fact, fix $y = \sum_{1\le i<j\le n} y_{i,j}e_{i,j} \in N_n(R)$, for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j}$, $x' = \sum_{1\le i<j\le n} x'_{i,j}e_{i,j}$, since $(xx')_{2,3} = 0$ and $(xx')_{1,3} = x_{1,2}x'_{2,3}$, we have that $\phi_a(y, xx') = ay_{2,3}x_{1,2}x'_{2,3}e_{1,n}$. Computing directly, we get that

$$\begin{aligned}\phi_a(y,x)x' + x\phi_a(y,x') =& a(y_{2,3}x_{1,3}e_{1,n} + y_{2,3}x_{2,3}e_{2,n} + y_{1,3}x_{2,3}e_{1,n})x' + \\ & ax(y_{2,3}x'_{1,3}e_{1,n} + y_{2,3}x'_{2,3}e_{2,n} + y_{1,3}x'_{2,3}e_{1,n}) \\ =& ax_{1,2}y_{2,3}x'_{2,3}e_{1,n}.\end{aligned}$$

Thus $\phi_a(y, xx') = \phi_a(y,x)x' + x\phi_a(y,x')$. This means that ϕ_a is a biderivation with respect to the second component. Since ϕ_a is a symmetric bilinear map, we have that ϕ_a is a derivation with respect to the first component. Therefore ϕ_a is a biderivation of $N_n(R)$. We call it the extremal biderivation of $N_n(R)$ induced by $a \in R$.

3. Biderivation of $N_n(R)$

In this section, we give the main result of this paper.

Theorem 3.1 *Let R be an arbitrary commutative ring with the identity and ϕ a biderivation of $N_n(R)$. If $n \ge 5$, then there are elements $a,\ d \in R$ and an $(n-1)\times(n-1)$ matrix A over R such that*

$$\phi(y,x) = d[y,x] + \phi_A(y,x) + \phi_a(y,x)$$

for all $x, y \in N_n(R)$.

In order to prove our main result, we need the following lemmas.

Lemma 3.2 ([2, Corollary 2.4]) *Let T be an algebra on a commutative ring R and $\phi : T\times T \longrightarrow T$ a biderivation. Then*

$$\phi(x,y)[u,v] = [x,y]\phi(u,v)$$

for all $x, y, u, v \in A$.

In what follows, for $1 \le i < j \le n$, since $\phi(e_{i,j}, x) : N_n(R) \longrightarrow N_n(R)$ is a derivation of $N_n(R)$, by Lemma 1.1, we suppose that

$$\phi(e_{i,j}, x) = [d^{i,j}, x] + [y^{i,j}, x] + \sum_{i=2}^{n-2} c_i^{i,j} x_{i,i+1}e_{1,n}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$, where $d^{i,j} = \mathrm{diag}(d_1^{i,j}, \dots, d_n^{i,j})$ is a diagonal matrix over R, $y^{i,j} = \sum_{1\le t<s\le n} y_{t,s}^{i,j}e_{i,j} \in N_n(R)$, and $(c_2^{i,j}, \dots, c_{n-2}^{i,j}) \in R^{n-3}$.

Lemma 3.3 *For any (i,j) $(1 \le i < j \le n)$, there is $d_{i,j} \in R$ such that $d^{i,j} = \mathrm{diag}(d_{i,j}, d_{i,j}, \dots, d_{i,j})$.*

Hence $[d^{i,j}, x] = 0$ *for all* $x \in N_n(R)$.

Proof

Step 1. First we will prove that if there is $e_{t,s}$ with $s - t < n - 2$ such that $[e_{i,j}, e_{t,s}] = 0$, then $d_t^{i,j} = d_s^{i,j}$.

Since $[e_{i,j}, e_{t,s}] = 0$, by Lemma 3.2, we get that

$$\phi(e_{i,j}, e_{t,s})[u, v] = [e_{i,j}, e_{t,s}]\phi(u, v) = 0 \text{ for all } u, v \in N_n(R).$$

Hence

$$\begin{aligned}\phi(e_{i,j}, e_{t,s}) =& [d^{i,j}, e_{t,s}] + [y^{i,j}, e_{t,s}] + c_t^{i,j}\delta_{t+1,s}e_{1,n} \\ =& (d_t^{i,j} - d_s^{i,j})e_{t,s} + [y^{i,j}, e_{t,s}] + c_t^{i,j}\delta_{t+1,s}e_{1,n} \in M_{n-2},\end{aligned}$$

where δ is the Kronecker's delta. Since $s - t < n - 2$, $e_{t,s} \notin M_{n-2}$. Since $e_{t,s} \in M_{s-t} \backslash M_{s-t+1}$ and $[y^{i,j}, e_{t,s}] + c_t^{i,j}\delta_{t+1,s}e_{1,n} \in M_{s-t+1}$, we have $d_t^{i,j} - d_s^{i,j} = 0$, i.e., $d_t^{i,j} = d_s^{i,j}$.

Henceforth fix (i, j) $(1 \le i < j \le n)$.

Step 2. (i) Clearly, $[e_{i,j}, e_{1,k}] = 0$ for each $1 < k \le n - 2$ with $k \ne i$. It follows from step 1 that $d_1^{i,j} = d_k^{i,j}$ for all $1 < k \le n - 2$ with $k \ne i$.

(ii) Since $[e_{i,j}, e_{k,n}] = 0$ for all $2 < k \le n - 1$ with $k \ne j$, by Step 1, we have that $d_k^{i,j} = d_n^{i,j}$ for all $2 < k \le n - 1$ with $k \ne j$.

(iii) Since $[e_{i,j}, e_{i,k}] = 0$ and $[e_{i,j}, e_{t,j}] = 0$, by Step 1, we get that $d_i^{i,j} = d_k^{i,j}$ for all $i < k \le \min\{n, n - 3 + i\}$ and $d_j^{i,j} = d_t^{i,j}$ for all $j > k \ge \max\{0, j - n + 2\}$. Thus $d_i^{i,j} = d_k^{i,j}$ for all $\min\{i, \max\{0, j - n + 2\}\} \le k \le \max\{j, \min\{n, n - 3 + i\}\}$.

Case 1 $i = 1$.

It follows from Step 2 (i) that $d_1^{1,j} = \cdots = d_{n-2}^{1,j}$.

If $j < n - 2$, by Step 2 (ii), we have that $d_{n-2}^{1,j} = d_{n-1}^{1,j} = d_n^{1,j}$. So $d_1^{1,j} = d_2^{1,j} = \cdots = d_n^{1,j}$.

If $j = n - 2$, then, by Step 2 (ii), $d_{n-1}^{1,n-2} = d_n^{1,n-2}$. In this case, to complete the proof, it suffices to prove that $d_2^{1,n-2} = d_{n-1}^{1,n-2}$. Since $[e_{1,n-2}, e_{2,n-1}] = 0$ and $(n - 1) - 2 < n - 2$, by Step 1, we get $d_2^{1,n-2} = d_{n-1}^{1,n-2}$. Thus $d_1^{1,n-2} = d_2^{1,n-2} = \cdots = d_n^{1,n-2}$.

If $j = n - 1$, then, by Step 2 (ii), $d_{n-2}^{1,n-1} = d_n^{1,n-1}$. Since $[e_{1,n-1}, e_{n-2,n-1}] = 0$ and $(n - 1) - (n - 2) < n - 2$, by Step 1, we get $d_{n-2}^{1,n-1} = d_{n-1}^{1,n-1}$. Thus $d_1^{1,n-1} = d_2^{1,n-1} = \cdots = d_n^{1,n-1}$.

When $j = n$, it follows from Step (ii) that $d_3^{1,n} = d_4^{1,n} = \cdots = d_n^{1,n}$. Thus $d_1^{1,n} = d_2^{1,n} = \cdots = d_n^{1,n}$.

Case 2 $i = 2$.

It follows from Step 2 (i) that $d_1^{2,j} = d_3^{2,j} = \cdots = d_{n-2}^{2,j}$, and from Step 2 (iii) that $d_2^{2,j} = d_3^{2,j} = \cdots = d_{n-1}^{2,j}$. Thus $d_1^{2,j} = d_2^{2,j} = \cdots = d_{n-1}^{2,j}$.

If $j \le n - 2$, by Step 2 (ii), we have that $d_{n-1}^{2,j} = d_n^{2,j}$. So $d_1^{2,j} = d_2^{2,j} = \cdots = d_n^{2,j}$.

If $j = n - 1$, then, by Step 2 (ii), $d_{n-2}^{1,n-1} = d_n^{1,n-1}$. Thus $d_1^{2,n-1} = d_2^{2,n-1} = \cdots = d_n^{2,n-1}$.

When $j = n$, it follows from Step (ii) that $d_3^{2,n} = d_4^{2,n} = \cdots = d_n^{2,n}$. Thus $d_1^{2,n} = d_2^{2,n} = \cdots = d_n^{2,n}$.

Case 3 $i \geq 3$.

It follows from Step 2 (i) that $d_1^{i,j} = d_2^{i,j} = \cdots = d_{i-1}^{i,j}$, and from Step 2 (iii) that $d_i^{i,j} = d_{i+1}^{i,j} = \cdots = d_n^{i,j}$. Since $[e_{i,j}, e_{i-1,i+1}] = 0$ and $(i+1)-(i-1) < n-2$, by Step 1, we get $d_{i-1}^{i,j} = d_{i+1}^{i,j}$. Thus $d_1^{i,j} = d_2^{i,j} = \cdots = d_n^{i,j}$. This completes the proof. □

Henceforth, for $1 \leq i < j \leq n$, we suppose that

$$\phi(e_{i,j}, x) = [y^{i,j}, x] + \sum_{i=2}^{n-2} c_i^{i,j} x_{i,i+1} e_{1,n}$$

for all $x = \sum_{1\leq i<j\leq n} x_{i,j} e_{i,j} \in N_n(R)$, where $y^{i,j} = \sum_{1\leq t<s\leq n} y_{t,s}^{i,j} e_{i,j} \in N_n(R)$, and $(c_2^{i,j}, \ldots, c_{n-2}^{i,j}) \in R^{n-3}$. Since $[e_{1,n}, x] = 0$ for all $x \in N_n(R)$, we suppose that $y_{1,n}^{i,j} = 0$ for all (i,j).

In the proof of the following lemmas, we often use the following fact: Suppose $c_{i,j} \in R$ for all $1 \leq i < j \leq n$. If $\sum_{1\leq i<j\leq n} c_{i,j} x_{i,j} = 0$ for all $x = \sum_{1\leq i<j\leq n} x_{i,j} e_{i,j} \in N_n(R)$, then $c_{i,j} = 0$ for all $1 \leq i < j \leq n$. And for every $z \in N_n(R)$,

$$\phi(e_{i,j}, x)z = ([y^{i,j}, x] + \sum_{i=2}^{n-2} c_i^{i,j} x_{i,i+1} e_{1,n})z = [y^{i,j}, x]z$$

and

$$z\phi(e_{i,j}, x) = z([y^{i,j}, x] + \sum_{i=2}^{n-2} c_i^{i,j} x_{i,i+1} e_{1,n}) = z[y^{i,j}, x].$$

Lemma 3.4 *There exists an element $d \in R$ such that*

(i) $y^{1,j} = de_{1,j} + y_{1,n-1}^{1,j} e_{1,n-1} + y_{2,n}^{1,j} e_{2,n}$ *for* $j < n-1$;

(ii) $y^{1,n} = y_{1,n-1}^{1,n} e_{1,n-1} + y_{2,n}^{1,n} e_{2,n}$;

(iii) $y^{1,n-1} = y_{1,n-2}^{1,n-1} e_{1,n-2} + y_{1,n-1}^{1,n-1} e_{1,n-1} + y_{2,n}^{1,n-1} e_{2,n}$;

(iv) $y^{i,j} = de_{i,j} + y_{1,n-1}^{i,j} e_{1,n-1} + y_{2,n}^{i,j} e_{2,n}$ *for* $j \neq n-1$ *and* $i > 2$;

(v) $y^{i,n-1} = y_{1,n-2}^{i,n-1} e_{1,n-2} + y_{1,n-1}^{i,n-1} e_{1,n-1} + y_{2,n}^{1,n-1} e_{2,n} + de_{i,n-1}$ *for* $i > 2$.

Proof (I) For $1 < j \leq n$, it follows from $e_{1,2}e_{1,j} = 0$ that

$$\begin{aligned} 0 &= \phi(e_{1,2}e_{1,j}, x) = e_{1,2}\phi(e_{1,j}, x) + \phi(e_{1,2}, x)e_{1,j} = e_{1,2}\phi(e_{1,j}, x) = e_{1,2}[y^{1,j}, x] \\ &= \sum_{s=4}^{n} (\sum_{2<t<s} y_{2,t}^{1,j} x_{t,s} - \sum_{2<t<s} x_{2,t} y_{t,s}^{1,j}) e_{1,s} \end{aligned}$$

for all $x = \sum_{1\leq i<j\leq n} x_{i,j} e_{i,j} \in N_n(R)$. Hence, for every $4 \leq s \leq n$, $\sum_{2<t<s} y_{2,t}^{1,j} x_{t,s} - \sum_{2<t<s} x_{2,t} y_{t,s}^{1,j} = 0$ for all $x = \sum_{1\leq i<j\leq n} x_{i,j} e_{i,j} \in N_n(R)$, which implies that $y_{t,s}^{1,j} = 0$ for $t > 2$ and $y_{2,t}^{1,j} = 0$ for $3 \leq t \leq n-1$. So $y^{1,j} = \sum_{t=2}^{n-1} y_{1,t}^{1,j} e_{1,t} + y_{2,n}^{1,j} e_{2,n}$.

(II) For $j < n-1$, since $e_{1,j}e_{n-1,n} = 0$, by (I), we have that

$$\begin{aligned} 0 &= \phi(e_{1,j}e_{n-1,n}, x) = e_{1,j}\phi(e_{n-1,n}, x) + \phi(e_{1,j}, x)e_{n-1,n} = e_{1,j}[y^{n-1,n}, x] + [y^{1,j}, x]e_{n-1,n} \\ &= \sum_{s=j+2}^{n} (\sum_{j<t<s} y_{j,t}^{n-1,n} x_{t,s} - \sum_{j<t<s} x_{j,t} y_{t,s}^{n-1,n}) e_{1,s} + \sum_{t=2}^{n-2} y_{1,t}^{1,j} x_{t,n-1} e_{1,n} \end{aligned}$$

for all $x = \sum_{1\leq i<j\leq n} x_{i,j} e_{i,j} \in N_n(R)$. Therefore $\sum_{j<t<n} y_{j,t}^{n-1,n} x_{t,n} - \sum_{j<t<s} x_{j,t} y_{t,n}^{n-1,n} + \sum_{t=2}^{n-2} y_{1,t}^{1,j} x_{t,n-1} = 0$ and $\sum_{j<t<s} y_{j,t}^{n-1,n} x_{t,s} - \sum_{j<t<s} x_{j,t} y_{t,s}^{n-1,n} = 0$ for $j+2 \leq s < n$, which

implies that $y_{1,t}^{1,j} = y_{n-1,n}^{n-1,n}$ for all $j < n-1$, and $y_{1,t}^{1,j} = 0$ for $2 \le t \le n-2$ with $t \ne j$. Suppose $d = y_{n-1,n}^{n-1,n}$. We have that $y_{1,j} = de_{1,j} + y_{1,n-1}^{1,j} e_{1,n-1} + y_{2,n}^{1,j} e_{2,n}$ for $j < n-1$. (i) is proved.

(III) Since $e_{1,n}e_{n-1,n} = 0$, by (I), we get that

$$\begin{aligned} 0 &= \phi(e_{1,n}e_{n-1,n}, x) = e_{1,n}\phi(e_{n-1,n}, x) + \phi(e_{1,n}, x)e_{n-1,n} \\ &= \phi(e_{1,n}, x)e_{n-1,n} = [y^{1,n}, x]e_{n-1,n} = \sum_{t=2}^{n-2} y_{1,t}^{1,n} x_{t,n-1} e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence we have that $y_{1,t}^{1,n} = 0$ for all $2 \le t \le n-2$. Thus $y^{1,n} = y_{1,n-1}^{1,n} e_{1,n-1} + y_{2,n}^{1,n} e_{2,n}$.

(IV) For $2 < i < n$, since $e_{1,2}e_{i,j} = 0$, by (i), we have that

$$\begin{aligned} 0 &= \phi(e_{1,2}e_{i,j}, x) = e_{1,2}\phi(e_{i,j}, x) + \phi(e_{1,2}, x)e_{i,j} = e_{1,2}[y^{i,j}, x] + [y^{1,2}, x]e_{i,j} \\ &= \sum_{s=4}^{n} \Big(\sum_{2<t<s} y_{2,t}^{i,j} x_{t,s} - \sum_{2<t<s} x_{2,t} y_{t,s}^{i,j} \Big) e_{1,s} + dx_{2,i}e_{1,j} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. These imply that, for every $4 \le s \le n$ with $s \ne j$, $\sum_{2<t<s} y_{2,t}^{i,j} x_{t,s} - \sum_{2<t<s} x_{2,t} y_{t,s}^{i,j} = 0$, and that $\sum_{2<t<j} y_{2,t}^{i,j} x_{t,j} - \sum_{2<t<j} x_{2,t} y_{t,j}^{i,j} + dx_{2,i} = 0$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $y_{i,j}^{i,j} = d$ and $y_{t,s}^{i,j} = 0$ unless $t = 1$ and $(t,s) = (i,j), (2,n)$. So $y^{i,j} = \sum_{t=2}^{n-1} y_{1,t}^{i,j} e_{1,t} + y_{2,n}^{i,j} e_{2,n} + de_{i,j}$.

(V) For any (i,j) ($i > 2$ and $j < n-1$), since $e_{i,j}e_{n-1,n} = 0$, by (IV), we have that

$$\begin{aligned} 0 &= \phi(e_{i,j}e_{n-1,n}, x) = e_{i,j}\phi(e_{n-1,n}, x) + \phi(e_{i,j}, x)e_{n-1,n} \\ &= e_{i,j}[y^{n-1,n}, x] + [y^{i,j}, x]e_{n-1,n} \\ &= -dx_{j,n-1}e_{i,n} + \sum_{t=2}^{n-2} y_{1,t}^{i,j} x_{t,n-1} e_{1,n} + dx_{j,n-1}e_{i,n} = \sum_{t=2}^{n-2} y_{1,t}^{i,j} x_{t,n-1} e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. This leads to $\sum_{t=2}^{n-2} y_{1,t}^{i,j} x_{t,n-1} = 0$, which implies that $y_{1,t}^{i,j} = 0$ for all $2 \le t \le n-2$. Thus $y^{i,j} = de_{i,j} + y_{1,n-1}^{i,j} e_{1,n-1} + y_{2,n}^{i,j} e_{2,n}$ for $j < n-1$.

(VI) For any i ($i > 2$), by (IV), it follows from $e_{i,n}e_{n-1,n} = 0$ that

$$\begin{aligned} 0 &= \phi(e_{i,n}e_{n-1,n}, x) = e_{i,n}\phi(e_{n-1,n}, x) + \phi(e_{i,n}, x)e_{n-1,n} = \phi(e_{i,n}, x)e_{n-1,n} \\ &= [\sum_{t=2}^{n-1} y_{1,t}^{i,n} e_{1,t} + y_{2,n}^{i,n} e_{2,n} + de_{i,n}, x]e_{n-1,n} = \sum_{t=2}^{n-2} y_{1,t}^{i,n} x_{t,n-1} e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence $y_{1,t}^{i,n} = 0$ for all $2 \le t \le n-2$. So $y^{i,n} = y_{1,n-1}^{i,n} e_{1,n-1} + y_{2,n}^{i,n} e_{2,n} + de_{i,n}$.

(VII) Since $e_{1,n-1}e_{n-2,n} = 0$, using (I) and (VI), we can get that

$$\begin{aligned} 0 &= \phi(e_{1,n-1}e_{n-2,n}, x) = e_{1,n-1}\phi(e_{n-2,n}, x) + \phi(e_{1,n-1}, x)e_{n-2,n} \\ &= e_{1,n-1}[y_{1,n-1}^{n-2,n} e_{1,n-1} + y_{2,n}^{n-2,n} e_{2,n} + de_{n-2,n}, x] + [\sum_{t=2}^{n-1} y_{1,t}^{1,n-1} e_{1,t} + y_{2,n}^{1,n-1} e_{2,n}, x]e_{n-2,n} \\ &= \sum_{t=2}^{n-3} y_{1,t}^{1,n-1} x_{t,n-2} e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $\sum_{t=2}^{n-3} y_{1,t}^{1,n-1}x_{t,n-2} = 0$ for all

$$x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R),$$

which implies that $y_{1,t}^{1,n-1} = 0$ for all $2 \le t \le n-3$. Thus

$$y^{1,n-1} = y_{1,n-2}^{1,n-1}e_{1,n-2} + y_{1,n-1}^{1,n-1}e_{1,n-1} + y_{2,n}^{1,n-1}e_{2,n}.$$

(VIII) For $i > 2$, since $e_{i,n-1}e_{n-2,n} = 0$, by (VI), we have that

$$\begin{aligned} 0 &= \phi(e_{i,n-1}e_{n-2,n}, x) = e_{i,n-1}\phi(e_{n-2,n}, x) + \phi(e_{i,n-1}, x)e_{n-2,n} \\ &= \sum_{t=2}^{n-3} y_{1,t}^{i,n-1}x_{t,n-2}e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $\sum_{t=2}^{n-3} y_{1,t}^{i,n-1}x_{t,n-2} = 0$ for all

$$x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R),$$

which leads to that $y_{1,t}^{1,n-1} = 0$ for all $2 \le t \le n-3$. So

$$y^{i,n-1} = y_{1,n-2}^{i,n-1}e_{1,n-2} + y_{1,n-1}^{i,n-1}e_{1,n-1} + y_{2,n}^{1,n-1}e_{2,n} + de_{i,n-1}. \quad \square$$

Lemma 3.5 *For $i = 2$, we have the following equalities:*

(i) $y^{2,j} = de_{2,j} + y_{1,n-1}^{2,j}e_{1,n-1} + y_{3,n}^{2,j}e_{3,n} + y_{2,n}^{2,j}e_{2,n}$ *for* $j < n-1$*;*

(ii) $y^{2,n-1} = y_{1,n-2}^{2,n-1}e_{1,n-2} + y_{1,n-1}^{2,n-1}e_{1,n-1} + y_{3,n}^{2,n-1}e_{3,n} + y_{2,n-1}^{2,n-1}e_{2,n-1} + y_{2,n}^{2,n-1}e_{2,n}$*;*

(iii) $y^{2,n} = y_{1,n-1}^{2,n}e_{1,n-1} + y_{2,n}^{2,n}e_{2,n} + y_{2,n-1}^{2,n}e_{2,n-1} + y_{3,n}^{2,n}e_{3,n}$.

Proof (I) For any $j > 2$, since $e_{1,3}e_{2,j} = 0$, by Lemma 3.4(i), we have that

$$\begin{aligned} 0 &= \phi(e_{1,3}e_{2,j}, x) = e_{1,3}\phi(e_{2,j}, x) + \phi(e_{1,3}, x)e_{2,j} \\ &= e_{1,3}[y^{2,j}, x] + [y^{1,3}, x]e_{2,j} \\ &= \sum_{s=5}^{n}\Big(\sum_{3<t<s} y_{3,t}^{2,j}x_{t,s} - \sum_{3<t<s} x_{3,t}y_{t,s}^{2,j}\Big)e_{1,s} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence, for every $5 \le s \le n$, $\sum_{3<t<s} y_{3,t}^{2,j}x_{t,s} - \sum_{3<t<s} x_{3,t}y_{t,s}^{2,j} = 0$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $y_{t,s}^{2,j} = 0$ for $t > 3$ and $y_{3,t}^{2,j} = 0$ for $4 \le t \le n-1$. So $y^{2,j} = \sum_{t=2}^{n-1} y_{1,t}^{2,j}e_{1,t} + \sum_{t=3}^{n} y_{2,t}^{2,j}e_{2,t} + y_{3,n}^{2,j}e_{3,n}$.

(II) For $3 \le j < n-1$, using (I) and Lemma 3.4 (iv), it follows from $e_{2,j}e_{n-1,n} = 0$ that

$$\begin{aligned} 0 &= \phi(e_{2,j}e_{n-1,n}, x) = e_{2,j}\phi(e_{n-1,n}, x) + \phi(e_{2,j}, x)e_{n-1,n} \\ &= e_{2,j}[y^{n-1,n}, x] + [y^{2,j}, x]e_{n-1,n} \\ &= -dx_{j,n-1}e_{2,n} + \sum_{t=2}^{n-2} y_{1,t}^{2,j}x_{t,n-1}e_{1,n} + \sum_{t=3}^{n-2} y_{2,t}^{2,j}x_{t,n-1}e_{2,n} - y_{2,n-1}^{2,j}x_{1,2}e_{1,n} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $\sum_{t=2}^{n-2} y_{1,t}^{2,j}x_{t,n-1} - y_{2,n-1}^{2,j}x_{1,2} = 0$ and $-dx_{j,n-1} + \sum_{t=3}^{n-2} y_{2,t}^{2,j}x_{t,n-1} = 0$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence $y_{1,t}^{2,j} = 0$ for all $2 \le t \le$

$n-2$, $d=y_{2,j}^{2,j}$, and $y_{2,t}^{2,j}=0$ for $2<t\leq n-1$ with $t\neq j$. Thus $y^{2,j}=de_{2,j}+y_{1,n-1}^{2,j}e_{1,n-1}+y_{3,n}^{2,j}e_{3,n}+y_{2,n}^{2,j}e_{2,n}$ for $j<n-1$.

(III) Since $e_{2,n-1}e_{n-2,n}=0$, by (I) and Lemma 3.4(iv), we have that

$$\begin{aligned}0=&\phi(e_{2,n-1}e_{n-2,n},x)=e_{2,n-1}\phi(e_{n-2,n},x)+\phi(e_{2,n-1},x)e_{n-2,n}\\=&e_{2,n-1}[y_{1,n-1}^{n-2,n}e_{1,n-1}+y_{2,n}^{n-2,n}e_{2,n}+de_{n-2,n},x]+\\&[\sum_{t=2}^{n-1}y_{1,t}^{2,n-1}e_{1,t}+\sum_{t=3}^{n}y_{2,t}^{2,n-1}e_{2,t}+y_{3,n}^{2,n-1}e_{3,n},x]e_{n-2,n}\\=&\sum_{t=3}^{n-3}y_{2,t}^{2,n-1}x_{t,n-2}e_{2,n}+\sum_{t=2}^{n-3}y_{1,t}^{2,n-1}x_{t,n-2}e_{1,n}-y_{2,n-2}^{2,n-1}x_{1,2}e_{1,n}\end{aligned}$$

for all $x=\sum_{1\leq i<j\leq n}x_{i,j}e_{i,j}\in N_n(R)$. Therefore $\sum_{t=2}^{n-3}y_{1,t}^{2,n-1}x_{t,n-2}-y_{2,n-2}^{2,n-1}x_{1,2}=0$ and $\sum_{t=3}^{n-3}y_{2,t}^{2,n-1}x_{t,n-2}=0$ for all $x=\sum_{1\leq i<j\leq n}x_{i,j}e_{i,j}\in N_n(R)$. It follows that $y_{1,t}^{2,n-1}=0$ for all $2\leq t\leq n-3$ and $y_{2,t}^{2,n-1}=0$ for all $3\leq t\leq n-2$. Thus $y^{2,n-1}=y_{1,n-2}^{2,n-1}e_{1,n-2}+y_{1,n-1}^{2,n-1}e_{1,n-1}+y_{3,n}^{2,n-1}e_{3,n}+y_{2,n-1}^{2,n-1}e_{2,n-1}+y_{2,n}^{2,n-1}e_{2,n}$.

(IV) Since $e_{2,n}e_{n-1,n}=0$, by (I), we have that

$$\begin{aligned}0=&\phi(e_{2,n}e_{n-1,n},x)=e_{2,n}\phi(e_{n-1,n},x)+\phi(e_{2,n},x)e_{n-1,n}\\=&\phi(e_{2,n},x)e_{n-1,n}=[\sum_{t=2}^{n-1}y_{1,t}^{2,n}e_{1,t}+\sum_{t=3}^{n}y_{2,t}^{2,n}e_{2,t}+y_{3,n}^{2,n}e_{3,n},x]e_{n-1,n}\\=&\sum_{t=2}^{n-2}y_{1,t}^{2,n}x_{t,n-1}e_{1,n}+\sum_{t=3}^{n-2}y_{2,t}^{2,n}x_{t,n-1}e_{2,n}-y_{2,n-1}^{2,n}x_{1,2}e_{1,n}\end{aligned}$$

for all $x=\sum_{1\leq i<j\leq n}x_{i,j}e_{i,j}\in N_n(R)$. Therefore $\sum_{t=2}^{n-2}y_{1,t}^{2,n}x_{t,n-1}-y_{2,n-1}^{2,n}x_{1,2}=0$ and $\sum_{t=3}^{n-2}y_{2,t}^{2,n}x_{t,n-1}=0$ for all $x=\sum_{1\leq i<j\leq n}x_{i,j}e_{i,j}\in N_n(R)$, which implies that $y_{1,t}^{2,n}=0$ for $2\leq t\leq n-2$ and $y_{2,t}^{2,n}=0$ for all $3\leq t\leq n-1$. Thus $y^{2,n}=y_{1,n-1}^{2,n}e_{1,n-1}+y_{2,n}^{2,n}e_{2,n}+y_{2,n-1}^{2,n}e_{2,n-1}+y_{3,n}^{2,n}e_{3,n}$. □

Lemma 3.6 *(i) For all $x\in N_n(R)$, we have $\phi(e_{1,n},x)=0$;*

(ii) For $3<j<n$, we get $y^{1,j}=de_{1,j}$ and $c_t^{1,j}=0$ for $2\leq t\leq n-2$;

(iii) Then $y^{1,3}=de_{1,3}$, $c_t^{1,j}=0$ for $3\leq t\leq n-2$ and $c_2^{1,3}=-y_{3,n}^{2,3}$.

Proof (I) By Lemma 3.4 (i) and (iv), it is easy to check that

$$\begin{aligned}\phi(e_{1,n},x)=&\phi(e_{1,3}e_{3,n},x)=e_{1,3}\phi(e_{3,n},x)+\phi(e_{1,3},x)e_{3,n}\\=&e_{1,3}[de_{3,n}+y_{1,n-1}^{3,n}e_{1,n-1}+y_{2,n}^{3,n}e_{2,n},x]+\\&[de_{1,3}+y_{1,n-1}^{1,3}e_{1,n-1}+y_{2,n}^{1,3}e_{2,n},x]e_{3,n}\\=&0\end{aligned}$$

for all $x\in N_n(R)$.

(II) Fix any i $(2<i<n-1)$, since $\phi(e_{1,n-1},x)=\phi(e_{1,i}e_{i,n-1},x)=e_{1,i}\phi(e_{i,n-1},x)+$

$\phi(e_{1,i},x)e_{i,n-1}$, using Lemma 3.4 (i) and (v), we have that

$$\begin{aligned}&[y_{1,n-2}^{1,n-1}e_{1,n-2}+y_{1,n-1}^{1,n-1}e_{1,n-1}+y_{2,n}^{1,n-1}e_{2,n},x]+\sum_{t=2}^{n-2}c_t^{1,n-1}x_{t,t+1}e_{1,n}\\&=e_{1,i}[y_{1,n-2}^{i,n-1}e_{1,n-2}+y_{1,n-1}^{i,n-1}e_{1,n-1}+y_{2,n}^{1,n-1}e_{2,n}+de_{i,n-1},x]+\\&\quad[de_{1,i}+y_{1,n-1}^{1,i}e_{1,n-1}+y_{2,n}^{1,i}e_{2,n},x]e_{i,n-1},\end{aligned}$$

i.e.,

$$\begin{aligned}&y_{1,n-2}^{1,n-1}x_{n-2,n-1}e_{1,n-1}+y_{1,n-2}^{1,n-1}x_{n-2,n}e_{1,n}+y_{1,n-1}^{1,n-1}x_{n-1,n}e_{1,n}-y_{2,n}^{1,n-1}x_{1,2}e_{1,n}+\\&\sum_{t=2}^{n-2}c_t^{1,n-1}x_{t,t+1}e_{1,n}=dx_{n-1,n}e_{1,n}\end{aligned}$$

for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$. Hence $y_{1,n-2}^{1,n-1}x_{n-2,n-1}=0$ and

$$y_{1,n-2}^{1,n-1}x_{n-2,n}+y_{1,n-1}^{1,n-1}x_{n-1,n}-y_{2,n}^{1,n-1}x_{1,2}+\sum_{t=2}^{n-2}c_t^{1,n-1}x_{t,t+1}=dx_{n-1,n}$$

for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$. Therefore $y_{1,n-2}^{1,n-1}=0$, $y_{2,n}^{1,n-1}=0$, $y_{1,n-1}^{1,n-1}=d$ and $c_t^{1,n-1}=0$ for $2\le t\le n-2$. So $y^{1,n-1}=de_{1,n-1}$.

(III) For $4\le j\le n-2$, since $\phi(e_{1,j},x)=\phi(e_{1,3}e_{3,j},x)=e_{1,3}\phi(e_{3,j},x)+\phi(e_{1,3},x)e_{3,j}$, we get, using Lemma 3.4 (i, iv), that

$$y_{1,n-1}^{1,j}x_{n-1,n}e_{1,n}-y_{2,n}^{1,j}x_{1,2}e_{1,n}+de_{1,j}x+\sum_{t=2}^{n-2}c_t^{1,j}x_{t,t+1}e_{1,n}=de_{1,j}x$$

for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$. Hence $y_{1,n-1}^{1,j}x_{n-1,n}-y_{2,n}^{1,j}x_{1,2}+\sum_{t=2}^{n-2}c_t^{1,j}x_{t,t+1}=0$ for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$, which implies that $y_{1,n-1}^{1,j}=y_{2,n}^{1,j}=0$ and $c_t^{1,j}=0$ for $2\le t\le n-2$. Thus $y^{1,j}=de_{1,j}$.

(IV) Since $\phi(e_{1,3},x)=\phi(e_{1,2}e_{2,3},x)=e_{1,2}\phi(e_{2,3},x)+\phi(e_{1,2},x)e_{2,3}$, we see using Lemma 3.4 (i) and Lemma 3.5 (i) that

$$y_{1,n-1}^{1,3}x_{n-1,n}e_{1,n}-y_{2,n}^{1,3}x_{1,2}e_{1,n}+de_{1,3}x+\sum_{t=2}^{n-2}c_t^{1,3}x_{t,t+1}e_{1,n}=de_{1,3}x-y_{3,n}^{2,3}x_{2,3}e_{1,n}$$

for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$. Hence $y_{1,n-1}^{1,3}x_{n-1,n}-y_{2,n}^{1,3}x_{1,2}+\sum_{t=2}^{n-2}c_t^{1,3}x_{t,t+1}=-y_{3,n}^{2,3}x_{2,3}$ for all $x=\sum_{1\le i<j\le n}x_{i,j}e_{i,j}\in N_n(R)$. Therefore $y_{1,n-1}^{1,3}=y_{2,n}^{1,3}=0$, $c_2^{1,3}=-y_{3,n}^{2,3}$ and $c_t^{1,3}=0$ for $3\le t\le n-2$. So $y^{1,3}=de_{1,3}$. □

Lemma 3.7 *For $3<j\le n$, we get that $y^{2,j}=de_{2,j}$ and $c_t^{2,j}=0$ for $2\le t\le n-2$.*

Proof (I) Since $\phi(e_{2,n},x)=\phi(e_{2,3}e_{3,n},x)=e_{2,3}\phi(e_{3,n},x)+\phi(e_{2,3},x)e_{3,n}$, by Lemma 3.4 (iv) and Lemma 3.5 (i, iii), we have that

$$\begin{aligned}&y_{1,n-1}^{2,n}x_{n-1,n}e_{1,n}-y_{2,n}^{2,n}x_{1,2}e_{1,n}-y_{2,n-1}^{2,n}x_{1,2}e_{1,n-1}+y_{2,n-1}^{2,n}x_{n-1,n}e_{2,n}-\\&y_{3,n}^{2,n}x_{1,3}e_{1,n}-y_{3,n}^{2,n}x_{2,3}e_{2,n}+\sum_{t=2}^{n-2}c_t^{2,n}x_{t,t+1}e_{1,n}\end{aligned}$$

$$= -dx_{1,2}e_{1,n}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $y^{2,n}_{2,n-1}x_{1,2} = 0$, $y^{2,n}_{2,n-1}x_{n-1,n} - y^{2,n}_{3,n}x_{2,3} = 0$ and $y^{2,n}_{1,n-1}x_{n-1,n} - y^{2,n}_{2,n}x_{1,2} - y^{2,n}_{3,n}x_{1,3} + \sum_{t=2}^{n-2} c^{2,n}_t x_{t,t+1} = -dx_{1,2}$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$, which means that $y^{2,n}_{2,n} = d$, $y^{2,n}_{1,n-1} = y^{2,n}_{2,n-1} = y^{2,n}_{3,n} = 0$, and $c^{2,n}_t = 0$ for $2 \le t \le n-2$. Thus $y^{2,n} = de_{2,n}$.

(II) Since $\phi(e_{2,n-1}, x) = \phi(e_{2,3}e_{3,n-1}, x) = e_{2,3}\phi(e_{3,n-1}, x) + \phi(e_{2,3}, x)e_{3,n-1}$, by Lemma 3.4 (v) and Lemma 3.5 (i, ii), we have that

$$\begin{aligned} &y^{2,n-1}_{1,n-2}x_{n-2,n-1}e_{1,n-1} + y^{2,n-1}_{1,n-2}x_{n-2,n}e_{1,n} + y^{2,n-1}_{1,n-1}x_{n-1,n}e_{1,n} + \\ &\quad y^{2,n-1}_{2,n-1}x_{n-1,n}e_{2,n} - y^{2,n-1}_{2,n-1}x_{1,2}e_{1,n-1} - y^{2,n-1}_{2,n}x_{1,2}e_{1,n} - \\ &\quad y^{2,n-1}_{3,n}x_{1,3}e_{1,n} - y^{2,n-1}_{3,n}x_{2,3}e_{2,n} + \sum_{t=2}^{n-2} c^{2,n-1}_t x_{t,t+1}e_{1,n} \\ &= dx_{n-1,n}e_{2,n} - dx_{1,2}e_{1,n-1} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore

$$y^{2,n-1}_{2,n-1}x_{n-1,n} - y^{2,n-1}_{3,n}x_{2,3} = dx_{n-1,n}, \quad y^{2,n-1}_{1,n-2}x_{n-2,n-1} - y^{2,n-1}_{2,n-1}x_{1,2} = -dx_{1,2},$$

$$y^{2,n-1}_{1,n-2}x_{n-2,n} + y^{2,n-1}_{1,n-1}x_{n-1,n} - y^{2,n-1}_{2,n}x_{1,2} - y^{2,n-1}_{3,n}x_{1,3} + \sum_{t=2}^{n-2} c^{2,n-1}_t x_{t,t+1}e_{1,n} = 0$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. This implies that $y^{2,n-1}_{3,n} = y^{2,n-1}_{1,n-2} = 0$, $y^{2,n-1}_{2,n-1} = d$, $y^{2,n-1}_{1,n-1} = y^{2,n-1}_{2,n} = 0$, and $c^{2,n-1}_t = 0$ for $2 \le t \le n-2$. So $y^{2,n-1} = de_{2,n-1}$.

(III) For $4 \le j < n-1$, since $\phi(e_{2,j}, x) = \phi(e_{2,3}e_{3,j}, x) = e_{2,3}\phi(e_{3,j}, x) + \phi(e_{2,3}, x)e_{3,j}$, using Lemma 3.4 (iv) and Lemma 3.5 (i), we can get that

$$y^{2,j}_{1,n-1}x_{n-1,n}e_{1,n} - y^{2,j}_{2,n}x_{1,2}e_{1,n} - dx_{1,2}e_{1,j} + de_{2,j}x$$

$$-y^{2,j}_{3,n}x_{1,3}e_{1,n} - y^{2,j}_{3,n}x_{2,3}e_{2,n} + \sum_{t=2}^{n-2} c^{2,j}_t x_{t,t+1}e_{1,n} = de_{2,j}x - dx_{1,2}e_{1,j}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore

$$y^{2,j}_{1,n-1}x_{n-1,n} - y^{2,j}_{2,n}x_{1,2} - y^{2,j}_{3,n}x_{1,3} + \sum_{t=2}^{n-2} c^{2,j}_t x_{t,t+1} = 0$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence $y^{2,j}_{1,n-1} = y^{2,j}_{2,n} = y^{2,j}_{3,n} = 0$, and $c^{2,j}_t = 0$ for $2 \le t \le n-2$. So $y^{2,j} = de_{2,j}$. □

Lemma 3.8 *For $i > 2$ and $j - i \le 2$, we have that $y^{i,j} = de_{i,j}$ and $c^{2,j}_t = 0$ for $2 \le t \le n-2$.*

Proof (I) Since $\phi(e_{i,n}, x) = \phi(e_{i,n-1}e_{n-1,n}, x) = e_{i,n-1}\phi(e_{n-1,n}, x) + \phi(e_{i,n-1}, x)e_{n-1,n}$, by Lemma 3.4 (iv) and (v), we have that

$$[y^{i,n}_{1,n-1}e_{1,n-1} + y^{i,n}_{2,n}e_{2,n} + de_{i,n}, x] + \sum_{t=2}^{n-2} c^{i,n}_t x_{t,t+1}e_{1,n}$$

$$= e_{i,n-1}[de_{n-1,n} + y_{1,n-1}^{n-1,n}e_{1,n-1} + y_{2,n}^{n-1,n}e_{2,n}, x]+$$
$$[de_{i,n-1} + y_{1,n-1}^{i,n-1}e_{1,n-1} + y_{2,n}^{i,n-1}e_{2,n} + y_{1,n-2}^{i,n-1}e_{1,n-2}, x]e_{n-1,n},$$

i.e.,

$$-dxe_{i,n} + y_{1,n-1}^{i,n}x_{n-1,n}e_{1,n} - y_{2,n}^{i,n}x_{1,2}e_{1,n} + \sum_{t=2}^{n-2} c_t^{i,n}x_{t,t+1}e_{1,n} = -dxe_{i,n} + y_{1,n-2}^{i,n-1}x_{n-2,n-1}e_{1,n}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore

$$y_{1,n-1}^{i,n}x_{n-1,n} - y_{2,n}^{i,n}x_{1,2} + \sum_{t=2}^{n-2} c_t^{i,n}x_{t,t+1} = y_{1,n-2}^{i,n-1}x_{n-2,n-1}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence $y_{1,n-1}^{i,n} = y_{2,n}^{i,n} = 0$, $y_{1,n-2}^{i,n-1} = c_{n-2}^{i,n}$ and $c_t^{2,n} = 0$ for $2 \le t \le n-3$. Thus $y^{i,n} = de_{i,n}$.

(II) Since $\phi(e_{i,n-1}, x) = \phi(e_{i,n-2}e_{n-2,n-1}, x) = e_{i,n-2}\phi(e_{n-2,n-1}, x) + \phi(e_{i,n-2}, x)e_{n-2,n-1}$ by Lemma 3.4 (iv) and (v), we have that

$$\begin{aligned} &dx_{n-1,n}e_{i,n} - dxe_{i,n-1} + y_{1,n-1}^{i,n-1}x_{n-1,n}e_{1,n} - y_{2,n}^{i,n-1}x_{1,2}e_{1,n}+ \\ &\quad y_{1,n-2}^{i,n-1}x_{n-2,n-1}e_{1,n-1} + y_{1,n-2}^{i,n-1}x_{n-2,n}e_{1,n} + \sum_{t=2}^{n-2} c_t^{i,n-1}x_{t,t+1}e_{1,n} \\ &= dx_{n-1,n}e_{i,n} - dxe_{i,n-1} \end{aligned}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $y_{1,n-2}^{i,n-1}x_{n-2,n-1} = 0$, and $y_{1,n-1}^{i,n-1}x_{n-1,n} - y_{2,n}^{i,n-1}x_{1,2} + y_{1,n-2}^{i,n-1}x_{n-2,n} + \sum_{t=2}^{n-2} c_t^{i,n-1}x_{t,t+1} = 0$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. This implies that $y_{1,n-2}^{i,n-1} = y_{1,n-1}^{i,n-1} = y_{2,n}^{i,n-1} = 0$, and $c_t^{2,n-1} = 0$ for $2 \le t \le n-2$. Thus $y^{i,n-1} = de_{i,n-1}$. By (I), we have that $c_{n-2}^{i,n} = y_{1,n-2}^{i,n-1} = 0$.

(III) For any (i,j) $(i > 2,\ i+2 < j < n-1)$, since $\phi(e_{i,j}, x) = \phi(e_{i,i+1}e_{i+1,j}, x) = e_{i,i+1}\phi(e_{i+1,j}, x) + \phi(e_{i,i+1}, x)e_{i+1,j}$, using Lemma 3.4 (iv), we can get that

$$de_{i,j}x - dxe_{i,j} + y_{1,n-1}^{i,j}x_{n-1,n}e_{1,n} - y_{2,n}^{i,j}x_{1,2}e_{1,n} + \sum_{t=2}^{n-2} c_t^{i,j}x_{t,t+1}e_{1,n} = de_{i,j}x - dxe_{i,j}$$

for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Therefore $y_{1,n-1}^{i,j}x_{n-1,n} - y_{2,n}^{i,j}x_{1,2} + \sum_{t=2}^{n-2} c_t^{i,j}x_{t,t+1} = 0$ for all $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$. Hence $y_{1,n-1}^{i,j} = y_{2,n}^{i,j} = 0$, and $c_t^{i,j} = 0$ for $2 \le t \le n-2$. So $y^{i,j} = de_{i,j}$. □

Summarizing Lemmas 3.4 (i), 3.5 (i) and 3.6–3.8, we have the following result.

Corollary 3.9 *The following equalities hold*

(i) $\phi(e_{1,n}, x) = 0$ *for all* $x = \sum_{1\le i<j\le n} x_{i,j}e_{i,j} \in N_n(R)$;

(ii) For $1 \le i \le n-1$ *with* $i \ne 2$, $\phi(e_{i,i+1}, x) = [de_{i,i+1} + y_{1,n-1}^{i,i+1}e_{1,n-1} + y_{2,n}^{i,i+1}e_{2,n}, x] + \sum_{t=2}^{n-2} c_t^{i,i+1}x_{t,t+1}e_{1,n}$ *for all* $x \in N_n(R)$;

(iii) For $i+2 \le j$, *except* $(i,j) = (1,3)$, $\phi(e_{i,j}, x) = [de_{i,j}, x]$ *for all* $x \in N_n(R)$;

(iv) $\phi(e_{1,3}, x) = [de_{1,3}, x] + c_2^{1,3}x_{2,3}e_{1,n}$ *for all* $x \in N_n(R)$;

(v) $\phi(e_{2,3}, x) = [de_{2,3} + y_{1,n-1}^{2,3}e_{1,n-1} + y_{2,n}^{2,3}e_{2,n} + y_{3,n}^{2,3}e_{3,n}, x] + \sum_{t=2}^{n-2} c_t^{i,i+1}x_{t,t+1}e_{1,n}$ *for all* $x \in N_n(R)$, *and* $y_{3,n}^{2,3} = -c_2^{1,3}$.

The Proof of Theorem 3.1 For $1 \le i \le n-1$, denote $y_{1,n-1}^{i,i+1}$ by $c_{n-1}^{i,i+1}$, $y_{2,n}^{i,i+1}$ by $-c_1^{i,i+1}$, and $c_2^{1,3} = -y_{3,n}^{2,3}$ by a. Then, for $i \neq 2$,

$$\phi(e_{i,i+1}, x) = [de_{i,i+1}, x] + \sum_{t=1}^{n-1} c_t^{i,i+1} x_{t,t+1} e_{1,n},$$

$$\phi(e_{2,3}, x) = [de_{2,3}, x] + [-ae_{3,n}, x] + \sum_{t=1}^{n-1} c_t^{i,i+1} x_{t,t+1} e_{1,n},$$

and

$$\phi(e_{1,3}, x) = [de_{1,3}, x] + ax_{2,3} e_{1,n}$$

for all $x \in N_n(R)$. Let A denote $\sum_{i,j=1}^{n-1} c_j^{i,i+1} e_{i,j}$. For $y = \sum_{1 \le i < j \le n} y_{i,j} e_{i,j}$, $x = \sum_{1 \le i < j \le n} x_{i,j} e_{i,j} \in N_n(R)$, it follows from Corollary 3.9 that

$$\begin{aligned}
\phi(y, x) = & \sum_{1 \le i < j \le n} y_{i,j} \phi(e_{i,j}, x) \\
= & \sum_{1 \le i < j \le n} y_{i,j} [de_{i,j}, x] + \sum_{i=1}^{n-1} (y_{i,i+1} \sum_{j=1}^{n-1} c_j^{i,i+1} x_{j,j+1} e_{1,n}) + \\
& y_{2,3} [-ae_{3,n}, x] + ay_{1,3} x_{2,3} e_{1,n} \\
= & d[\sum_{1 \le i < j \le n} y_{i,j} e_{i,j}, x] + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (y_{i,i+1} c_j^{i,i+1} x_{j,j+1} e_{1,n}) + \\
& a(y_{2,3} x_{1,3} e_{1,n} + y_{2,3} x_{2,3} e_{2,n} + y_{1,3} x_{2,3} e_{1,n}) \\
= & d[y, x] + \phi_A(y, x) + \phi_a(y, x).
\end{aligned}$$

We thus complete the proof of Theorem 3.1. □

Acknowledgments The authors would like to thank the referees for their comments.

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