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Identities Involving Powers and Inverse of Binomial Coefficients

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Abstract In this paper, we give several identities of finite sums and some infinite series involving powers and inverse of binomial coefficients, which extends the results of T. Trif.

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1. Introduction and preliminaries

Binomial coefficients play an important role in many areas of mathematics, including combinatorial analysis, graph theory, number theory, statistics and probability. Inverses of binomial coefficients are also prolific in the mathematical literature and many results on the inverses of binomial coefficient identities can be found in the papers [1–4].

In [1], Sury first used the identity

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

to observe that

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}\mathrm{d}t. \tag{1}$$

In this paper, we obtain several identities of summations involving powers and inverses of binomial coefficients by the integral identity (1), which extends the results of Trif [2].

Lemma 1.1 ([5]) *For each $r\ge 0$, the power series with coefficients'r-th powers'equals:*

$$\sum_{k\ge 0}k^r x^k=\sum_{j=0}^{r}j!S(r,j)\frac{x^j}{(1-x)^{j+1}}=\frac{A_r(x)}{(1-x)^{r+1}}\,,\quad |x|<1,$$

where $S(r,j)$ and $A_n(x)$ are Stirling numbers of the second kind and Eulerian polynomials respectively.

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Lemma 1.2 ([6]) *Let r be nonnegative integer. Then*

$$\sum_{k=0}^{n} k^r x^k = \frac{A_r(x)}{(1-x)^{r+1}} - x^{n+1} \sum_{k=0}^{r} \binom{r}{k} \frac{A_k(x)}{(1-x)^{k+1}} (n+1)^{r-k},$$

where $A_k(x)$ are Eulerian polynomials.

Lemma 1.3 *Let r be nonnegative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} (-x)^k k^r = \sum_{h=0}^{r} (n)_h S(r,h)(-x)^h (1-x)^{n-h},$$

where $S(r,h)$ are Stirling numbers of the second kind and x is a real number.

Proof Let $f(z) = \sum_{k=0}^{\infty} (-xz)^k k^r = \sum_{h=0}^{r} (n)_h S(r,h) \frac{(-xz)^h}{(1+xz)^{h+1}}$. Then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (-x)^k k^r &= [z^n] \frac{1}{1-z} f(\frac{z}{1-z}) \\
&= [z^n] \sum_{h=0}^{r} h! S(r,h)(-x)^h \sum_{i=0}^{\infty} (1-x)^i \binom{h+i}{i} z^{h+i} \\
&= \sum_{h=0}^{r} (n)_h S(r,h)(-x)^h (1-x)^{n-h}. \quad \square
\end{aligned}$$

By the same way, we can get the following Lemma.

Lemma 1.4 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{n} \binom{n}{k} x^k k^r = \sum_{h=0}^{r} (n)_h S(r,h) x^h (1+x)^{n-h},$$

where $S(r,h)$ are Stirling numbers of the second kind.

2. Finite sums involving inverses of binomial coefficients

In this section, we present some finite sums involving the inverses of binomial coefficients and powers.

Theorem 2.1 *Let $r, m \geq 0, j \geq 1$ be any integers. Then*

$$\begin{aligned}
\sum_{k=j}^{n} \binom{n+m}{m+k}^{-1} (-1)^k k^r =& \frac{m+n+1}{n+2-j} \sum_{k=0}^{r} \binom{r}{k} j^{r-k} \sum_{i=0}^{k} i!(-1)^{i+j} S(k,i) \binom{n+m+i+2}{m+i+j}^{-1} + \\
& (-1)^n \sum_{k=0}^{r} \binom{r}{k} (n+1)^{r-k} \sum_{i=0}^{k} i!(-1)^i S(k,i) \frac{n+m+1}{n+2+m+i}
\end{aligned}$$

where $S(r,h)$ are Stirling numbers of the second kind.

Proof By the integral identity (1), we have

$$\sum_{k=j}^{n}\binom{n+m}{m+k}^{-1}(-1)^k k^r = (n+m+1)\int_0^1 (1-t)^n t^m \sum_{k=j}^{n}(\frac{-t}{1-t})^k k^r \mathrm{d}t$$

$$= (n+m+1)\sum_{k=0}^{r}\binom{r}{k} j^{r-k}\sum_{i=0}^{k} i! S(k,i)\int_0^1 (1-t)^{n+1-j} t^{m+j+i}(-1)^{i+j}\mathrm{d}t -$$

$$(n+m+1)\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r-k}\sum_{i=0}^{k} i! S(k,i)\int_0^1 t^{m+n+i+1}(-1)^{n+1+i}\mathrm{d}t$$

$$= \frac{m+n+1}{n+2-j}\sum_{k=0}^{r}\binom{r}{k} j^{r-k}\sum_{i=0}^{k} i!(-1)^{i+j} S(k,i)\binom{n+m+i+2}{m+i+j}^{-1} +$$

$$(-1)^n\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r-k}\sum_{i=0}^{k} i!(-1)^i S(k,i)\frac{n+m+1}{n+2+m+i},$$

which completes the proof. □

In a similar way, we can get the following Corollary.

Corollary 2.1 *Let r, m be any nonnegative integers. Then*

$$\sum_{k=0}^{n}\binom{n+m}{m+k}^{-1}(-1)^k k^r = \frac{m+n+1}{n+2}\sum_{h=0}^{r} h!(-1)^h S(r,h)\binom{n+m+h+2}{m+h}^{-1} +$$

$$(-1)^n\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r-k}\sum_{i=0}^{k} i! S(k,i)\frac{(-1)^i(m+n+1)}{n+2+m+i},$$

where $S(r,h)$ are Stirling numbers of the second kind.

Remark 2.1 It is the conclusion of [2] when $r=0$.

By setting $r=0, r=1$ in Theorem 2.1, we get the following Corollary.

Corollary 2.2 *Let j be any nonnegative integer. Then*

$$\sum_{k=j}^{n}\binom{n+m}{m+k}^{-1}(-1)^k = \frac{m+n+1}{n+m+2}((-1)^n + (-1)^j\binom{n+m+1}{m+j}^{-1}),$$

$$\sum_{k=j}^{n}\binom{n+m}{m+k}^{-1}(-1)^k k = \frac{m+n+1}{n+m+2}(\binom{n+m+1}{m+j}^{-1}(j-\frac{m+1+j}{n+m+3})(-1)^j +$$

$$\frac{(-1)^n(n^2+n(m+3)+1)}{(m+n+3)}),$$

where $S(r,h)$ are Stirling numbers of the second kind.

Setting $m=n$ in Theorem 2.1 gives another Corollary.

Corollary 2.3 *Let r be any nonnegative integer. Then*

$$\sum_{k=j}^{n}\binom{2n}{n+k}^{-1}(-1)^k k^r = \frac{2n+1}{n+2-j}\sum_{k=0}^{r}\binom{r}{k}j^{r-k}\sum_{i=0}^{k}i!(-1)^{i+j}S(k,i)\binom{2n+i+2}{n+i+j}^{-1}+$$
$$(-1)^n\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r-k}\sum_{i=0}^{k}i!(-1)^iS(k,i)\frac{(2n+1)}{2n+2+i},$$

where $j\geq 1$ is integer, $S(r,h)$ are Stirling numbers of the second kind.

By setting $m=n$ in Corollary 2.1, we get the following conclusion.

Corollary 2.4 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{n}\binom{2n}{n+k}^{-1}(-1)^k k^r = \frac{2n+1}{n+2}\sum_{h=0}^{r}h!(-1)^hS(r,h)\binom{2n+h+2}{n+h}^{-1}+$$
$$(-1)^n\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r-k}\sum_{i=0}^{k}i!(-1)^iS(k,i)\frac{(2n+1)}{2n+2+i},$$

where $S(r,h)$ are Stirling numbers of the second kind.

Corollary 2.5 *Let j be any nonnegative integer. Then*

$$\sum_{k=j}^{n}\binom{2n}{n+k}^{-1}(-1)^k = \frac{2n+1}{2n+2}((-1)^n+(-1)^j\binom{2n+1}{n+j}^{-1}),$$
$$\sum_{k=j}^{n}\binom{2n}{n+k}^{-1}(-1)^k k = \frac{2n+1}{2n+2}\binom{2n+1}{n+j}^{-1}(j-\frac{n+1+j}{2n+3})(-1)^j+\frac{(-1)^n(2n+1)^2}{2(2n+3)}.$$

Theorem 2.2 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{2n}\binom{2n}{k}\binom{4n}{2k}^{-1}(-1)^k k^r$$
$$=(4n+1)\sum_{h=0}^{r}(2n)_h(-1)^hS(r,h)\sum_{i=0}^{2n-h}\binom{2n-h}{i}\frac{(-2)^i}{2h+i+1},$$

where $S(r,h)$ are Stirling numbers of the second kind and $(n)_h=n(n-1)\cdots(n-h+1)$.

Proof By the integral identity (1) and Lemma 1.1, we have

$$\sum_{k=0}^{2n}\binom{2n}{k}\binom{4n}{2k}^{-1}(-1)^k k^r = (4n+1)\int_0^1(1-t)^{4n}\sum_{k=0}^{2n}\binom{2n}{k}(\frac{-t^2}{(1-t)^2})^k k^r\mathrm{d}t$$
$$=(4n+1)\sum_{h=0}^{r}(2n)_hS(r,h)(-1)^h\int_0^1(1-2t)^{2n-h}t^{2h}\mathrm{d}t$$
$$=(4n+1)\sum_{h=0}^{r}(2n)_h(-1)^hS(r,h)\sum_{i=0}^{2n-h}\binom{2n-h}{i}\frac{(-2)^i}{2h+i+1},$$

which completes the proof. □

Remark 2.2 It is the conclusion of [2] when $r=0$.

Theorem 2.3 *If r, m, n and p are nonnegative integers with $p \le n$, then*

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}k^r=\sum_{h=0}^{r}(m)_hS(r,h)\binom{n+h}{p+h}^{-1}\frac{m+n+1}{n+h+1},$$

where $S(r,h)$ are Stirling numbers of the second kind.

Proof By the integral identity(1) and Lemma 1.2, we have

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}k^r&=(n+m+1)\int_0^1 t^p(1-t)^{n+m-p}\sum_{k=0}^{m}\binom{m}{k}(\frac{t}{1-t})^kk^r\mathrm{d}t\\&=(n+m+1)\sum_{h=0}^{r}(m)_hS(r,h)\int_0^1(1-t)^{n-p}t^{p+h}\mathrm{d}t\\&=\sum_{h=0}^{r}(m)_hS(r,h)\binom{n+h}{p+h}^{-1}\frac{m+n+1}{n+h+1},\end{aligned}$$

which completes the proof. □

Remark 2.3 It is the conclusion of [2] when $r=0$.

Theorem 2.4 *If r, m, n and p are nonnegative integers with $p \le n$, then*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}\binom{n+m}{p+k}^{-1}(-1)^kk^r\\&=\sum_{h=0}^{r}(m)_hS(r,h)(-1)^h\sum_{i=0}^{m-h}\binom{m-h}{i}\binom{n+h+i}{h+p+i}^{-1}\frac{(-2)^i(m+n+1)}{h+n+i+1},\end{aligned}$$

where $S(r,h)$ are Stirling numbers of the second kind.

Theorem 2.4 can be proved in the same way and the proof is omitted here.

3. Infinite sums involving inverses of binomial coefficients

In this section, we present some infinite sums involving the inverses of binomial coefficients and powers.

Theorem 3.1 *Let $r \ge 0, j \ge 1$ be any integers. Then*

$$\begin{aligned}\sum_{k=j}^{\infty}\binom{mk}{nk}^{-1}(-1)^kk^r=&(-1)^j\sum_{k=0}^{r+1}\binom{r+1}{k}j^{r+1-k}(m+\frac{r+1-k}{j(r+1)})\times\\&\sum_{i=0}^{k}i!S(k,i)(-1)^i\int_0^1\frac{(t^n(1-t)^{m-n})^{i+j}\mathrm{d}t}{(1+t^n(1-t)^{m-n})^{i+1}},\end{aligned}$$

where $S(r,h)$ are Stirling numbers of the second kind.

Proof By the integral identity (1), we get

$$\sum_{k=j}^{\infty}\binom{mk}{nk}^{-1}(-1)^k k^r=\sum_{k=j}^{\infty}(mk+1)\int_0^1(1-t)^{(m-n)k}t^{nk}(-1)^k k^r\mathrm{d}t$$

$$=m\int_0^1\sum_{k=j}^{\infty}(-t^n(1-t)^{m-n})^k k^{r+1}\mathrm{d}t-\int_0^1\sum_{k=j}^{\infty}(-t^n(1-t)^{m-n})^k k^r\mathrm{d}t$$

$$=(-1)^j\sum_{k=0}^{r+1}\binom{r+1}{k}j^{r+1-k}(m+\frac{r+1-k}{j(r+1)})\sum_{i=0}^{k}i!S(k,i)(-1)^i\int_0^1\frac{(t^n(1-t)^{m-n})^{i+j}\mathrm{d}t}{(1+t^n(1-t)^{m-n})^{i+1}},$$

which completes the proof. □

By the same way, we can get the following Corollary.

Corollary 3.1 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}(-1)^k k^r$$

$$=\sum_{h=0}^{r+1}h!(mS(r+1,h)+S(r,h))\sum_{i=0}^{h}\binom{h}{i}(-1)^{h-i}\int_0^1\frac{\mathrm{d}t}{(1+t^n(1-t)^{m-n})^{i+1}},$$

where $S(r,h)$ are Stirling numbers of the second kind.

By setting $m=2,n=1$ in Corollary 3.1, we obtain the following results.

Corollary 3.2 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}(-1)^k k^r$$

$$=\frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2}((-1)^{r+1}+\sum_{h=1}^{r+1}h!(2S(r+1,h)+S(r,h))\sum_{i=1}^{h}\binom{h}{i}\binom{2i}{i}\frac{(-1)^{h-i}}{5^i})+$$

$$\sum_{h=2}^{r+1}h!(2S(r+1,h)+S(r,h))\sum_{i=2}^{h}\binom{h}{i}\sum_{s=1}^{i-1}\frac{(-1)^{h-i}(2i)_{2s+1}}{5^{s+1}[(i)_{s+1}]^2}+\frac{2(r+2)(-1)^r}{5},$$

where $S(r,h)$ are Stirling numbers of the second kind, $(i)_s=i(i-1)\cdots(i-s+1)$.

Proof By the integral identity (1) and Corollary 3.1, we get

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}(-1)^k k^r=2\sum_{h=0}^{r+1}h!S(r+1,h)\sum_{i=0}^{h}\binom{h}{i}(-1)^{h-i}\int_0^1\frac{\mathrm{d}t}{(1+t(1-t))^{i+1}}+$$

$$\sum_{h=0}^{r}h!S(r,h)\sum_{i=0}^{h}\binom{h}{i}(-1)^{h-i}\int_0^1\frac{\mathrm{d}t}{(1+t(1-t))^{i+1}}$$

$$=(-1)^{r+1}\int_0^1\frac{\mathrm{d}t}{1+t(1-t)}+\sum_{h=1}^{r+1}(2S(r+1,h)+S(r,h))h!\sum_{i=1}^{h}\binom{h}{i}(-1)^{h-i}\times$$

$$(\frac{2}{5i}+\sum_{s=1}^{i-1}(\frac{2}{5})^{s+1}\frac{(2i-1)(2i-3)\cdots(2(i-s)+1)}{(i)_{s+1}}+(\frac{2}{5})^i\frac{(2i-1)!!)}{i!}\frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2})$$

$$= \frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2}((-1)^{r+1}+\sum_{h=1}^{r+1}h!(2S(r+1,h)+S(r,h))\sum_{i=1}^{h}\binom{h}{i}\binom{2i-1}{i}\frac{(-1)^{h-i}}{5^i})+$$

$$\sum_{h=2}^{r+1}h!(2S(r+1,h)+S(r,h))\sum_{i=2}^{h}\binom{h}{i}\sum_{s=1}^{i-1}\frac{(-1)^{h-i}(2i)_{2s+1}}{5^{s+1}[(i)_{s+1}]^2}+\frac{2}{5}(r+2)(-1)^r,$$

which completes the proof. □

Corollary 3.3 *Let $i \geq 1$ be any integer. Then*

$$\sum_{s=0}^{\infty}\binom{s+i}{i}\binom{2s+1}{s}^{-1}\frac{(-1)^s}{s+1} = \int_0^1 \frac{dt}{(1+t(1-t))^{i+1}}$$

$$= \frac{2}{5i}+\sum_{s=1}^{i-1}\frac{(2i)_{2s+1}}{5^{s+1}[(i)_{s+1}]^2}+\binom{2i}{i}\frac{4\sqrt{5}}{5^{i+1}}\ln\frac{1+\sqrt{5}}{2},$$

where $(i)_s = i(i-1)\cdots(i-s+1)$.

By Corollary 3.3, we obtain the following Corollary 3.4.

Corollary 3.4 *Let i be any nonnegative integer. Then*

$$\sum_{s=0}^{\infty}\binom{s+i}{i}\binom{2(s+i)+1}{s+i}^{-1}\frac{(-1)^{s+i}}{s+i+1} = \sum_{s=2}^{i}\binom{i}{s}\sum_{j=1}^{s-1}\frac{(-1)^{i-s}(2s)_{2j+1}}{5^{j+1}[(s)_{j+1}]^2}+$$

$$\frac{2(-1)^{i+1}H_i}{5}+\frac{4\sqrt{5}}{5}\ln\frac{1+\sqrt{5}}{2}((-1)^i+\sum_{s=1}^{i}\binom{i}{s}\binom{2s}{s}\frac{(-1)^{i-s}}{5^s}).$$

Remark 3.1 By setting $r = 0, 1, 2$ in Corollary 3.2, we have

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}(-1)^k = \frac{4}{5}-\frac{4\sqrt{5}}{5^2}\ln\frac{1+\sqrt{5}}{2},$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}(-1)^k k = -\frac{6}{5^2}-\frac{4\sqrt{5}}{5^3}\ln\frac{1+\sqrt{5}}{2},$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1}(-1)^k k^2 = -\frac{4}{5^2}+\frac{4\sqrt{5}}{5^3}\ln\frac{1+\sqrt{5}}{2}.$$

Theorem 3.2 *Let $r \geq 0, j \geq 1$ be any integers. Then*

$$\sum_{k=j}^{\infty}\binom{mk}{nk}^{-1}k^r$$

$$= \sum_{k=0}^{r+1}\binom{r+1}{k}j^{r+1-k}(m+\frac{r+1-k}{j(r+1)})\sum_{i=0}^{k}i!S(k,i)\int_0^1\frac{(t^n(1-t)^{m-n})^{i+j}\mathrm{d}t}{(1+t^n(1-t)^{m-n})^{i+1}},$$

where $S(r,h)$ are Stirling numbers of the second kind.

The proof of Theorem 3.2 is the same as that of Theorem 3.1.

Corollary 3.5 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{\infty}\binom{mk}{nk}^{-1}k^r$$

$$= \sum_{h=0}^{r+1} h!(mS(r+1,h)+S(r,h)) \sum_{i=0}^{h} \binom{h}{i} (-1)^{h-i} \int_0^1 \frac{\mathrm{d}t}{(1-t^n(1-t)^{m-n})^{i+1}},$$

where $S(r,h)$ are Stirling numbers of the second kind.

Remark 3.2 It is the conclusion of [2] when $r=0$.

By setting $m=2, n=1$ in Corollary 3.5, we get the following conclusion.

Corollary 3.6 *Let r be any nonnegative integer. Then*

$$\sum_{k=0}^{\infty} \binom{2k}{k}^{-1} k^r$$
$$= \frac{2\pi\sqrt{3}}{9}((-1)^{r+1} + \sum_{h=1}^{r+1} h!(2S(r+1,h)+S(r,h)) \sum_{i=1}^{h} \binom{h}{i}\binom{2i}{i} \frac{(-1)^{h-i}}{3^i}) +$$
$$\sum_{h=2}^{r+1} h!(2S(r+1,h)+S(r,h)) \sum_{i=2}^{h} \binom{h}{i} \sum_{s=1}^{i-1} \frac{(-1)^{h-i}(2i)_{2s+1}}{3^{s+1}[(i)_{s+1}]^2} + \frac{2(r+2)(-1)^r}{3},$$

where $S(r,h)$ are Stirling numbers of the second kind and $(i)_s = i(i-1)\cdots(i-s+1)$.

The proof of Corollary 3.6 is the same as that of Corollary 3.2.

Corollary 3.7 *Let $i \geq 1$ be any integer. Then*

$$\sum_{s=0}^{\infty} \binom{s+i}{s}\binom{2s+1}{s}^{-1} \frac{1}{s+1} = \int_0^1 \frac{\mathrm{d}t}{(1-t(1-t))^{i+1}}$$
$$= \frac{2}{3i} + \frac{2\pi\sqrt{3}}{3^{i+2}}\binom{2i}{i} + \sum_{s=1}^{i-1} \frac{(2i)_{2s+1}}{3^{s+1}[(i)_{s+1}]^2},$$

where $(i)_s = i(i-1)\cdots(i-s+1)$.

By Corollary 3.7, we can get the following Corollary 3.8.

Corollary 3.8 *Let i be any nonnegative integer. Then*

$$\sum_{s=0}^{\infty} \binom{s+i}{s}\binom{2(s+i)+1}{s+i}^{-1} \frac{1}{s+i+1}$$
$$= \frac{2\pi\sqrt{3}}{9}((-1)^i + \sum_{s=1}^{i} \binom{i}{s}\binom{2s}{s}\frac{(-1)^{i-s}}{3^s}) + \sum_{s=2}^{i}\binom{i}{s}\sum_{j=1}^{s-1} \frac{(-1)^{i-s}(2s)_{2j+1}}{3^{j+1}[(s)_{j+1}]^2} + \frac{2(-1)^{i+1}H_i}{3},$$

where $(i)_s = i(i-1)\cdots(i-s+1)$.

Remark 3.3 By setting $r=0,1,2$ in Corollary 3.6, we can establish the following identities.

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} = \frac{4}{3} + \frac{2\pi\sqrt{3}}{3^3},$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}^{-1} k = \frac{2}{3} + \frac{2\pi\sqrt{3}}{3^3},$$

$$\sum_{k=0}^{\infty} \binom{2k}{k}^{-1} k^2 = \frac{4}{3} + \frac{10\pi\sqrt{3}}{3^4}.$$

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