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Hyponormality of Toeplitz Operators on the Dirichlet Space

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Abstract In this paper, we prove that the necessary and sufficient condition for a Toeplitz operator T_u on the Dirichlet space to be hyponormal is that the symbol u is constant for the case that the projection of u in the Dirichlet space is a polynomial and for the case that u is a class of special symbols, respectively. We also prove that a Toeplitz operator with harmonic polynomial symbol on the harmonic Dirichlet space is hyponormal if and only if its symbol is constant.

Keywords Toeplitz operator; hyponormality; Dirichlet space; harmonic Dirichlet space.

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1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$ and $\mathrm{d}A$ denote the normalized area measure on D. We denote by $W^{1,2}(D)$ the Sobolev space which consists of all functions $u : D \to \mathbb{C}$ with the weak partial derivatives of order 1, where the norm satisfies

$$\|u\| = \Big(\Big|\int_D u\mathrm{d}A\Big|^2 + \int_D \Big(\Big|\frac{\partial u}{\partial z}\Big|^2 + \Big|\frac{\partial u}{\partial \overline{z}}\Big|^2\Big)\mathrm{d}A\Big)^{\frac{1}{2}} < \infty.$$

The Sobolev space $W^{1,2}(D)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_D u\mathrm{d}A \int_D \overline{v}\mathrm{d}A + \Big\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\Big\rangle_{L^2(D)} + \Big\langle \frac{\partial u}{\partial \overline{z}}, \frac{\partial v}{\partial \overline{z}}\Big\rangle_{L^2(D)},$$

where the symbol $\langle\cdot,\cdot\rangle_{L^2(D)}$ denotes the inner product in the Hilbert space $L^2(D, \mathrm{d}A)$.

The Dirichlet space $\mathfrak{D}$ consisting of all the analytic functions vanishing at 0 is the closed subspace of $W^{1,2}(D)$, and is a Hilbert space with inner product

$$\big\langle h, g\big\rangle = \big\langle h', g'\big\rangle_{L^2(D)}.$$

The Sobolev space $W^{1,\infty}(D)$ is defined by

$$W^{1,\infty}(D) = \Big\{u \in W^{1,2}(D) : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \overline{z}} \in L^\infty(D)\Big\},$$

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where $L^\infty(D)$ is the space of essentially bounded measurable functions on D. The norm in $W^{1,\infty}(D)$ is defined by

$$\|u\|_{1,\infty} = \max\Big\{\|u\|_\infty, \Big\|\frac{\partial u}{\partial z}\Big\|_\infty, \Big\|\frac{\partial u}{\partial \overline{z}}\Big\|_\infty\Big\}.$$

Let P be the orthogonal projection from $W^{1,2}(D)$ onto $\mathfrak{D}$, and P is an integral operator represented by

$$P(u)(z) = \int_D \frac{\partial u}{\partial w}\overline{\frac{\partial K_z(w)}{\partial w}}\mathrm{d}A, \quad u \in W^{1,2}(D), \tag{1}$$

where $K_z(w) = \sum_{k=1}^\infty \frac{\overline{z}^k w^k}{k}$ is the reproducing kernel of $\mathfrak{D}$. By the reproducing property of K_z, we can see that

$$f(z) = \langle f, K_z\rangle, \quad \forall\, f \in \mathfrak{D}. \tag{2}$$

Given $u \in W^{1,\infty}(D)$, the Toeplitz operator T_u with symbol u is the linear operator on $\mathfrak{D}$ defined by

$$T_u f = P(uf), \quad \forall\, f \in \mathfrak{D}. \tag{3}$$

We know that the Toeplitz operator $T_u : \mathfrak{D} \to \mathfrak{D}$ is always bounded for every $u \in W^{1,\infty}(D)$.

A bounded linear operator T on a Hilbert space is said to be hyponormal if its self-commutator $T^*T - TT^*$ is a positive operator. The hyponormality of the Toeplitz operators on the Hardy space has been studied by [1–8]. The corresponding problem for Toeplitz operators on the Bergman space has been studied by Hwang [9, 10], Lu and Liu [11], Lu and Shi [12] and others. Cowen [1] gave an elegant characterization of the hyponormality of Toeplitz operator with a bounded measurable symbol on the unit circle in the complex plane. In [10], Hwang and Lee gave some necessary conditions for the hyponormality of T_φ under certain assumptions about the coefficients of φ, where $\varphi = f + \overline{g}$, f and g are polynomials.

From the definition of the Dirichlet space, we know it has a direct connection with the Bergman space. However, the hyponormality of the Toeplitz operators on the Dirichlet space is very different from the one on the Bergman space. In this paper, we investigate the hyponormality of the Toeplitz operators on the Dirichlet space. In Section 2 we first prove that the Toeplitz operator with a harmonic symbol whose projection in the Dirichlet space is a polynomial is hyponormal if and only if its symbol is constant. Furthermore, we study the same property of Toeplitz operators with a special symbol.

In Section 3 we discuss Toeplitz operators with harmonic polynomial symbols on the harmonic Dirichlet space, and prove a Toeplitz operator is hyponormal if and only if its symbol is constant. That u is a harmonic polynomial means that u has a form as $u(z) = \sum_{l=1}^n a_l \bar{z}^l + \sum_{l=0}^m b_l z^l$.

2. Case on the Dirichlet space

Let $\mathcal{P}_0$ be the set consisting of all polynomials on the unit disk D in variables z and $\bar{z}$ which have the following form:

$$\sum_{j\geq -l}\sum_{l\geq 0} a_{l+j,l} z^{l+j}\bar{z}^l,$$

where j and l run over a finite subset of $\mathbb{Z}$ (the set of integers), and $\sum_{l\geq 0} a_{l+j,l} = 0$.

Let $\mathcal{A}_0$ denote the closure of $\mathcal{P}_0$ in $W^{1,2}(D)$, and let $\mathcal{A}$ denote $\mathcal{A}_0 + \mathbb{C}$. Then both $\mathcal{A}_0$ and $\mathcal{A}$ are orthogonal to $\mathfrak{D}$ and $\overline{\mathcal{D}}$, the space of the conjugates of functions in $\mathfrak{D}$. Then we have the following results about the Sobolev space [13].

Lemma 1 *Let $\mathcal{A}_0$ and $\mathcal{A}$ as above. Then*

1) $W^{1,2}(D) = \mathcal{A} \oplus \mathfrak{D} \oplus \overline{\mathcal{D}}$.

2) $\phi\mathcal{A}_0 \subset A_0$ *for* $\phi \in W^{1,\infty}(D)$.

For $\phi \in W^{1,2}(D)$, the Toeplitz operator T_ϕ is well defined on $\mathfrak{D} \cap W^{1,\infty}(D)$, a dense subspace of Dirichlet space. By Lemma 1 we write $\phi = u + \phi_0$ such that u is harmonic and $\phi_0 \in \mathcal{A}_0$. Then (2) of Lemma 1 implies that T_ϕ equals to the Toeplitz operator T_u. So we need only to discuss Toeplitz operators with harmonic symbols.

For a harmonic function $u \in W^{1,\infty}(D)$, let $u = u_+ + \bar{u}_-$ with u_+ and u_- being analytic. Then, by the definition of Toeplitz operator on the Dirichlet space, we have a bounded sesquilinear form

$$\langle T_u f, g\rangle = \langle uf, g\rangle = \langle u_+ f, g\rangle + \langle \bar{u}_- f, g\rangle, \quad \forall\ f, g \in \mathfrak{D}.$$

Now suppose that u is a harmonic function. If u_+ is bounded and $|u_+|^2 dA$ is a $\mathfrak{D}$-Carleson measure, then the first item of the right side of above equation is a bounded sesquilinear form [14], and the boundedness of u_- ensures the boundedness of the second item.

Let $H^\infty(D)$ be the space of all bounded analytic functions on D. Denote

$$\Omega = \{u \text{ is harmonic on } D : u = f + \bar{g}, f, g \in H^\infty(D), \text{ and } |f|^2 \mathrm{d}A \text{ is a } \mathfrak{D}\text{-Carleson measure}\}.$$

By the discussion above, the Toeplitz operator T_u is bounded for $u \in \Omega$.

Lemma 2 *We fix $m, k \geq 1$, then*

(a) $$T^*_{z^m}(z^k) = \begin{cases} \dfrac{k}{k-m} z^{k-m}, & \text{if } m < k, \\ 0, & \text{if } m \geq k; \end{cases}$$

(b) $$T^*_{\overline{z}^m}(z^k) = \frac{k}{k+m} z^{k+m};$$

(c) $$\|T_{z^m}(z^k)\|^2 - \|T^*_{z^m}(z^k)\|^2 = \begin{cases} \dfrac{-m^2}{k-m}, & \text{if } m < k, \\ k+m, & \text{if } m \geq k; \end{cases}$$

(d) $$\|T_{\overline{z}^m}(z^k)\|^2 - \|T^*_{\overline{z}^m}(z^k)\|^2 = \begin{cases} \dfrac{-m^2}{k+m}, & \text{if } m < k, \\ \dfrac{-k^2}{k+m}, & \text{if } m \geq k. \end{cases}$$

Proof (a) For any m and k, by (2) and (1) we have

$$\begin{aligned} T^*_{z^m}(z^k) &= \langle T^*_{z^m}(z^k), K_z\rangle = \langle z^k, T_{z^m} K_z\rangle = \langle z^k, z^m K_z\rangle = \int_D \frac{\partial w^k}{\partial w} \overline{\frac{\partial w^m K_z(w)}{\partial w}} \mathrm{d}A(w) \\ &= \int_D kmw^{k-1}\overline{w}^{m-1}\overline{K_z(w)}\mathrm{d}A(w) + \int_D kw^{k-1}\overline{w}^m \overline{\frac{\partial K_z(w)}{\partial w}} \mathrm{d}A(w). \end{aligned}$$

A simple calculation shows that

$$\int_D kmw^{k-1}\overline{w}^{m-1}\overline{K_z(w)}\mathrm{d}A(w)=\begin{cases}\dfrac{m}{k-m}z^{k-m}, & \text{if } m<k,\\ 0, & \text{if } m\geq k,\end{cases}$$

and

$$\int_D kw^{k-1}\overline{w}^{m}\overline{\frac{\partial K_z(w)}{\partial w}}\mathrm{d}A(w)=\begin{cases}z^{k-m}, & \text{if } m<k,\\ 0, & \text{if } m\geq k.\end{cases}$$

Hence, we have

$$T^*_{z^m}(z^k)=\begin{cases}\dfrac{k}{k-m}z^{k-m}, & \text{if } m<k,\\ 0, & \text{if } m\geq k.\end{cases}$$

The proof of (b) is similar to (a), so we can use (a) and (b) to obtain the results of (c) and (d) directly.

Now, we can state and prove our result.

Theorem 1 *Let $u\in\Omega$, such that the analytic part of u is a polynomial. Then T_u is hyponormal on $\mathfrak{D}$ if and only if u is a constant function.*

Proof The sufficiency is trivial. Now, we give the proof of the necessity.

Fix $k\geq 1$, we can obtain the following results:

(a) $\{T_{z^m}(z^k)\}_{m=0}^{\infty}$, $\{T_{\overline{z}^m}(z^k)\}_{m=1}^{\infty}$, $\{T^*_{z^m}(z^k)\}_{m=0}^{\infty}$, and $\{T^*_{\overline{z}^m}(z^k)\}_{m=1}^{\infty}$ are orthogonal sequences respectively;

(b) $T_{z^m}(z^k)$ and $T_{\overline{z}^l}(z^k)$ are orthogonal, for all $m\geq 0$, $l\geq 1$;

(c) $T^*_{z^m}(z^k)$ and $T^*_{\overline{z}^l}(z^k)$ are orthogonal, for all $m\geq 0$, $l\geq 1$.

The results above can be acquired by computation of inner product.

Suppose T_u is hyponormal on $\mathfrak{D}$, then we have

$$\|T_u(z^k)\|^2-\|T_u^*(z^k)\|^2\geq 0,\quad \forall\ k\geq 1.$$

By the hypothesis, write u as $u(z)=\sum_{l=1}^{\infty}b_l\overline{z}^l+\sum_{l=0}^{N}a_lz^l$, where $N\in\mathbb{N}$. By Lemma 2, we have

$$\begin{aligned}\|T_uz^k\|^2-\|T_u^*z^k\|^2&=\|\sum_{l=0}^{N}T_{a_lz^l}(z^k)+\sum_{l=1}^{\infty}T_{b_l\overline{z}^l}(z^k)\|^2-\|\sum_{l=0}^{N}T^*_{a_lz^l}(z^k)+\sum_{l=1}^{\infty}T^*_{b_l\overline{z}^l}(z^k)\|^2\\&=\sum_{l=0}^{N}(\|T_{a_lz^l}(z^k)\|^2-\|T^*_{a_lz^l}(z^k)\|^2)+\sum_{l=1}^{\infty}(\|T_{b_l\overline{z}^l}(z^k)\|^2-\|T^*_{b_l\overline{z}^l}(z^k)\|^2)\\&=\begin{cases}\displaystyle\sum_{l=1}^{N}\Big(\frac{-l^2|a_l|^2}{k-l}+\frac{-l^2|b_l|^2}{k+l}\Big)+\sum_{l=N+1}^{\infty}\frac{-k^2|b_l|^2}{k+l}, & \text{if } N<k,\\ \displaystyle\sum_{l=1}^{k-1}\Big(\frac{-l^2|a_l|^2}{k-l}+\frac{-l^2|b_l|^2}{k+l}\Big)+\sum_{l=k}^{N}|a_l|^2(k+l)+\sum_{l=k}^{\infty}\frac{-k^2|b_l|^2}{k+l}, & \text{if } N\geq k.\end{cases}\end{aligned}$$

Let $k = N + 1$. Then

$$\|T_u z^k\|^2 - \|T_u^* z^k\|^2 = \sum_{l=1}^{N} \Big(\frac{-l^2|a_l|^2}{k-l} + \frac{-l^2|b_l|^2}{k+l}\Big) + \sum_{l=N+1}^{\infty} \frac{-k^2|b_l|^2}{k+l} \geq 0,$$

so we can get $a_l = 0$ $(l = 1, 2, \ldots, N)$ and $b_l = 0$ $(l = 1, 2, \ldots)$ because of the hyponormality of T_u. Hence, we know that u is a constant function. This completes the proof. □

Theorem 2 *Let $u \in \Omega$ and write $u(z) = \sum_{l=1}^{\infty} b_l \overline{z}^l + \sum_{l=0}^{\infty} a_l z^l$ $(z \in D)$. If there exist a constant $\delta > 0$ and $N \in \mathbb{N}$, such that $|b_l| \geq \delta l |a_l|$, for $l \geq N$, then T_u is hyponormal on $\mathfrak{D}$ if and only if u is a constant function.*

Proof Similarly to the proof of Theorem 1, we have, for all $k \in \mathbb{N}$,

$$\|T_u z^k\|^2 - \|T_u^* z^k\|^2 = \sum_{l=1}^{k-1} \Big(\frac{-l^2|a_l|^2}{k-l} + \frac{-l^2|b_l|^2}{k+l}\Big) + \sum_{l=k}^{\infty} \Big(|a_l|^2(k+l) - \frac{k^2|b_l|^2}{k+l}\Big).$$

Let $k \geq N$ and $k \geq \frac{2}{\delta}$. We have $k^2|b_l|^2 \geq 4l^2|a_l|^2$ by hypothesis, and so

$$\sum_{l=k}^{\infty} \Big(|a_l|^2(k+l) - \frac{k^2|b_l|^2}{k+l}\Big) \leq \sum_{l=k}^{\infty} |a_l|^2\Big(k + l - \frac{4l^2}{k+l}\Big) \leq 0.$$

Since $\|T_u z^k\|^2 - \|T_u^* z^k\|^2 \geq 0$, we get

$$\sum_{l=1}^{N} \Big(\frac{-l^2|a_l|^2}{k-l} + \frac{-l^2|b_l|^2}{k+l}\Big) \geq 0,$$

then $a_l = b_l = 0$, for $l < k$. From the arbitrariness of k, we complete the proof of the theorem. □

3. Case on the harmonic Dirichlet space

In this section we discuss Toeplitz operators on the harmonic Dirichlet space. The harmonic Dirichlet space, denoted by $\mathfrak{D}_h$, is a closed subspace of the Sobolev space $W^{1,2}(D)$ consisting of all the harmonic functions. It is well known that

$$\mathfrak{D}_h = \overline{\mathfrak{D}} \oplus \mathbb{C} \oplus \mathfrak{D},$$

where $\overline{\mathfrak{D}} = \{\bar{f} : f \in \mathfrak{D}\}$, and $\mathfrak{D}_h$ is also a reproducing Hilbert space with reproducing kernel

$$R_z(w) = \overline{K_z(w)} + K_z(w) + 1 = \log \frac{1}{1 - z\overline{w}} + \log \frac{1}{1 - \bar{z}w} + 1. \tag{3}$$

Let Q be the orthogonal projection from $W^{1,2}(D)$ onto $\mathfrak{D}_h$. By means of the reproducing kernel R_z, the projection Q can be written as an integral operator

$$(Qf)(z) = \langle f, R_z \rangle = \int_D \frac{\partial f}{\partial w}(w) \frac{z}{1 - z\overline{w}} \mathrm{d}A(w) + \int_D \frac{\partial f}{\partial \overline{w}}(w) \frac{\bar{z}}{1 - \bar{z}w} \mathrm{d}A(w) + \int_D f \mathrm{d}A \tag{4}$$

for $f \in W^{1,2}(D)$. For $u \in W^{1,\infty}(D)$, the Toeplitz operator on the harmonic Dirichlet space with symbol u denoted by $\tilde{T}_u$ is the linear operator defined by

$$\tilde{T}_u f = Q(uf), \quad \forall\, f \in \mathfrak{D}_h.$$

It is easy to see that the Toeplitz operator $\tilde{T}_u$ is bounded for every $u \in W^{1,\infty}(D)$ (see [15]). If u is harmonic in D, let $u = u_+ + u_-$ with u_+ and u_- being analytic. Then, by (4), we have

$$\begin{aligned}\tilde{T}_u f(z) = Q(uf)(z) = &\int_D (u'_+(w)f(w) + u(w)f'_+(w))\frac{z}{1-z\bar{w}}\mathrm{d}A(w) + \\ &\int_D (\bar{u}'_-(w)f(w) + u(w)\bar{f}'_-(w))\frac{\bar{z}}{1-\bar{z}w}\mathrm{d}A(w) + \int_D uf\mathrm{d}A,\end{aligned} \tag{5}$$

where $f \in \mathfrak{D}_h$ and $f = f_+ + f_-$ with f_+ and f_- being analytic.

Lemma 3 *For $m, k \in \mathbb{N}$, we have*

(a) $\tilde{T}_{z^m}(z^k) = z^{k+m}$, $\tilde{T}_{\bar{z}^m}(z^k) = \begin{cases} z^{k-m}, & \text{if } m < k, \\ \frac{1}{k+1}, & \text{if } m = k, \\ \bar{z}^{m-k}, & \text{if } m > k; \end{cases}$

(b) $\tilde{T}^*_{z^m}(z^k) = \begin{cases} \dfrac{k}{k-m}z^{k-m}, & \text{if } m < k, \\ k, & \text{if } m = k, \\ \frac{k}{m-k}\bar{z}^{m-k}, & \text{if } m > k; \end{cases}$ $\quad \tilde{T}^*_{\bar{z}^m}(z^k) = \dfrac{k}{k+m}z^{k+m}$.

Proof (a) The proof is a direct calculation and can consult [15].

(b) For any m and k in $\mathbb{N}$, by (2) and (1) we have

$$\begin{aligned}\tilde{T}^*_{z^m}(z^k) &= \langle \tilde{T}^*_{w^m}(w^k), R_z\rangle = \langle w^k, \tilde{T}_{w^m}R_z\rangle = \langle w^k, w^m R_z\rangle \\ &= \int_D kw^{k-1}m\bar{w}^{m-1}\overline{R_z(w)}\mathrm{d}A(w) + \int_D kw^{k-1}\bar{w}^m\overline{\frac{\partial R_z(w)}{\partial w}}\mathrm{d}A(w).\end{aligned}$$

By the expression of R_z in (3), a simple calculation shows that

$$T^*_{z^m}(z^k) = \begin{cases} \dfrac{k}{k-m}z^{k-m}, & \text{if } m < k, \\ k, & \text{if } m = k, \\ \frac{k}{m-k}\bar{z}^{m-k}, & \text{if } m \geq k. \end{cases}$$

Similarly we can prove that $\tilde{T}^*_{\bar{z}^m}(z^k) = \dfrac{k}{k+m}z^{k+m}$.

Theorem 3 *Let u be a harmonic polynomial. Then $\tilde{T}_u$ is hyponormal on $\mathfrak{D}_h$ if and only if u is a constant function.*

Proof The sufficiency is trivial. Now, we give the proof of the necessity.

Suppose $\tilde{T}_u$ is hyponormal on $\mathfrak{D}_h$, then we have

$$\|\tilde{T}_u(z^k)\|^2 - \|\tilde{T}^*_u(z^k)\|^2 \geq 0, \quad \forall k \geq 1. \tag{6}$$

Let $u(z) = \sum_{l=1}^{N_1} b_l\bar{z}^l + \sum_{l=0}^{N_2} a_l z^l$, where $N_1, N_2 \in \mathbb{N}$. Fix $k > \max\{N_1, N_2\}$, by Lemma 3, we have

$$\|\tilde{T}_u(z^k)\|^2 = \Big\|\sum_{l=1}^{N_1} b_l\tilde{T}_{z^l}(z^k) + \sum_{l=1}^{N_2} a_l\tilde{T}_{\bar{z}^l}(z^k)\Big\|^2 = \Big\|\sum_{l=1}^{N_1} b_l z^{k-l} + \sum_{l=0}^{N_2} a_l z^{k+l}\Big\|^2$$

$$= \sum_{l=1}^{N_1} |b_l|^2(k-l) + \sum_{l=0}^{N_2} |a_l|^2(k+l),$$

and

$$\|\tilde{T}_u^* z^k\|^2 = \Big\| \sum_{l=1}^{N_1} \bar{b}_l \frac{k}{k+l} z^{k+l} + \sum_{l=0}^{N_2} \bar{a}_l \frac{k}{k-l} z^{k-l} \Big\|^2 = \sum_{l=1}^{N_1} \frac{|b_l|^2 k^2}{k+l} + \sum_{l=0}^{N_2} \frac{|a_l|^2 k^2}{k-l}.$$

Thus we have

$$\|\tilde{T}_u z^k\|^2 - \|\tilde{T}_u^* z^k\|^2 = \sum_{l=1}^{N_1} \frac{-|b_l|^2 l^2}{k+l} + \sum_{l=1}^{N_2} \frac{-|a_l|^2 l^2}{k-l}. \tag{7}$$

Using (6) and (7), we can get $b_l = 0$ $(l = 1, 2, \ldots, N_1)$ and $a_l = 0$ $(l = 1, 2, \ldots, N_2)$, hence, we know that u is a constant function. This completes the proof. □

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