A Note on the Exponential Diophantine Equation

\[(a^m - 1)(b^n - 1) = x^2\]

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Abstract Let \(a\) and \(b\) be fixed positive integers. In this paper, using some elementary methods, we study the diophantine equation \((a^m - 1)(b^n - 1) = x^2\). For example, we prove that if \(a \equiv 2 \pmod{6}\), \(b \equiv 3 \pmod{12}\), then \((a^n - 1)(b^m - 1) = x^2\) has no solutions in positive integers \(n, m\) and \(x\).

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1. Introduction

Let \(a\) and \(b\) be fixed positive integers. There are many works concerning the diophantine equation \((a^m - 1)(b^n - 1) = x^2\). In [5], Szalay proved that the diophantine equation \((2^n - 1)(3^n - 1) = x^2\) has no solutions in positive integers \(n\) and \(x\), \((2^n - 1)(5^n - 1) = x^2\) has the only solution \(n = 1, x = 2\) in positive integers \(n\) and \(x\), and \((2^n - 1)((2^k)^n - 1) = x^2\) has the only solution \(k = 2, n = 3, x = 21\) in positive integers \(k \geq 2, n\) and \(x\). In 2000, Hajdu and Szalay [1] proved the equation \((2^n - 1)(6^n - 1) = x^2\) has no solutions in positive integers \((n, x)\), while the only solutions to the equation \((a^n - 1)(a^{kn} - 1) = x^2\), with \(a > 1, k > 1, kn > 2\) are \((a, n, k, x) = (2, 3, 2, 21), (3, 1, 5, 22), (7, 1, 4, 120)\). In 2000, Walsh [6] proved that \((2^n - 1)(3^m - 1) = x^2\) has no solutions in positive integers \(n, m\) and \(x\).

Following these works, Luca and Walsh [4] showed that the diophantine equation \((a^k - 1)(b^k - 1) = x^n\) has finite solutions in positive integers \((k, x, n)\) with \(n > 1\). Moreover, they showed how one can determine all integers \((k, x, 2)\) of the equation above with \(k \geq 1\), for almost all pairs \((a, b)\) with \(2 \leq b < a \leq 100\). In 2009, Le [3] proved that if \(3 \mid b\), then \((2^n - 1)(b^n - 1) = x^2\) has no solutions in positive integers \(n\) and \(x\). Recently, Li and Lzalay [2] proved that if \(a \equiv 2 \pmod{6}\) and \(b \equiv 0 \pmod{3}\), then the equation \((a^n - 1)(b^n - 1) = x^2\) has no positive integer solution \((n, x)\).

In this paper, using some elementary methods, we obtain the following results:

**Theorem 1** If \(a \equiv 0 \pmod{2}\), \(b \equiv 15 \pmod{20}\), then the equation

\[(a^n - 1)(b^n - 1) = x^2\]

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A note on the exponential diophantine equation \((a^m - 1)(b^n - 1) = x^2\) has no solutions in positive integers \(n\) and \(x\).

**Theorem 2** If \(a \equiv 2 \pmod{6}\), \(b \equiv 3 \pmod{12}\), then the equation

\[
(a^n - 1)(b^m - 1) = x^2
\]

(2)

has no solutions in positive integers \(n, m\) and \(x\).

2. Proofs of Theorems

Let \(d\) be a positive integer which is not a square. It is well known that the Pell’s equation

\[x^2 - dy^2 = 1\]

has infinitely many positive solutions. If \((x_1, y_1)\) is the smallest positive integer solution, then for \(n = 1, 2, 3, \ldots\), define \(x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^n\). The pairs \((x_n, y_n)\) are all positive solutions of the Pell’s equation. Moreover, the \(x_n’s\) and \(y_n’s\) satisfy the following recurrence relations

\[x_{2n} = 2x_n^2 - 1, \quad x_{n+2} = 2x_1x_{n+1} - x_n,\]

(3)

and

\[y_{2n} = 2x_ny_n, \quad y_{n+2} = 2x_1y_{n+1} - y_n.\]

(4)

**Proof of Theorem 1** If Eq.(1) has a solution \((n, x)\), then we have

\[a^n - 1 = dy^2,\]

(5)

and

\[b^n - 1 = dz^2,\]

(6)

where \(d, y\) and \(z\) are positive integers satisfying \(dyz = x\), and \(d\) is square-free. Note that \(a \equiv 0 \pmod{2}\). By (5) we know that \(d\) is odd. Thus \(b^n - 1\) is properly divisible by an even power of \(2\). Hence \(b^n - 1 \equiv 3^n - 1 \equiv 0 \pmod{4}\), and we know that \(n\) must be even.

Let \((x_1, y_1)\) denote the smallest positive integer solution, and \(x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^n\) for \(n \geq 1\). By (4), if \(n\) is even, then \(y_n = 2x_n/2y_n/2\) is even. Since \((x_n, y_n) = 1\) \((n \geq 1)\), we have \(x_n\) is odd for all even values of \(n\). Hence

\[a^{n/2} + y\sqrt{d} = x_r + y_r \sqrt{d}\]

(7)

holds for some odd positive integer \(r\). By (3), we know that \(x_n\) is even for all odd positive integers \(n\). Thus

\[b^{n/2} + y\sqrt{d} = x_s + y_s \sqrt{d}\]

(8)

holds for some positive even integers. Let \(s = 2t\). Then by (3) we have \(b^{n/2} = x_{2t} = 2x^2 - 1 \equiv 0 \pmod{5}\). It follows that \(x^2 \equiv 3 \pmod{5}\), which is impossible.

This completes the proof of Theorem 1.

**Proof of Theorem 2** If Eq.(2) has a solution \((n, m, x)\), then we have

\[a^n - 1 = dy^2,\]

(9)
and

\[ b^m - 1 = dz^2, \tag{10} \]

where \( d, y \) and \( z \) are positive integers satisfying \( dyz = x \), and \( d \) is square-free. Since \( b \equiv 3 \pmod{12} \), by (10) we have \( dz^2 \equiv 2 \pmod{3} \), thus \( 3 \nmid d \), \( 3 \nmid z \), hence \( z^2 \equiv 1 \pmod{3} \), \( d \equiv 2 \pmod{3} \).

If \( 3 \nmid y \), then \( y^2 \equiv 1 \pmod{3} \), thus

\[ a^n = dy^2 + 1 \equiv 0 \pmod{3}, \]

hence \( m = 4 \). By (9), we know that \( d \) is odd, thus \( b^m - 1 \) is properly divisible by an even power of 2. Hence

\[ b^m - 1 \equiv 3^m - 1 \equiv 0 \pmod{4}, \]

and we know that \( m \) must be even.

Let \((x_1, y_1)\) denote the smallest positive integer solution, and \( x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^n \) for \( n \geq 1 \). By (4), if \( n \) is even, then \( y_n = 2x_n/2y_n/2 \) is even. Noting that \((x_n, y_n) = 1 \) \((n \geq 1)\), we have \( x_n \) is odd for all even values of \( n \). Hence

\[ a^{n/2} + y \sqrt{d} = x_r + y \sqrt{d}\] (11)

holds for some odd positive integer \( r \). By (3), we know that \( x_n \) is even for all odd positive integers \( n \), thus

\[ b^{m/2} + y \sqrt{d} = x_s + y \sqrt{d}\] (12)

holds for some positive even integers. Let \( s = 2t \). Then by (3) we have \( b^{m/2} = 2x_t^2 - 1 \equiv 0 \pmod{3} \). It follows that \( x_t^2 \equiv 2 \pmod{3} \), which is impossible.

This completes the proof of Theorem 2. \( \Box \)

References


