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On the Closure of Complex Exponential System in Weighted Banach Space

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Abstract In this paper, closure of the linear span on complex exponential system in weighted Banach space L^p_α is studied. Each function in the closure of complex exponential system can be extended to an entire function represented by Taylor-Dirichlet series.

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1. Introduction and main result

Let $E=\{e_k: k=1,2,\ldots\}$ be a system in a Banach space B, $\operatorname{span} E$ be the linear span of E, and $\overline{\operatorname{span}}\, E$ be the closure of span of E in B. If $\overline{\operatorname{span}}\, E\neq B$, E is called incomplete in B (see [1, 2]).

Suppose that $\alpha(t)$ is a continuous function (called a weight on $\mathbb{R}$) such that

$$\lim_{t\to+\infty} t^{-1}\alpha(t)=\infty,\quad a_0=\limsup_{t\to-\infty}|t|^{-1}|\alpha(t)|<\infty. \tag{1}$$

Given a weight $\alpha(t)$, let

$$C_\alpha=\{f\in C(\mathbb{R}): \lim_{|t|\to+\infty} f(t)e^{-\alpha(t)}=0\}$$

with the norm

$$\| f\|_\alpha=\sup\{|f(t)e^{-\alpha(t)}|: t\in\mathbb{R}\}$$

and

$$L^p_\alpha=\{f: \| f\|_{p,\alpha}=(\int_{-\infty}^{+\infty}|f(t)e^{-\alpha(t)}|^p\mathrm{d}t)^{\frac{1}{p}}<+\infty\},\quad 1\leq p<+\infty.$$

Thus, C_α and L^p_α equipped with these norms are weighted Banach spaces.

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In addition, assume that $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in the right half plane $\mathbb{C}_{a_0} = \{z = x + iy : x > a_0\}$ satisfying the following conditions:

$$a_1(\Lambda) = \sup_n |\theta_n| < \frac{\pi}{2} \tag{2}$$

and the space condition [3]

$$a_2(\Lambda) = \liminf_{n\to\infty} \frac{\inf\{\log|\lambda_k - \lambda_n| : k \neq n\}}{\log|\lambda_n|} > -\infty. \tag{3}$$

Let $M = \{m_n : n = 1, 2, \ldots\}$ be a sequence of positive integers and $q(r)$ be an increasing positive function on $[0, +\infty)$ with

$$a_3(q) = \limsup_{r\to+\infty} q(r)r^{-1}\log r < +\infty. \tag{4}$$

Suppose Λ, M and $q(r)$ satisfy the upper density condition [3]

$$D(q) = \limsup_{r\to+\infty} \frac{n(r + q(r)) - n(r)}{q(r)} < +\infty, \tag{5}$$

where $n(t) = \sum_{|\lambda_n|\le t} m_n$ is the counting function of (Λ, M). Define now

$$\lambda_\Lambda(r) = \begin{cases} 2\sum_{|\lambda_n|\le r} \frac{m_n\cos\theta_n}{|\lambda_n|}, & \text{if } \ r \ge |\lambda_1|, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Denote by $E(\Lambda, M)$ the complex exponential system of the form

$$E(\Lambda, M) = \{t^{k-1}e^{\lambda_n t} : k = 1, 2, \ldots, m_n; n = 1, 2, \ldots\}.$$

The condition (1) guarantees that $E(\Lambda, M)$ is a subset of C_α and L^p_α.

One of the most important problems in the approximation theory is the completeness of some special function system. Many results are obtained by the method of complex analysis. For example, the famous Müntz theorem [4] and the Bernstein problem [5–7] on weighted polynomial approximation were studied by many authors with complex methods. Recently, one of the authors [3, 8] and the others obtained some results about the completeness of the exponential system in some weighted Banach space. In [3], the author has obtained some results on incompleteness of $E(\Lambda, M)$ in C_α. In [9], the necessary and sufficient condition for $E(\Lambda, M)$ to be dense in L^p_α was obtained, but the another problem was not considered, i.e., if span $E(\Lambda, M)$ is not dense in L^p_α, what is the $\overline{\text{span}}E(\Lambda, M)$ in L^p_α? The aim of the present paper is to give a solution to this problem. Our main conclusion is as follows.

Theorem *Suppose $\alpha(t)$ is a continuous function on R satisfying (1) and convex function on $(t_0, +\infty)$ for some constant t_0, $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{\alpha_0}$ satisfying (2) and (3), $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of positive integers, and $q(r)$ is an increasing positive function on $[0, +\infty)$ such that (4) and (5) hold. If $E(\Lambda, M)$ is incomplete in L^p_α, then for each $f \in \overline{\text{span}}E(\Lambda, M)$, there exists an entire function $g(z)$ represented by a Taylor-Dirichlet series*

$$g(z) = \sum_{n=1}^{\infty}\sum_{k=0}^{m_n-1} a_{n,k}z^k e^{\lambda_n z} \tag{7}$$

such that $g(x) = f(x)$ almost everywhere for $x \in \mathbb{R}$.

2. Proof of Theorem

Hereafter, A denotes a positive constant, not necesarily the same at each occurrence in the paper.

In order to prove the Theorem, we need the following lemmas:

Lemma 1 ([3]) *Let $\beta(x)$ be a convex function on $[0, +\infty)$ and*

$$\beta^*(t) = \sup\{xt - \beta(x) : x > 0\}, \quad t \in \mathbb{R}$$

be the Legendre transform [10] (or the Young dual function) of $\beta(x)$. If $\lambda(r)$ is an increasing function on $[0, +\infty)$ satisfying

$$\lambda(R) - \lambda(r) \leq A(\log R - \log r + 1), \quad R > r > 1,$$

then there exists an analytic function $f(z) \neq 0$ in $\mathbb{C}_0 = \{z = x + iy : x > 0\}$ with

$$|f(z)| \leq A \exp\{Ax + \beta(x) - x\lambda(|z|)\}, \quad z = x + iy \in \mathbb{C}_0$$

if and only if there exists $a \in \mathbb{R}$ such that

$$\int_1^{+\infty} \frac{\beta^*(\lambda(t) + a)}{1 + t^2} \mathrm{d}t < +\infty.$$

Lemma 2 ([3]) *Suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{a_0}$ satisfying (2) and $M = \{m_n : n = 1, 2, \ldots\}$ is a sequence of distinct integers. If there exists a positive and increasing function $q(r)$ on $[0, \infty)$ such that (4) and (5) hold, then for each positive number b, the function*

$$G_b(z) = Q_b(z) \prod_{\operatorname{Re}\lambda_n > b} \left(\frac{1 - \frac{z}{\lambda_n}}{1 + \frac{z}{\overline{\lambda}_n}}\right)^{m_n} \exp\left(\frac{2zm_n \cos\theta_n}{|\lambda_n|}\right) \tag{8}$$

is meromorphic and analytic in the half-plane $\mathbb{C}_{-b} = \{z = x + iy : x > -b\}$ with zeros of order m_n at each point λ_n $(n = 1, 2 \ldots)$ and satisfies the following inequality

$$| G_b(z) | \leq \exp\{|x|\lambda(2r) + A|x| + A\}, \quad z \in \mathbb{C}_{-b}, \tag{9}$$

where

$$Q_b(z) = \prod_{|\operatorname{Re}\lambda_n| \leq b} \left(\frac{z - \lambda_n}{z + b + 1}\right)^{m_n}.$$

Moreover, for each positive constant A_0 and $\epsilon_0 > 0$,

$$|G_b(z)| \geq \exp\{x\lambda(r) - A|x| - A\}, \quad z \in C(A_0, \epsilon_0), \tag{10}$$

where $C(A_0, \epsilon_0) = \{z \in \mathbb{C}_{-b} : |z - \lambda_n| \geq \delta_n, n = 1, 2, \ldots\}$, $\delta_n = \epsilon_0 |\lambda_n|^{-A_0}$, $n = 1, 2, \ldots$.

Lemma 3 ([9]) *Let $\alpha(t)$ be a continuous function on $\mathbb{R}$ satisfying (1) and convex function on the real line $R = (t_0, +\infty)$ for some constant t_0. Suppose that $\Lambda = \{\lambda_n = |\lambda_n| e^{i\theta_n} : n = 1, 2, \ldots\}$ is a sequence of distinct complex numbers in $\mathbb{C}_{a_0}$ satisfying (2) and $M = \{m_n : n = 1, 2, \ldots\}$ is* *a sequence of distinct integers. If there exists a positive and increasing function $q(r)$ on $[0,\infty)$ such that (4) and (5) hold, then $E(\Lambda, M)$ is incomplete in L^p_α if and only if there exists a real number a such that*

$$J(a) = \int_o^\infty \frac{\alpha(\lambda(t)+a)}{1+t^2}\mathrm{d}t < \infty \tag{11}$$

holds, where $\lambda(r)$ is defined in (6).

Proof of Theorem For the proof of incompleteness of $E(\Lambda, M)$ in L^p_α, see Section 3.2 in [9]. By (3), there exist postive constants ϵ_0 and A_0 such that open disks $D_n = \{z : |z-\lambda_n| < \delta_n\}$ $(n = 1,2,\ldots)$ are disjoint, where $\delta_n = \frac{\epsilon_0}{|\lambda_n|^{A_0}} : n = 1,2,\ldots$. If $E(\Lambda, M)$ is incomplete in L^p_α, then by Lemma 3, there exists a real number a, such that (11) holds. If for any constant a, $J(a) < \infty$ (For example, if $\lambda(r)$ is bounded, then $J(a) < \infty$ for any constant a), then there exists an infinite sequence $\{r_n : n = 1,2,\ldots\}$ satisfying $r_{n+1} > 1 + 2r_n$ $(n = 1,2,\ldots)$ and

$$\int_{r_n}^{+\infty} \frac{\alpha(\lambda(r)+n)}{1+t^2}\mathrm{d}t < \frac{1}{2^n}, \quad n = 1,2,\ldots .$$

Thus there exists positive sequence $\tilde\Lambda = \{\tilde\lambda_n : n = 1,2,\ldots\}$ satisfying the following four conditions:

i) $\tilde\lambda(r)$ is unbounded in $(\tilde\lambda_1, \infty)$, where

$$\tilde\lambda(r) = 2\Sigma_{\tilde\lambda_n \le r}\frac{1}{\tilde\lambda_n}(r \ge \tilde\lambda_1);$$

ii) $\liminf_{n\to\infty}(\tilde\lambda_{n+1} - \tilde\lambda_n) > 0$;

iii) $\inf\{|\tilde\lambda_m - \tilde\lambda_n| : m,n = 1,2,\ldots\} > 0$;

iv)

$$\int_o^\infty \frac{\alpha(\lambda(t)+\tilde\lambda(t))}{1+t^2}\mathrm{d}t < \infty.$$

Therefore, without loss of generality, we may suppose $\lambda(r)$ is unbounded and (11) holds with $a = 0$. Let $\varphi(t)$ be an even function such that $\varphi(t) = \alpha(\lambda(t))$ with $t \ge 0$. By the proof of the necessity of Theorem in [9], there exists an analytic function $g(z)$ on $\mathbb{C}_0 = \{z = x+iy : x > 0\}$ such that $g(z)$ satisfies $\operatorname{Re} g(z) \ge \varphi(|z|)$. For any $b > 0$ and $\epsilon \in (0, \frac{\pi}{2})$, we have

$$\lim_{r\to\infty} r^{-1}\max\{\operatorname{Re} g(b+re^{i\theta}) : |\theta| \le \frac{\pi}{2} - \epsilon\} = 0. \tag{12}$$

Let $b > 2 + a_0$ and $\varphi_b(z)$ be defined by

$$\varphi_b(z) = \frac{G_b(z)}{(1+z+b)^N}\exp\{-g(z+b)\}, \tag{13}$$

where $G_b(z)$ is defined by (8), then there exists a positive constant A_2 such that

$$|\varphi_b(z)| \le \frac{1}{1+|z|^2}\exp\{\alpha^*(x-1) + A_2x\}, \quad x > -b, \tag{14}$$

where

$$\alpha^*(x) = \sup\{xt - \alpha(t) : t \in R\}, \quad x > x_0$$

is the Legendre transform [10] (or Young transform) of the convex function $\alpha(x)$. We may assume, without loss of generality, that $\alpha(0)=0$. As is known [10], $\alpha^*(0)=0$ and $(\alpha^*)^*=\alpha$. Hence for $t>t_0$,

$$\sup\{xt-\alpha^*(x):x\geq 0\}=\alpha(t).$$

Let $A_{n,j}$ be the coefficient of the principal part of the Laurent series of the function $\frac{e^{A_2z}}{\varphi_b(z)}$ in $D_n-\{\lambda_n\}$, i.e.,

$$\frac{e^{A_2z}}{\varphi_b(z)}=\sum_{j=1}^{m_n}\frac{A_{n,j}}{(z-\lambda_n)^j}+\tilde{\varphi}_n(z), \tag{15}$$

where $\tilde{\varphi}_n(z)$ is analytic in disk D_n. By Cauchy's formula,

$$A_{n,j}=\frac{1}{2\pi i}\int_{|z-\lambda_n|=\delta_n}\frac{e^{A_2z}}{\varphi_b(z)}(z-\lambda_n)^{j-1}\mathrm{d}z. \tag{16}$$

According to (10) and (13),

$$\max\{|A_{n,j}|:1\leq j\leq m_n\}\leq\exp\{-\mathrm{Re}\,\lambda_n\lambda(|\lambda_n|)+A\mathrm{Re}\,\lambda_n+A\}. \tag{17}$$

Note that A is independent of x and λ_n. Consider the analytic function

$$H_{n,k}(z)=\varphi_b(z)\exp\{-A_2z\}\sum_{l=1}^{m_n-k}\frac{A_{n,k+l}}{k!(z-\lambda_n)^l},\quad k=0,1,\dots,m_n;\ n=1,2,\dots,\ z\in\mathbb{C}_{-b}. \tag{18}$$

For every $x>-b$, combining (14) with (17) gives

$$|H_{n,k}(z)|\leq\frac{A}{1+|z|^2}\exp\{\alpha^*(x-1)-\mathrm{Re}\,\lambda_n\lambda(|\lambda_n|)+A\mathrm{Re}\,\lambda_n\}. \tag{19}$$

Let

$$h_{n,k}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}H_{n,k}(iy)e^{-iyt}\mathrm{d}y$$

be the Fourier transform of $H_{n,k}(iy)$. Then $h_{n,k}(t)$ is bounded and continuous on $(-\infty,+\infty)$. By Cauchy's formula,

$$h_{n,k}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}H_{n,k}(x+iy)e^{-(x+iy)t}\mathrm{d}y,\quad x>-b.$$

From (18) and the formula of Legendre transform $(\alpha^*)^*=\alpha$, one gets that

$$|h_{n,k}(t)e^{\alpha(t)}|\leq\exp\{A+A\mathrm{Re}\,\lambda_n-Re\lambda_n\lambda(|\lambda_n|)-|t|\}. \tag{20}$$

For every $x>-b$, the function $h_{n,k}(t)e^{xt}$ can be regarded as the Fourier transform of $H_{n,k}(x+iy)$, consquently,

$$H_{n,k}(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}h_{n,k}(t)e^{tz}\mathrm{d}z,\ \mathrm{Re}\,z>-b.$$

Next we will prove that

$$H_{n,k}^{(l)}(\lambda_j)=\begin{cases}1, & \text{if}\ \ n=j\ \ \text{and}\ \ k=l,\\ 0, & \text{otherwise.}\end{cases}$$

i.e.,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\lambda_j t} t^l h_{n,k}(t)\mathrm{d}t = \begin{cases} 1, & \text{if } \ n=j \ \text{ and } \ k=l, \\ 0, & \text{otherwise.}\end{cases}$$

It is obvious that if $j \neq n$, then $H_{n,k}^{(l)}(\lambda_j) = 0$ $(l = 0, 1, \ldots, m_j - 1)$. If $j = n$, then by (17), one gets that, for $z \in D_n$

$$\begin{aligned} H_{n,k}(z) &= \varphi_b(z) e^{-A_2 z} \sum_{l=k+1}^{m_n} \frac{A_{n,l}}{k!(z-\lambda_n)^l} \\ &= \varphi_b(z) e^{-A_2 z} \frac{(z-\lambda_n)^k}{k!} \left(\frac{e^{A_2 z}}{\varphi_b(z)} - \sum_{l=1}^{k} \frac{A_{n,l}}{(z-\lambda_n)^l} - \varphi_n(z)\right) \\ &= \frac{(z-\lambda_n)^k}{k!} + \sum_{l=m_n}^{+\infty} B_{n,l}(z-\lambda_n)^l, \end{aligned}$$

where $k = 0, 1, \ldots, m_n - 1$; $n = 1, 2, \ldots$. Thus we obtain

$$H_{n,k}^{(l)}(\lambda_j) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\lambda_j t} t^l h_{n,k}(t)\mathrm{d}t = \delta_{nj}\delta_{kl}.$$

Define a linear function $T_{n,k}$ on $\mathrm{span}E(\Lambda, M)$ by

$$T_{n,k}(P) = a_{n,k} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sum a_{j,l} h_{n,k}(t) t^l e^{\lambda_j t}\mathrm{d}t$$

for each exponential polynomial $P(t) = \sum a_{j,l} t^l e^{\lambda_j t} \in \mathrm{span}\, E(\Lambda, M)$. By (20),

$$|T_{n,k}(P)| \leq 2 \parallel P \parallel_{L^p_\alpha} \exp\{A + A\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_n \lambda(|\lambda_n|)\}.$$

Hence, $T_{n,k}$ is a bounded linear functional on $E(\Lambda, M)$ and can be extended to a bounded linear functional (denoted by $\overline{T}_{n,k}$) on L^p_α, by Hahn-Banach theorem with

$$\parallel \overline{T}_{n,k} \parallel = \parallel T_{n,k} \parallel \leq C_n = 2\exp\{A + A\mathrm{Re}\,\lambda_n - \mathrm{Re}\,\lambda_n\lambda(|\lambda_n|)\}. \tag{21}$$

If $f \in \overline{\mathrm{span}}E(\Lambda, M)$, then there exists a sequence of exponential polynomials

$$P_j(t) = \sum_{n=1}^{j} \sum_{k=0}^{m_n - 1} a^j_{n,k} t^k e^{\lambda_n t} \in \mathrm{span}\, E(\Lambda, M)$$

such that

$$\lim_{j\to\infty} \parallel f - P_j \parallel_{p,\alpha} = 0.$$

Since

$$|\overline{T}_{n,k}(f)| \leq \parallel \overline{T}_{n,k} \parallel \parallel f \parallel_{p,\alpha}$$

and $\lambda(r)$ is unbounded on $(0, \infty)$, (21) implies that the function

$$g(z) = \sum_{n=1}^{\infty} \sum_{k=0}^{m_n - 1} a_{n,k} z^k e^{\lambda_n z}$$

is an entire function, where $a_{n,k} = \overline{T}_{n,k}(f)$. Note that

$$|a_{n,k} - a^j_{n,k}| = |\overline{T}_{n,k}(f) - \overline{T}_{n,k}(P_j)| \leq C_n \parallel f - P_j \parallel_{p,\alpha}.$$

For any real number a and b $(a < b)$, one gets that

$$\begin{aligned}
&(\int_a^b |(f(t)-g(t))e^{-\alpha(t)}|^p dt)^{\frac{1}{p}} \leq \| f - P_j \|_{p,\alpha} + \| P_j - g \|_{p,\alpha} \\
&\leq \| f - P_j \|_{p,\alpha} + \sum_{n=1}^{j} \sum_{k=0}^{m_n-1} |a_{n,k} - a_{n,k}^j| (\int_a^b e^{\mathrm{Re}\,\lambda_n t - p\alpha(t)} t^{kp} \mathrm{d}t)^{\frac{1}{p}} + \\
&\quad \sum_{j+1}^{+\infty} \sum_{k=0}^{m_n-1} |a_{n,k}| (\int_a^b e^{\mathrm{Re}\,\lambda_n t - p\alpha(t)} t^{kp} \mathrm{d}t)^{\frac{1}{p}} \\
&\leq \| f - P_j \|_{p,\alpha} [1 + \sum_{n=1}^{j} \sum_{k=0}^{m_n-1} e^{\mathrm{Re}\,\lambda_n b} (1 + |a| + |b|)^{K(\Lambda)} (\int_{-\infty}^{+\infty} e^{-p\alpha(t)} \mathrm{d}t)^{\frac{1}{p}}] + \\
&\quad \sum_{j+1}^{+\infty} \sum_{k=0}^{m_n-1} \| f \|_{p,\alpha} \|\bar{T}_{n,k}\| e^{\mathrm{Re}\,\lambda_n b} (1 + |a| + |b|)^{K(\Lambda)} (\int_{-\infty}^{+\infty} e^{-p\alpha(t)} \mathrm{d}t)^{\frac{1}{p}}.
\end{aligned}$$

Let $j \to \infty$. (21) and (22) imply that $f(t) = g(t)$ a.e for $t \in \mathbb{R}$. This completes the proof of Theorem. $\square$

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