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Uniform Attractors for a Non-Autonomous Brinkman-Forchheimer Equation

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Abstract This paper is concerned with the three-dimensional non-autonomous Brinkman-Forchheimer equation. By Galerkin approximation method, we give the existence and uniqueness of weak solutions for non-autonomous Brinkman-Forchheimer equation. And we investigate the asymptotic behavior of the weak solution, the existence and structures of the (H,H)-uniform attractor and (H,V)-uniform attractor. Then we prove that an L^2-uniform attractor is actually an H^1-uniform attractor.

Keywords Galerkin approximation; uniform attractor; non-autonomous Brinkman-Forchheimer equation.

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1. Introduction

Consider now the non-autonomous Brinkman-Forchheimer equation:

$$\begin{cases} u_t-\nu\Delta u+au+b|u|^{\beta}u+\nabla p=g(t,x), & \text{in } \Omega\times(\tau,T);\\ \nabla\cdot u=0, & \text{in } \Omega\times(\tau,T);\\ u(x,\tau)=u_\tau(x), & \text{in } \Omega;\\ u(x,t)=0, & \text{in } \partial\Omega\times(\tau,T), \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $u=(u_1,u_2,u_3)$ is the fluid velocity vector, ν is the Brinkman coefficient, $a>0$ is the Darcy coefficient, $b>0$ is the Forchheimer coefficient, p is the pressure, and $\beta>1$ is a constant.

The model equations (Brinkman, Darcy and Forchheimer equations) describing the flow in a porous medium have been extensively studied in [1], and several papers have been published [2–10]. We should note that most of these papers have been focused on the question of continuous dependence of solutions on the coefficients ν,b. In [11] and [12], Davut, Ouyang and Yang proved the existence of global attractor in H_0^1 for autonomous Brinkman-Forchheimer equation, respectively, with respect to initial data $u_0\in V$.

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In this paper, we suppose the external force $g(t,x)$ is uniformly bounded in $L^2(\Omega)$ with respect to $t \in \mathbb{R}$, i.e., there exists a positive constant K, such that,

$$\sup_{t\in\mathbb{R}} \| g(t,x) \|_{L^2(\Omega)} \leq K, \tag{2}$$

then $g(t) \in L^2_b(\mathbb{R}, L^2(\Omega))$. And furthermore, suppose the weak differential of g with respect to t, denoted by $h(t)$, is in the space $L^2_b(\mathbb{R}, L^2(\Omega))$. Here $L^2_b(\mathbb{R}, L^2(\Omega))$ is the translation bounded subspace in $L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))$, i.e., $g(t) \in L^2_b(\mathbb{R}, L^2(\Omega))$,

$$\| g \|^2_{L^2_b} = \| g \|^2_{L^2_b(\mathbb{R}, L^2(\Omega))} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \| g \|^2_{L^2(\Omega)} \,\mathrm{d}s < +\infty. \tag{3}$$

In this paper, we focus on the existence and the structures of the (H,H) and (H,V)-uniform attractor. First, by the Galerkin approximation method, we give the existence of weak solutions for the non-autonomous three dimensional Brinkman-Forchheimer equation. After that, we explore the asymptotic behavior of the solutions. The existence and structures of the (H,H)-uniform attractor and (H,V)-uniform attractor are obtained. Finally, the asymptotic smoothing effect of the solutions is addressed.

The mathematical setting of our problem is similar to that of the Navier-Stokes equations. Let us introduce the following spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \text{div} u = 0\},\ H = \text{cl}_{(L^2(\Omega))^3}\mathcal{V}, V = \text{cl}_{(H_0^1(\Omega))^3}\mathcal{V},$$

where cl_X denotes the closure in the space X. H and V endowed, respectively, with the inner products

$$(u,v) = \int_\Omega u \cdot v \mathrm{d}x, \quad u, v \in H,$$

and

$$((u,v)) = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i \mathrm{d}x, \quad u, v \in V,$$

and norm $|\cdot|_2 = (\cdot,\cdot)^{1/2}, \|\cdot\| = ((\cdot,\cdot))^{1/2}$.

In this paper, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and we use $|\cdot|_p$ to denote the norm in $\mathbf{L}^p(\Omega)$.

Let $\tilde{P}$ be the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto H. Then applying $\tilde{P}$ to (1), we obtain

$$\begin{aligned} &\frac{\partial u}{\partial t} + \nu A u + a u + B(u) = g, \\ &u(\tau) = u_\tau, \end{aligned} \tag{4}$$

where $A = \tilde{P}(-\Delta)$ is the Stokes operator with the domain $D(A) = (H^2(\Omega))^3 \cap V$ and $B(u) = \tilde{P}F(u)$, while $F(u) = b|u|^\beta u$.

Throughout this paper, we use the following notations: let X be a Banach space, X^* be the dual space of X, $|u|$ the modular of u, $(\cdot,\cdot)$ be the inner product in $L^2(\Omega)$, and $\langle\cdot,\cdot\rangle$ be the duality product between X and X^*, and C an arbitrary positive constant, which may be different from line to line.

2. Existence and uniqueness of weak solution

The following lemma is a compactness result, whose proof can be found in [13].

Lemma 1 *Let X_0, X be Hilbert spaces satisfying a compact imbedding $X_0 \hookrightarrow X$. Let $0 < \gamma \leq 1$ and $\{v_j\}_{j=1}^{\infty}$ be a sequence in $L^2(\mathbb{R}; X_0)$ satisfying*

$$\sup_j \Big(\int_{-\infty}^{+\infty} \| v_j \|_{X_0}^2 \, \mathrm{d}t \Big) < \infty, \ \sup_j \Big(\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \| \hat{v}_j \|_X^2 \, \mathrm{d}\tau \Big) < \infty,$$

where $\hat{v}(\tau) = \int_{-\infty}^{+\infty} v(t) \exp(-2\pi i \tau t) \mathrm{d}t$ is the Fourier transformation of $v(t)$ on the time variable. Then there exists a subsequence of $\{v_j\}_{j=1}^{\infty}$ which converges strongly in $L^2(\mathbb{R}; X)$ to some $v \in L^2(\mathbb{R}; X)$.

Lemma 2 ([14]) *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n \times \mathbb{R}$. Given a sequence $\{g_n\}$ with $\{g_n\} \in L^q(\mathcal{O})$ and $1 < q < \infty$. Assume that $\| g_n \|_{L^q(\mathcal{O})} \leq C$, where C is independent of n, $g_n \to g$ $(n \to \infty)$ almost everywhere in $\mathcal{O}$, and $g \in L^q(\mathcal{O})$. Then $g_n \rightharpoonup g$ $(n \to \infty)$ weakly in $L^q(\mathcal{O})$.*

Theorem 1 *For any τ, $T \in \mathbb{R}$, suppose Ω is a bounded domain of $\mathbb{R}^3$, $g(t) \in L_b^2(\mathbb{R}, L^2(\Omega))$, and $u_\tau \in H$. Then there exists a unique solution $u(\cdot) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \cap L^{\beta+2}(\tau, T; \mathbf{L}^{\beta+2}(\Omega))$.*

Proof We employ the Galerkin approximation to prove the theorem. For simplicity, we take $\tau = 0$, and $u(x, 0) = u_0(x)$. Since V is separable and $\mathcal{V}$ is dense in V, there exists a sequence $\omega_1, \omega_2, \ldots, \omega_m$ of elements of $\mathcal{V}$, which is free and total in V. For each m we define an approximate solution u_m as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t) \omega_i(x),$$

and

$$(u_m'(t), \omega_j) + \nu(\nabla u_m(t), \nabla \omega_j) + a(u_m(t), \omega_j) + (b|u_m|^\beta u_m(t), \omega_j) = (g(t), \omega_j), \tag{5}$$

$t \in [0, T]$, $j = 1, 2, \ldots, m$, and $u_{0m} \to u_0$ in H, as $m \to \infty$.

Multiplying on both sides of (5) by $g_{jm}(t)$ and summing over $j = 1, \ldots, m$, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u_m|_2^2 + \nu \| u_m \|^2 + a|u_m|_2^2 + b|u_m|_{\beta+2}^{\beta+2} \leq \frac{1}{2a} |g(t)|_2^2 + \frac{a}{2} |u_m|_2^2,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m|_2^2 + 2\nu \| u_m \|^2 + a|u_m|_2^2 + 2b|u_m|_{\beta+2}^{\beta+2} \leq \frac{1}{a} |g(t)|_2^2, \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} |u_m|_2^2 + a|u_m|_2^2 \leq \frac{1}{a} |g(t)|_2^2. \tag{7}$$

By Gronwall's Lemma, we obtain

$$\begin{aligned} |u_m(t)|_2^2 &\leq |u_m(0)|_2^2 e^{-at} + \frac{1}{a} \int_0^t e^{-a(t-s)} |g(s)|_2^2 \mathrm{d}s \\ &\leq |u_m(0)|_2^2 e^{-at} + C \| g \|_{L_b^2}^2, \end{aligned} \tag{8}$$

and

$$\int_0^t e^{-a(t-s)} |g(s)|_2^2 \mathrm{d}s \leq \int_{t-1}^t e^{-a(t-s)} |g(s)|_2^2 \mathrm{d}s + \int_{t-2}^{t-1} e^{-a(t-s)} |g(s)|_2^2 \mathrm{d}s + \cdots$$

$$
\begin{aligned}
&\le \int_{t-1}^{t} |g(s)|_2^2 \mathrm{d}s + e^{-a}\int_{t-2}^{t-1} |g(s)|_2^2 \mathrm{d}s + e^{-2a}\int_{t-3}^{t-2} |g(s)|_2^2 \mathrm{d}s + \cdots \\
&\le (1+e^{-a}+e^{-2a}+\cdots)\parallel g \parallel_{L_b^2}^2 \\
&\le \frac{1}{1-e^{-a}} \parallel g \parallel_{L_b^2}^2 \\
&\le C \parallel g \parallel_{L_b^2}^2 .
\end{aligned}
$$

Integrating (6) in s from 0 to T, $T \ge 0$, we obtain

$$
\begin{aligned}
&\sup_{s\in[0,T]} |u_m(s)|_2^2 + 2\nu \int_0^T \parallel u_m(s) \parallel^2 \mathrm{d}s + a\int_0^T |u_m(s)|_2^2 \mathrm{d}s + 2b\int_0^T |u_m(s)|_{\beta+2}^{\beta+2} \mathrm{d}s \\
&\le \frac{1}{a}\int_0^T |g(s)|_2^2 \mathrm{d}s + |u_{0m}|_2^2. \qquad (9)
\end{aligned}
$$

From (8) and (9), we deduce that the sequence $\{u_m\}$ is a bounded set of $L^\infty(0,T;H)\cap L^2(0,T;V)\cap L^{\beta+2}(0,T;\mathbf{L}^{\beta+2}(\Omega))$.

Denote by $\tilde{u}_m$ the function from $\mathbb{R}$ into V, which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. Similarly, we prolong $g_{im}(t)$ to $\mathbb{R}$ by defining $\tilde{g}_{im}(t) = 0$ for $t \in \mathbb{R}\backslash[0,T]$. The Fourier transforms on time variable of $\tilde{u}_m$ and $\tilde{g}_{im}$ are denoted by $\hat{\tilde{u}}_m$ and $\hat{\tilde{g}}_{im}$, respectively.

Note that the approximate solution $\tilde{u}_m$ satisfies

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}_m,\omega_j) = &- \nu(\nabla\tilde{u}_m,\nabla\omega_j) - (a\tilde{u}_m,\omega_j) - (b|\tilde{u}_m|^\beta\tilde{u}_m,\omega_j) + (\tilde{g},\omega_j) + \\
&(u_{0m},\omega_j)\delta_0 - (u_m(T),\omega_j)\delta_T, \quad j = 1,2,\dots,m, \qquad (10)
\end{aligned}
$$

where δ_0 and δ_T are the Dirac distributions at 0 and T, respectively.

Taking the Fourier transform about the time variable in (10) gives

$$
2\pi i\tau(\hat{\tilde{u}}_m,\omega_j) = (\hat{\tilde{h}}_m,\omega_j) - (b\widehat{|\tilde{u}_m|^\beta\tilde{u}_m},\omega_j) + (u_{0m},\omega_j) - (u_m(T),\omega_j)\exp(-2\pi iT\tau), \qquad (11)
$$

where $\hat{\tilde{h}}_m$ denotes the Fourier transform of $\tilde{h}_m$,

$$
(\tilde{h}_m,\omega_j) = (\tilde{g},\omega_j) - \nu(\nabla\tilde{u}_m,\nabla\omega_j) - (a\tilde{u}_m,\omega_j).
$$

Multiplying (11) by $\hat{\tilde{g}}_{jm}(\tau)$ and summing the results for $j = 1,\dots,m$, one finds that

$$
2\pi i\tau|\hat{\tilde{u}}_m(\tau)|_2^2 = (\hat{\tilde{h}}_m,\hat{\tilde{u}}_m) - b(\widehat{|\tilde{u}_m|^\beta\tilde{u}_m},\hat{\tilde{u}}_m) + (u_{0m},\hat{\tilde{u}}_m) - (u_m(T),\hat{\tilde{u}}_m)\exp(-2\pi iT\tau). \qquad (12)
$$

For any $v \in L^2(0,T;V)\cap L^{\beta+2}(0,T;\mathbf{L}^{\beta+2}(\Omega))$, we have

$$
(h_m(t),v) = (g(t),v) - \nu(\nabla u_m,\nabla v) - (au_m,v) \le C(|g(t)|_2 + \parallel u_m \parallel + |u_m|_2) \parallel v \parallel .
$$

It follows that for any given $T > 0$

$$
\int_0^T \parallel h_m(t) \parallel_{V'} \mathrm{d}t \le \int_0^T C(|g(t)|_2 + \parallel u_m \parallel + |u_m|_2)\mathrm{d}t \le C,
$$

and hence

$$
\sup_{s\in\mathbb{R}} \parallel \hat{\tilde{h}}_m(s) \parallel_{V'} \le \int_0^T \parallel h_m(t) \parallel_{V'} \mathrm{d}t \le C. \qquad (13)
$$

Moreover,

$$\int_0^T \Big| |u_m|^\beta u_m \Big|_{\frac{\beta+2}{\beta+1}} \mathrm{d}t \le \int_0^T |u_m|_{\beta+2}^{\beta+1} \mathrm{d}t \le C,$$

which implies that

$$\sup_{s\in\mathbb{R}} \Big| \widehat{|u_m|^\beta u_m}(s) \Big|_{\frac{\beta+2}{\beta+1}} \le C. \tag{14}$$

From (9), we have

$$|u_m(0)|_2 \le C, \quad |u_m(T)|_2 \le C. \tag{15}$$

So we deduce from (13)–(15) that

$$|\tau||\hat{\tilde{u}}_m(\tau)|_2^2 \le C(\| \hat{\tilde{u}}_m(\tau) \| + |\hat{\tilde{u}}_m(\tau)|_{\beta+2}). \tag{16}$$

For any γ fixed, $0 < \gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \le C \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \quad \forall \tau \in \mathbb{R}.$$

Thus

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{\tilde{u}}_m(\tau)|_2^2 \mathrm{d}\tau \le& C \int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}} |\hat{\tilde{u}}_m(\tau)|_2^2 \mathrm{d}\tau \\ \le& C \int_{-\infty}^{+\infty} |\hat{\tilde{u}}_m(\tau)|_2^2 \mathrm{d}\tau + C \int_{-\infty}^{+\infty} \frac{\| \hat{\tilde{u}}_m(\tau) \|}{1+|\tau|^{1-2\gamma}} \mathrm{d}\tau + \\ & C \int_{-\infty}^{+\infty} \frac{|\hat{\tilde{u}}_m(\tau)|_{\beta+2}}{1+|\tau|^{1-2\gamma}} \mathrm{d}\tau. \end{aligned} \tag{17}$$

Thanks to the Parseval equality and (9), the first integral on the right-hand side of (17) is bounded uniformly on m.

By the Schwarz inequality, Parseval equality and (9), we have

$$\int_{-\infty}^{+\infty} \frac{\| \hat{\tilde{u}}_m \|}{1+|\tau|^{1-2\gamma}} \mathrm{d}\tau \le \Big(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{(1+|\tau|^{1-2\gamma})^2} \Big)^{\frac{1}{2}} \Big(\int_0^T \| u_m(\tau) \|^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} \le C$$

for $0 < \gamma < \frac{1}{4}$.

Similarly, when $0 < \gamma < \frac{1}{2(\beta+2)}$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{|\hat{\tilde{u}}_m(\tau)|_{\beta+2}}{1+|\tau|^{1-2\gamma}} \mathrm{d}\tau \le& \Big(\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{(1+|\tau|^{1-2\gamma})^{\frac{\beta+2}{\beta+1}}} \Big)^{\frac{\beta+1}{\beta+2}} \Big(\int_{-\infty}^{+\infty} |\hat{\tilde{u}}_m(\tau)|_{\beta+2}^{\beta+2} \mathrm{d}\tau \Big)^{\frac{1}{\beta+2}} \\ \le& C \Big(\int_{-\infty}^{+\infty} |\tilde{u}_m(\tau)|_{\beta+2}^{\frac{\beta+2}{\beta+1}} \mathrm{d}\tau \Big)^{\frac{\beta+1}{\beta+2}} \\ \le& C T^{\frac{\beta}{\beta+2}} \Big(\int_0^T |u_m(\tau)|_{\beta+2}^{\beta+2} \mathrm{d}\tau \Big)^{\frac{1}{\beta+2}}. \end{aligned}$$

It follows from (17) that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{\tilde{u}}_m(\tau)|_2^2 \mathrm{d}\tau \le C. \tag{18}$$

Now, since $F(u_m)u_m = b|u_m|^{\beta+2} \leq b|u_m|^{\beta+2} + \delta|u_m|^2$, and $\delta > 0$ is a positive constant, we have $|F(u_m)| \leq C(|u_m|^{\beta+1} + |u_m|)$. Since $\mathbf{L}^{\beta+2}(\Omega) \hookrightarrow \mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega)$, we obtain

$$\begin{aligned}\int_0^T \int_\Omega |F(u_m)|^{\frac{\beta+2}{\beta+1}} \mathrm{d}x\mathrm{d}t &\leq C\int_0^T \int_\Omega [|u_m|^{\beta+1} + |u_m|]^{\frac{\beta+2}{\beta+1}} \mathrm{d}x\mathrm{d}t \\ &\leq C\int_0^T \int_\Omega |u_m|^{\beta+2}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_\Omega |u_m|^{\frac{\beta+2}{\beta+1}}\mathrm{d}x\mathrm{d}t \\ &\leq C\int_0^T \int_\Omega |u_m|^{\beta+2}\mathrm{d}x\mathrm{d}t + C\int_0^T \int_\Omega |u_m|^{\beta+2}\mathrm{d}x\mathrm{d}t.\end{aligned}$$

From (9) we know that $\{u_m\}$ is bounded in $L^{\beta+2}(0,T;\mathbf{L}^{\boldsymbol{\beta+2}}(\boldsymbol{\Omega}))$, so $\{F(u_m)\}$ is bounded in $L^{\frac{\beta+2}{\beta+1}}(0,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$.

Since $\forall v \in L^2(0,T;V)$, we have

$$\begin{aligned}\int_0^T \int_\Omega -\nu\Delta u_m \cdot v\mathrm{d}x\mathrm{d}t &= \nu\int_0^T \int_\Omega \nabla u_m \cdot \nabla v\mathrm{d}x\mathrm{d}t \\ &\leq \nu\Big(\int_0^T \| u_m \|^2\Big)^{\frac{1}{2}}\Big(\int_0^T \| v \|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

It follows from (9) that $\{u_m\}$ is bounded in $L^2(0,T;V)$, so $\{-\nu\Delta u_m\} \in L^2(0,T;V')$. Therefore, by taking a subsequence when necessary, we can assume that there exists a function $u(\cdot) \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+2}(0,T;\mathbf{L}^{\beta+2}(\Omega))$ such that $u_m(s) \rightharpoonup u(s)$ weakly in $L^2(0,T;V)$, weakly in $L^{\beta+2}(0,T;\mathbf{L}^{\beta+2}(\Omega))$, and weak-star in $L^\infty(0,T;H)$ as $n \to \infty$. So, as $n \to \infty$, $-\nu\Delta u_m(s) \rightharpoonup -\nu\Delta u(s)$ weakly in $L^2(0,T;V')$, and $\partial_t u_m(s) \rightharpoonup \partial_t u(s)$ weakly in $L^{\frac{\beta+2}{\beta+1}}(0,T;H^{-s}(\Omega))$ for some $s > 0$. Assume $F(u_m(s)) \rightharpoonup \eta(s)$ weakly in $L^{\frac{\beta+2}{\beta+1}}(0,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$ for some $\eta(s) \in L^{\frac{\beta+2}{\beta+1}}(0,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$. Passing to the limit in $(1)_1$ with respect to u_m, we obtain the equality

$$\partial_t u - \nu\Delta u + au + \eta = g(t)$$

in the space $L^{\frac{\beta+2}{\beta+1}}(0,T;H^{-s}(\Omega))$.

By Lemma 1 and (18), there exists a subsequence of $\{u_m\}_{m=1}^\infty$, still denoted by itself, such that $u_m \to u$ strongly in $L^2(0,T;H)$, and so $u_m(x,s) \to u(x,s)$ for almost every $(x,s) \in \Omega\times[0,T]$ as $m \to \infty$. Since $F(u) \in C^0(\mathbb{R})$, $F(u_m(x,s)) \to F(u(x,s))$ $(m \to \infty)$ for almost every $(x,s) \in \Omega \times [0,T]$. On the other hand, the sequence $F(u_m)$ is bounded in $L^{\frac{\beta+2}{\beta+1}}(0,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$. From Lemma 2, we conclude that $F(u_m) \rightharpoonup F(u)$ $(m \to \infty)$ weakly in $L^{\frac{\beta+2}{\beta+1}}(0,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$, hence $\eta(s) = F(u(x,s))$. So $u \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+2}(0,T;\mathbf{L}^{\beta+2}(\Omega))$ is a solution of (1).

Finally, let us verify the uniqueness of the solution. Let u_1, u_2 be two solutions of (1) with the initial data $u_1|_{t=0} = u_1(0)$, $u_2|_{t=0} = u_2(0)$, respectively. Subtracting the corresponding to equation $(4)_1$, we obtain

$$\partial_t(u_1 - u_2) + \nu A(u_1 - u_2) + a(u_1 - u_2) + B(u_1) - B(u_2) = 0. \tag{19}$$

Taking the inner product of (19) with $u_1 - u_2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_1 - u_2|_2^2 + \nu \| u_1 - u_2 \|^2 + a|u_1 - u_2|_2^2 = -\langle F(u_1) - F(u_2), u_1 - u_2\rangle. \tag{20}$$

Since the function $F(u)$ is monotone, $(F(u)-F(v),u-v)\geq 0$, and hence from (20) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_1-u_2|_2^2+2a|u_1-u_2|_2^2\leq 0.$$

Using the Gronwall Lemma, we obtain

$$|u_1-u_2|_2^2\leq e^{-2at}|u_1(0)-u_2(0)|_2^2.$$

So the uniqueness of the solution is proved.

Lemma 3 *Let $F(u)=\alpha|u|^{\beta}u$. Then*

(i) F is continuously differentiable in $\mathbb{R}^3$ and for $u=(u_1,u_2,u_3)$ in $\mathbb{R}^3$ the Jacobian matrix is given by:

$$F'(u)=\alpha|u|^{\beta-2}\begin{pmatrix}\beta u_1^2+|u|^2 & \beta u_1u_2 & \beta u_1u_3\\ \beta u_1u_2 & \beta u_2^2+|u|^2 & \beta u_2u_3\\ \beta u_1u_3 & \beta u_2u_3 & \beta u_3^2+|u|^2\end{pmatrix}.$$

Further, $F'(u)$ is positive definite and for any $u,v,w\in\mathbb{R}^3$:

$$|(F'(u)v)\cdot w|\leq c|u|^{\beta}|v||w|,$$

where c is a positive constant depending on β and α.

(ii) F is monotonic in $\mathbb{R}^3$, i.e., for any $u,v\in\mathbb{R}^3$:

$$(F(u)-F(v),u-v)\geq 0.$$

Proof (i) can be obtained by simple calculations, and (ii) is an immediate consequence of (i).

3. Preliminaries about processes

Let Σ be a metric space, X,Y be two spaces, and $Y\subset X$ continuously. $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is a family of processes in Banach space X. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X. $\mathbb{R}^\tau=[\tau,+\infty)$.

Definition 1 *A set $B_0\in\mathcal{B}(Y)$ is said to be (X,Y)-uniformly absorbing for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if, for any $\tau\in\mathbb{R}$ and every $B\in\mathcal{B}(X)$, there exists $t_0=t_0(\tau,B)\geq\tau$ such that $\bigcup_{\sigma\in\Sigma}U_\sigma(t,\tau)B\subset B_0$ for all $t\geq t_0$. A set P belonging to Y is said to be (X,Y)-uniformly attracting for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if, for an arbitrary fixed $\tau\in\mathbb{R}$ and $B\in\mathcal{B}(X)$,* $\lim_{t\to+\infty}(\sup_{\sigma\in\Sigma}\mathrm{dist_Y(U_\sigma(t,\tau)B,P)})=0$.

Definition 2 *A closed set $A_\Sigma\subset Y$ is said to be (X,Y)-uniformly attractor of the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if it is (X,Y)-uniformly attracting and it is contained in any closed (X,Y)-uniformly attracting set $\mathcal{A}'$ of the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}:\mathcal{A}_\Sigma\subset\mathcal{A}'$.*

Definition 3 *Define the uniform ω-limit set of B by $\omega_{\tau,\,\Sigma}(B)=\bigcap_{t\geq\tau}\overline{\bigcup_{\sigma\in\Sigma}\bigcup_{s\geq t}U_\sigma(s,\tau)B}$. This can be characterized by the following: $y\in\omega_{\tau,\,\Sigma}(B)\Leftrightarrow$ there are sequences $\{x_n\}\subset B,\{\sigma_n\}\subset\Sigma,\{t_n\}\subset\mathbb{R}^\tau$, $t_n\to\infty$ such that $U_{\sigma_n}(t_n,\tau)x_n\to y$ $(n\to\infty)$.*

Definition 4 *A family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ possessing a compact (X,Y)-uniformly absorbing set is called (X,Y)-uniformly compact. And a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is called (X,Y)-uniformly asymptotically compact if it possesses a compact (X,Y)-uniformly attracting set.*

Now let us consider the most interesting case where $U_\sigma(t,\tau)$ satisfies the following cocycle property: there is a dynamical system $\{T(h)|h\geq 0\}$ on Σ such that:

(C1) $T(h)\Sigma=\Sigma$, $\forall h\in\mathbb{R}^+$; (C2) translation identity:

$$U_\sigma(t+h,\tau+h)=U_{T(h)\sigma}(t,\tau),\quad \forall\sigma\in\Sigma,\ t\geq\tau,\ \tau\in\mathbb{R},\ h\geq 0.$$

Definition 5 *The kernel $\mathcal{K}$ of the process $\{U(t,\tau)\}$ acting on X consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$: $\mathcal{K}=\{u(\cdot)|U(t,\tau)u(\tau)=u(t),\mathrm{dist}(u(t),u(0))\leq C_u,\ \forall t\geq\tau,\ \tau\in\mathbb{R}\}$. The set $\mathcal{K}(s)=\{u(s)|u(\cdot)\in\mathcal{K}\}$ is said to be kernel section at time $t=s, s\in\mathbb{R}$.*

Definition 6 *$\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ is said to be $(X\times\Sigma,Y)$-weakly continuous if, for any fixed $t\geq\tau$, $\tau\in\mathbb{R}$, the mapping $(u,\sigma)\to U_\sigma(t,\tau)u$ is weakly continuous from $X\times\Sigma$ to Y.*

Assumption 1 *Let Σ be a weakly compact set and $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ be $(X\times\Sigma,Y)$-weakly continuous.*

Theorem 2 ([15]) *Under (C1), (C2) and Assumption 1 with $\{T(h)\}_{h\geq 0}$, which is a weakly continuous semigroup, if $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ acting on X is (X,Y)-uniformly asymptotically compact, then it possesses an (X,Y)-uniform attractor $\mathcal{A}_\Sigma$, $\mathcal{A}_\Sigma$ is compact in Y, and attracts the bounded subset of X in the topology of Y; moreover,*

$$\mathcal{A}_\Sigma=\omega_{\tau,\ \Sigma}(B_0)=\bigcup_{\sigma\in\Sigma}\mathcal{K}_\sigma(s),\quad \forall s\in\mathbb{R},$$

where B_0 is a bounded neighborhood of the compact (X,Y)-uniformly attracting set in Y, i.e., B_0 is a bounded (X,Y)-uniformly absorbing set of $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$, and $\mathcal{K}_\sigma(s)$ is the section at $t=s$ of kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$ with symbol $\sigma\in\Sigma$. Furthermore, $\mathcal{K}_\sigma$ is nonempty for all $\sigma\in\Sigma$.

4. $(H,\ H)$-uniform attractor

We denote by $L^{2,w}_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$ the space $L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ endowed with a local weak convergence topology. Let $\mathcal{H}_w(g)$ be the hull of g in $L^{2,w}_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$, i.e., the closure of the set $\{g(h+s)|h\in\mathbb{R}\}$ in $L^{2,w}_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$, and $g(x,s)\in L^2_b(\mathbb{R};L^2(\Omega))$.

Proposition 1 ([16]) *If X is reflective separable, $\varphi\in L^2_b(\mathbb{R};X)$, then*

(i) For all $\varphi_1\in\mathcal{H}_w(\varphi)$, $\|\varphi_1\|^2_{L^2_b}\leq\|\varphi\|^2_{L^2_b}$;

(ii) The translation group $\{T(h)\}$ is weakly continuous on $\mathcal{H}_w(\varphi)$;

(iii) $T(h)\mathcal{H}_w(\varphi)=\mathcal{H}_w(\varphi)$ for $h\geq 0$;

(iv) $\mathcal{H}_w(\varphi)$ is weakly compact.

Because of the uniqueness of solution, the following translation identity holds

$$U_\sigma(t+h,\tau+h)=U_{T(h)\sigma}(t,\tau),\quad \forall\sigma\in\mathcal{H}_w(g),\ t\geq\tau,\ \tau\in\mathbb{R},\ h\geq 0. \tag{21}$$

Theorem 3 *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ corresponding to problem (i) is $(H\times\mathcal{H}_w(g),H)$-weakly continuous, and $(H\times\mathcal{H}_w(g),V\cap\mathbf{L}^{\beta+2}(\Omega))$-weakly continuous.*

Proof For any fixed t_1 and τ, $t_1\geq\tau$, $\tau\in\mathbb{R}$, let $u_{\tau m}\rightharpoonup u_\tau$ weakly in H, and $\sigma_m\rightharpoonup\sigma_0$ weakly in $\mathcal{H}_w(g)$ as $m\to\infty$. Denote by $u_m(t)=U_{\sigma_m}(t,\tau)u_{\tau m}$. The same estimate for u_m given in the Galerkin approximations in Section 2 is valid for the $u_m(t)$ here. Therefore, for some subsequence $\{n\}\subset\{m\}$ and $w(t)$, we have for any t_1, $\tau\leq t_1\leq T$, $u_n(t_1)\rightharpoonup w(t_1)$ weakly in H and $V\cap\mathbf{L}^{\beta+2}(\Omega)$. And the sequence $\{u_n(s)\}$, $\tau\leq s\leq T$, is bounded in the class $L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\cap L^{\beta+2}(\tau,T;\mathbf{L}^{\beta+2}(\Omega))$. Denote by $\eta_1(s)$, and $\eta_0(s)$ the weak limits of $-\Delta u_n(s)$ and $F(u_n(s))$ in $L^2(\tau,T;V')$ and $L^{\frac{\beta+2}{\beta+1}}(\tau,T;\mathbf{L}^{\frac{\beta+2}{\beta+1}}(\Omega))$, respectively. So we get the equation for $w(s)$

$$\partial_t w+\nu\eta_1+aw+\eta_0=\sigma_0.$$

By the same method as in Theorem 1.3.1 in [14] and the proof of the Theorem 1, we know that $\eta_1=-\Delta w$ and $\eta_0=F(w)$, which means that $w(s)$ in $L^\infty(\tau,T;H)\cap L^2(\tau,T;V)\cap L^{\beta+2}(\tau,T;\mathbf{L}^{\beta+2}(\Omega))$ is the weak solution of (1) with initial condition u_τ. Due to the uniqueness of the solution, we state that $U_{\sigma_n}(t_1,\tau)u_{\tau n}\rightharpoonup U_{\sigma_0}(t_1,\tau)u_\tau$ weakly in H and $V\cap\mathbf{L}^{\beta+2}(\boldsymbol{\Omega})$. For any other subsequences $\{u_{\tau n'}\}$ and $\{\sigma_{n'}\}$, we have $u_{\tau n'}\rightharpoonup u_\tau$ weakly in H and $\sigma_{n'}\rightharpoonup\sigma_0$. By the same process we obtain the analogous relation $U_{\sigma n'}(t_1,\tau)u_{\tau n'}\rightharpoonup U_{\sigma_0}(t_1,\tau)u_\tau$ weakly in H and $V\cap\mathbf{L}^{\beta+2}(\Omega)$ holds. Then it can be easily seen that for any weakly convergent initial sequence $\{u_{\tau m}\}\in H$ and weakly convergent sequence $\{\sigma_m\}\in\mathcal{H}_w(g)$, we have $U_{\sigma_m(t_1,\tau)}u_{\tau m}\rightharpoonup U_{\sigma_0}(t_1,\tau)u_\tau$ weakly in H and $V\cap\mathbf{L}^{\beta+2}(\Omega)$.

Theorem 4 *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ corresponding to problem (i) has a bounded $(H,V\cap\mathbf{L}^{\beta+2}(\Omega))$-uniformly absorbing set.*

Proof Taking the inner product of $(\mathrm{i})_1$ with u, with respect to an external force $\sigma\in\mathcal{H}_w(g)$, yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2+\nu\parallel u\parallel^2+a|u|_2^2+b|u|_{\beta+2}^{\beta+2}=\int_\Omega\sigma(t)u\leq\frac{1}{2a}|\sigma(t)|_2^2+\frac{a}{2}|u|_2^2, \tag{22}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2+2\nu\parallel u\parallel^2+a|u|_2^2+2b|u|_{\beta+2}^{\beta+2}\leq\frac{1}{a}|\sigma(t)|_2^2. \tag{23}$$

Applying the Gronwall Lemma, we get

$$\begin{aligned}|u(t)|_2^2&\leq|u_\tau|_2^2e^{-a(t-\tau)}+\frac{1}{a}\int_\tau^t e^{-a(t-s)}|\sigma(s)|_2^2\mathrm{d}s\\&\leq|u_\tau|_2^2e^{-a(t-\tau)}+C\parallel g\parallel_{L_b^2}^2.\end{aligned}$$

From this inequality, we know that the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ has an (H,H)-uniformly absorbing set, i.e., for an arbitrary bounded subset B in H, there exists $T_1=T_1(B,\tau)$

such that

$$|u(t)|_2^2 \leq \rho_0(\| g \|_{L_b^2}^2), \text{ for all } t \geq T_1, \ u_\tau \in B, \sigma \in \mathcal{H}_w(g). \tag{24}$$

Taking $t \geq T_1$, integrating (23) on $[t, t+1]$ and combining with (24), we have

$$\int_t^{t+1} [\| u(s) \|^2 + |u(s)|_2^2 + |u(s)|_{\beta+2}^{\beta+2}] \mathrm{d}s \leq C(\rho_0, \| g \|_{L_b^2}^2), \text{ for all } t \geq T_1. \tag{25}$$

On the other hand, taking the inner product of $(\mathrm{i})_1$ with u_t yields

$$|u_t|_2^2 + \frac{\nu}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| u \|^2 + \frac{a}{2} \frac{\mathrm{d}}{\mathrm{d}t} |u|_2^2 + \frac{b}{\beta+2} \frac{\mathrm{d}}{\mathrm{d}t} |u|_{\beta+2}^{\beta+2} = \int_\Omega \sigma(t) u_t \mathrm{d}x \leq \frac{1}{2}|u_t|_2^2 + \frac{1}{2}|\sigma(t)|_2^2.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\| u \|^2 + |u|_2^2 + |u|_{\beta+2}^{\beta+2}] \leq C|\sigma(t)|_2^2. \tag{26}$$

From (25) and (26), by virtue of the uniform Gronwall Lemma, we get

$$\| u(t) \|^2 + |u(t)|_2^2 + |u(t)|_{\beta+2}^{\beta+2} \leq \rho, \text{ for all } t \geq T_1 + 1, \tag{27}$$

where ρ is a positive constant. From (27), we get the $(H, V \cap \mathbf{L}^{\beta+2}(\boldsymbol{\Omega}))$-uniformly absorbing set and thus complete the proof. □

From Theorem 4 and the compactness of the Sobolev embedding $V \hookrightarrow H$, and Theorem 2 we have the following result:

Corollary 1 *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma \in \mathcal{H}_w(g)}$ generated by (i) with initial data $u_\tau \in H$ has an (H, H)-uniform attractor $\mathcal{A}_0$, which is compact in H and attracts every bounded subset of H in the topology of H. Moreover,*

$$\mathcal{A}_0 = \omega_{\tau, \mathcal{H}_w(g)}(B_0) = \bigcup_{\sigma \in \mathcal{H}_w(g)} \mathcal{K}_\sigma(s), \ \forall s \in \mathbb{R},$$

where B_0 is the (H, H)-uniformly absorbing set in H, and $\mathcal{K}_\sigma(s)$ is the section at $t = s$ of kernel $\mathcal{K}_\sigma$ of the processes $\{U_\sigma(t, \tau)\}$ with symbol $\sigma \in \mathcal{H}_w(g)$.

5. (H, V)-uniform attractor

In this section, we prove the existence of the (H, V)-uniform attractor. For this purpose, first we will give a priori estimate about u_t endowed with an H-norm.

Lemma 4 *For any bounded subset $B \subset H$, any $\tau \in \mathbb{R}$ and $\sigma \in \mathcal{H}_w(g)$, there exists a positive constant $T = T(B, \tau) \geq \tau$, and a positive constant ρ_1, such that*

$$|u_t(s)|_2^2 \leq \rho_1, \ \textit{for any } u_\tau \in B, \ s \geq T, \ \sigma \in \mathcal{H}_w(g),$$

where $u_t(s) = \frac{\mathrm{d}}{\mathrm{d}t}(U_\sigma(t,\tau)u_\tau)|_{t=s}$ and ρ_1 is a positive constant which is independent of B and σ.

Proof By differentiating $(\mathrm{i})_1$ with the external force σ in time, we get

$$u_{tt} - \nu \Delta u_t + a u_t + F'(u) u_t = \sigma'(t).$$

Taking the inner product of above equation with u_t yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2 + \nu \parallel u_t \parallel^2 + a|u_t|_2^2 = -\int_\Omega (F'(u)u_t)\cdot u_t \mathrm{d}x + \int_\Omega \sigma'(t)u_t \mathrm{d}x.$$

By Lemma 3, $-\int_\Omega (F'(u)u_t)\cdot u_t \mathrm{d}x$ is non-positive definite, hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_t|_2^2 \le \frac{1}{4a}|\sigma'(t)|_2^2 + 2a|u_t|_2^2. \tag{28}$$

Taking the inner product of (i)$_1$ with u_t yields

$$\begin{aligned}&|u_t|_2^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel u \parallel^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \frac{b}{\beta+2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+2}^{\beta+2}\\ &= \int_\Omega \sigma(t)u_t \mathrm{d}x \le \frac{1}{2}|\sigma(t)|_2^2 + \frac{1}{2}|u_t|_2^2.\end{aligned}$$

Therefore,

$$\frac{1}{2}|u_t|_2^2 + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t} \parallel u \parallel^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_2^2 + \frac{b}{\beta+2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{\beta+2}^{\beta+2} \le \frac{1}{2}|\sigma(t)|_2^2. \tag{29}$$

Integrating (29) from t to $t+1$, and according to Theorem 4, we have

$$\int_t^{t+1} |u_t|_2^2 \le C, \tag{30}$$

for t large enough.

Combining (28) with (30), and using the uniform Gronwall Lemma, we get

$$|u_t|_2^2 \le \rho_1$$

for t large enough, where ρ_1 is a positive constant independent of σ.

Theorem 5 *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ corresponding to problem (1) with initial data $u_\tau \in H$ is (H,V)-uniformly asymptotically compact, i.e., there exists a compact uniformly attracting set in V, which attracts any bounded subset $B \subset H$ in the topology of V.*

Proof Let B_0 be an (H,V)-uniformly absorbing set obtained in Theorem 4. Then we need only to show that: for any $\{u_{\tau n}\} \subset B_0$, $\{\sigma_n\} \subset \mathcal{H}_w(g)$ and $t_n \to \infty$, $\{U_{\sigma_n}(t_n,\tau)u_{\tau n}\}_{n=1}^\infty$ is precompact in V.

In fact, from Corollary 1, we know that $\{U_{\sigma_n}(t_n,\tau)u_{\tau n}\}_{n=1}^\infty$ is precompact in H. Without loss of generality, we assume that $\{U_{\sigma_n}(t_n,\tau)u_{\tau n}\}_{n=1}^\infty$ is a Cauchy sequence in H. Now, we prove that

$$\{U_{\sigma_n}(t_n,\tau)u_{\tau n}\}_{n=1}^\infty \text{ is a Cauchy sequence in } V. \tag{31}$$

Let $u_n^{\sigma_n}(t_n) = U_{\sigma_n}(t_n,\tau)u_{\tau n}$. We have

$$\begin{aligned}&\nu \parallel u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m) \parallel^2 = \nu(Au_n^{\sigma_n}(t_n) - Au_m^{\sigma_m}(t_m), u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m))\\ &\quad = (-\frac{\mathrm{d}}{\mathrm{d}t}u_n^{\sigma_n}(t_n) + \frac{\mathrm{d}}{\mathrm{d}t}u_m^{\sigma_m}(t_m) - au_n^{\sigma_n}(t_n) + au_m^{\sigma_m}(t_m) -\\ &\qquad B(u_n^{\sigma_n}(t_n)) + B(u_m^{\sigma_m}(t_m)) + \sigma_n - \sigma_m, u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m))\\ &\quad \le \int_\Omega \left|\frac{\mathrm{d}}{\mathrm{d}t}u_n^{\sigma_n}(t_n) - \frac{\mathrm{d}}{\mathrm{d}t}u_m^{\sigma_m}(t_m)\right| |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)| + a\int_\Omega |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)|^2 +\end{aligned}$$

$$\int_\Omega |\sigma_n - \sigma_m|\ |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)|$$
$$\leq \Big|\frac{\mathrm{d}}{\mathrm{d}t}u_n^{\sigma_n}(t_n) - \frac{\mathrm{d}}{\mathrm{d}t}u_m^{\sigma_m}(t_m)\Big|_2 |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)|_2 + |\sigma_n - \sigma_m|_2 |u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)|_2 +$$
$$a|u_n^{\sigma_n}(t_n) - u_m^{\sigma_m}(t_m)|_2^2,$$

which, combined with Lemma 4, yields (31) immediately.

Theorem 6 *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ corresponding to problem (i) with initial data $u_\tau \in H$ has an (H,V)-uniform attractor $\mathcal{A}_1$, where $\mathcal{A}_1$ is compact in V and attracts every bounded subset B of H in the topology of V. Moreover,*

$$\mathcal{A}_1 = \omega_{\tau,\mathcal{H}_w(g)}(B_0) = \bigcup_{\sigma\in\mathcal{H}_w(g)} \mathcal{K}_\sigma(s), \quad \forall s \in \mathbb{R},$$

where B_0 is the (H,V)-uniformly absorbing set, and $\mathcal{K}_\sigma(s)$ is the section at $t = s$ of kernel $\mathcal{K}_\sigma$ of the processes $\{U_\sigma(t,\tau)\}$ with symbol $\sigma \in \mathcal{H}_w(g)$.

6. Asymptotic smoothing effect

Theorem 7 *The (H,H)-uniform attractor $\mathcal{A}_0$ is equivalent with the (H,V)-uniform attractor $\mathcal{A}_1$, i.e., $\mathcal{A}_0 = \mathcal{A}_1$.*

Proof First, let us prove $\mathcal{A}_0 \subset \mathcal{A}_1$. Since $\mathcal{A}_1$ is bounded in V and the imbedding $V \hookrightarrow H$ is continuous, we see $\mathcal{A}_1$ is bounded in H. Also we deduce from Theorem 6 that $\mathcal{A}_1$ attracts uniformly all bounded sets of H and thus $\mathcal{A}_1$ can be regarded as a bounded uniformly attracting set for $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(g)}$ in H. By the minimality property of $\mathcal{A}_0$, we obtain $\mathcal{A}_0 \subset \mathcal{A}_1$.

Now, let us prove $\mathcal{A}_1 \subset \mathcal{A}_0$. First, we want to prove $\mathcal{A}_0$ is (H,V)-uniform attracting for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$. That is to say, we will prove for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(H)$,

$$\lim_{t\to+\infty} \big(\sup_{\sigma\in\Sigma} \mathrm{dist}_V(U_\sigma(t,\tau)B, \mathcal{A}_0)\big) = 0. \tag{32}$$

Suppose (32) is not true. Then there are $\tau \in \mathbb{R}$, $B \in \mathcal{B}(H)$, $\varepsilon_0 > 0$, $\sigma_n \in \mathcal{H}_w(g)$ and $t_n \to +\infty$ as $n \to +\infty$, such that, for all $n \geq 1$,

$$\mathrm{dist}_V(U_{\sigma_n}(t_n,\tau)B, \mathcal{A}_0) \geq 2\varepsilon_0,$$

which implies that, there exists $v_n \in B$ such that

$$\mathrm{dist}_V(U_{\sigma_n}(t_n,\tau)v_n, \mathcal{A}_0) \geq \varepsilon_0. \tag{33}$$

By Theorem 5, there are $w \in V$ and a subsequence of $U_{\sigma_n}(t_n,\tau)v_n$ (not relabeled) such that

$$U_{\sigma_n}(t_n,\tau)v_n \to w \text{ in } V. \tag{34}$$

On the other hand, by Corollary 1, there are $v \in H$ and a subsequence of $U_{\sigma_n}(t_n,\tau)v_n$ (not relabeled) such that

$$U_{\sigma_n}(t_n,\tau)v_n \to v \text{ in } H. \tag{35}$$

By (34) and (35), we find that $v = w$, and hence by (34) we have

$$U_{\sigma_n}(t_n, \tau)v_n \to v \text{ in } V. \tag{36}$$

Since $\mathcal{A}_0$ attracts B in H by Corollary 1, we get

$$\lim_{n\to+\infty} \text{dist}_H(U_{\sigma_n}(t_n, \tau)v_n, \mathcal{A}_0) = 0. \tag{37}$$

By (35), (37) and the compactness of $\mathcal{A}_0$ in H, we must have $v \in \mathcal{A}_0$, which along with (36) shows that

$$\lim_{n\to+\infty} \text{dist}_V(U_{\sigma_n}(t_n, \tau)v_n, \mathcal{A}_0) \le \lim_{n\to+\infty} \text{dist}_V(U_{\sigma_n}(t_n, \tau)v_n, v) = 0, \tag{38}$$

a contradiction with (33). So $\mathcal{A}_0$ is (H, V)-uniform attracting for the family of processes $\{U_\sigma(t, \tau)\}_{\sigma\in\Sigma}$. By minimality property of $\mathcal{A}_1$, we obtain $\mathcal{A}_1 \subset \mathcal{A}_0$.

Theorem 7 shows that the L^2-uniform attractor is actually the H^1-uniform attractor.

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