Journal of Mathematical Research with Applications
Jan., 2012, Vol. 32, No. 1, pp. 108–118
DOI:10.3770/j.issn:2095-2651.2012.01.012
Http://jmre.dlut.edu.cn

The Strong Law of Large Numbers for $\tilde{\rho}$-Mixing Random Variables

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Abstract In this paper we present some results for the general strong laws of large numbers of $\tilde{\rho}$-mixing random variables by a maximal inequality of Utev and Peligrad. These results extend and improve the related known works in the literature.

Keywords strong laws of large numbers; $\tilde{\rho}$-mixing random variables; Utev and Peligrad's maximal inequality.

MR(2010) Subject Classification 60F15

1. Introduction

Throughout this paper, we suppose that $\{\Omega, \Im, P\}$ is a probability space, and all random variables are assumed to be defined on $\{\Omega, \Im, P\}$. For a sequence of random variables $\{X_n, n \geq 1\}$, we denote $\Im_S = \sigma(X_n : n \in S \subset \mathbb{N})$. Given two σ-subalgebras $\Im_1, \Im_2 \subset \Im$, denote

$$\rho(\Im_1, \Im_2) = \sup\{|\mathrm{corr}(\zeta, \eta)|, \zeta \in L_2(\Im_1), \eta \in L_2(\Im_2)\},$$

where the correlation coefficient is defined in the usual way

$$\mathrm{corr}(\zeta, \eta) = \frac{E(\zeta\eta) - E\zeta E\eta}{\sqrt{\mathrm{Var}(\zeta)\mathrm{Var}(\eta)}}$$

and by $L_2(\Im)$ we denote the space of all $\Im$-measurable random variables ζ such that $E(\zeta^2) < \infty$.

Stein [1] introduced the following coefficients of dependence (with slightly different notations):

$$\tilde{\rho}(k) = \sup\{\rho(\Im_S, \Im_T) : \text{ all finite subsets } S, T \subset \mathbb{N} \text{ such that } \mathrm{dist}(S, T) \geq k\}, k \geq 0.$$

Obviously, $0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1, k \geq 0$, and $\tilde{\rho}(0) = 1$.

Definition *A sequence of random variables $\{X_n, n \geq 1\}$ are said to be a $\tilde{\rho}$-mixing sequence if there exists $k \in N$ such that $\tilde{\rho}(k) < 1$.*

The notion of $\tilde{\rho}$-mixing assumption is similar to ρ-mixing, but they are quite different from each other. A number of publications are devoted to $\tilde{\rho}$-mixing sequence. We refer to Bradley [2, 3] for the central limit theorem, Bryc and Smolenski [4] for moment inequalities and almost sure convergence, Gan [5], Gut and Peligrad [6] and Wu [7, 8] for almost sure convergence, Qiu

Received April 3, 2010; Accepted January 10, 2011
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and Gan [9, 10] for complete convergence, Qiu and Gan [11] for Hájeck-Rényi inequality and strong law of large numbers, Utev and Peligrad [12] for maximal inequalities and the invariance principle, Yang [13] for moment inequalities and strong law of large numbers.

Various limit properties for sums of independent random variables have been studied by many authors, more precisely, Chung [14] proved the following result:

Theorem A *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$. Let g be a positive, even and continuous function on $\mathbb{R}$ such that*

$$\frac{g(x)}{x} \nearrow, \quad \frac{g(x)}{x^2} \searrow, \quad \text{as} \quad |x| \to \infty.$$

If

$$\sum_{n=1}^{\infty} Eg(|X_n|)/g(n) < \infty,$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{n} = 0 \quad a.s.$$

Teicher [15] proved that:

Theorem B *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$. Suppose that*

(i) $\sum_{n=2}^{\infty} (EX_n^2/n^4) \sum_{j=1}^{n-1} EX_j^2 < \infty$;

(ii) $\sum_{i=1}^{n} EX_i^2/n^2 \to 0$;

(iii) $\sum_{n=1}^{\infty} P(|X_n| > C_n) < \infty$, *for some positive constants* $\{C_n, n \geq 1\}$ *with*

$$\sum_{n=1}^{\infty} C_n^2 EX_n^2/n^4 < \infty.$$

Then

$$\sum_{i=1}^{n} X_i/n \to 0 \quad a.s..$$

In this paper, we study strong law of large numbers for $\tilde{\rho}$-mixing random variables inspired by Kuczmaszewska and Szynal [16] and Sung [17], and present some sufficient conditions for the general strong law of large numbers which extend and improve Theorems A and B to $\tilde{\rho}$-mixing random variables.

Throughout this paper, we assume that C is a positive constant which may vary from one place to another.

2. Main results

In order to prove our main result, we need the following lemmas. The proof of the first lemma could be found in [12].

Lemma 1 *For a positive integer J and $0 \leq r < 1$ and $u \geq 2$, there exists a positive constant*

$C = C(u, J, r)$ such that if $\{X_n, n \geq 1\}$ is a sequence of random variables with $\tilde{\rho}(J) \leq r, EX_k = 0$, and $E|X_k|^u < \infty$ for every $k \geq 1$, then for all $n \geq 1$,

$$E \max_{1\leq i\leq n} \Big| \sum_{k=1}^{i} X_k \Big|^u \leq C\Big\{ \sum_{k=1}^{n} E|X_k|^u + \Big(\sum_{k=1}^{n} EX_k^2 \Big)^{u/2} \Big\}.$$

The following lemma can be found in [18].

Lemma 2 *Let $\{b_n, n \geq 1\}$ be a sequence of positive numbers with $b_n \nearrow \infty$. For any $M > 0$, there exists a sequence $\{m_k, k \geq 1\} \subset \mathbb{N}$*

$$Mb_{m_k} \leq b_{m_{k+1}} \leq M^3 b_{m_k+1}, \quad k = 1, 2, \ldots .$$

With the lemmas, we now state and prove one of our main results.

Theorem 1 *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables and let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n \neq 0, n \geq 1, 0 < b_n \nearrow \infty$. Assume that $\{g_n(x), n \geq 1\}$ is a sequence of positive and Borel measurable functions on $[0, +\infty)$ such that*

$$\frac{g_n(x)}{x} \nearrow, \quad \frac{g_n(x)}{x^p} \searrow, \text{ as } x \to \infty \text{ for some } 1 < p \leq 2, \ \forall\, n \geq 1. \tag{1}$$

Suppose that

(i) $\sum_{n=2}^{\infty} b_n^{-p} \frac{Eg_n(|X_n|)}{g_n(b_n/|a_n|)} \sum_{i=1}^{n-1} b_i^p \frac{Eg_i(|X_i|)}{g_i(b_i/|a_i|)} < \infty$;

(ii) $\sum_{n=1}^{\infty} P(|a_n X_n| > C_n) < \infty$, for some sequence $\{C_n, n \geq 1\}$ of positive numbers such that

(iii) $\sum_{n=1}^{\infty} (\frac{C_n}{b_n})^p \frac{Eg_n(|X_n|)}{g_n(b_n/|a_n|)} < \infty$. Then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i \left[X_i - EX_i I(|a_i X_i| \leq b_i) \right] = 0 \ \text{ a.s.}. \tag{2}$$

Proof Let $Y_n = a_n X_n I(|a_n X_n| \leq b_n), T_n = Y_n - EY_n, n \geq 1$. Since

$$\frac{1}{b_n}\Big| \sum_{i=1}^{n} a_i [X_i - EX_i I(|a_i X_i| \leq b_i)] \Big| \leq \frac{1}{b_n} \sum_{i=1}^{n} |a_i X_i| I(|a_i X_i| > b_i) + \frac{1}{b_n} \Big| \sum_{i=1}^{n} T_i \Big|, \tag{3}$$

noting that $g_n(x)$ is nondecreasing on $[0, +\infty)$, by conditions (ii) and (iii), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P(|a_n X_n| > b_n) \\
&\quad = \sum_{n=1}^{\infty} E\left[I(|a_n X_n| > b_n) I(|a_n X_n| > C_n) + I(|a_n X_n| > b_n) I(|a_n X_n| \leq C_n) \right] \\
&\quad \leq \sum_{n=1}^{\infty} \left[P(|a_n X_n| > C_n) + EI(|a_n X_n|^p g_n(|X_n|) > b_n^p g_n(b_n/|a_n|)) I(|a_n X_n| \leq C_n) \right] \\
&\quad \leq C + \sum_{n=1}^{\infty} E\Big[\frac{|a_n X_n|^p g_n(|X_n|)}{b_n^p g_n(b_n/|a_n|)} I(|a_n X_n| \leq C_n) \Big] \\
&\quad \leq C + \sum_{n=1}^{\infty} (\frac{C_n}{b_n})^p \frac{Eg_n(|X_n|)}{g_n(b_n/|a_n|)} < \infty.
\end{aligned}$$

Therefore, by Borel-Cantelli lemma, we have

$$\frac{1}{b_n}\sum_{i=1}^{n}|a_iX_i|I(|a_iX_i|>b_i)\to 0 \ \text{ a.s. } n\to\infty.$$

To prove (2), by (3), it suffices to show that

$$\frac{1}{b_n}\sum_{i=1}^{n}T_i\to 0 \ \text{ a.s. } n\to\infty. \tag{4}$$

By virtue of Lemma 2, we can choose $\{m_k, k\geq 1\}\subset\mathbb{N}$ such that

$$2b_{m_k}\leq b_{m_{k+1}}\leq 8b_{m_k+1}, \quad k=1,2,\ldots .$$

Note that $0<b_n\nearrow\infty$ and there exists a corresponding positive integer number k such that $m_k<n\leq m_{k+1}$ for every $n\in\mathbb{N}$, then

$$\frac{1}{b_n}\Big|\sum_{i=1}^{n}T_i\Big|\leq\max\Big\{\frac{1}{b_{m_k}}\max_{1\leq j\leq m_k}|\sum_{i=1}^{j}T_i|,\frac{8}{b_{m_{k+1}}}\max_{m_k<j\leq m_{k+1}}\Big|\sum_{i=1}^{j}T_i\Big|\Big\}.$$

Thus, to prove (4), by Borel-Cantelli Lemma, it suffices to show that

$$\sum_{k=1}^{\infty}P\Big(\frac{1}{b_{m_k}}\max_{1\leq j\leq m_k}\Big|\sum_{i=1}^{j}T_i\Big|>\epsilon\Big)<\infty. \tag{5}$$

By Markov's inequality and Lemma 1, we get

$$\begin{aligned}\sum_{k=1}^{\infty}P\Big(\frac{1}{b_{m_k}}\max_{1\leq j\leq m_k}\Big|\sum_{i=1}^{j}T_i\Big|>\epsilon\Big)&\leq C\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}E\max_{1\leq j\leq m_k}\Big|\sum_{i=1}^{j}T_i\Big|^4\\ &\leq C\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=1}^{m_k}ET_i^4+C\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\Big(\sum_{i=1}^{m_k}ET_i^2\Big)^2.\end{aligned}$$

By C_r-inequality, Jensen's inequality and the conditions (ii) and (iii) and (1), we have

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=1}^{m_k}ET_i^4\leq C\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=1}^{m_k}E|a_iX_i|^4I(|a_iX_i|\leq b_i)\\ &=C\sum_{i=1}^{\infty}E|a_iX_i|^4I(|a_iX_i|\leq b_i)\sum_{k:m_k\geq i}\frac{1}{b_{m_k}^4}\\ &\leq C\sum_{i=1}^{\infty}\frac{1}{b_i^4}E|a_iX_i|^4I(|a_iX_i|\leq b_i)\\ &\leq C\sum_{i=1}^{\infty}\frac{1}{b_i^4}E\left[|a_iX_i|^4I(|a_iX_i|\leq b_i)I(|a_iX_i|\leq C_i)\right]+C\sum_{i=1}^{\infty}P(|a_iX_i|>C_i)\\ &\leq C\sum_{i=1}^{\infty}E\Big[(\frac{|a_iX_i|}{b_i})^{2p}I(|a_iX_i|\leq b_i)I(|a_iX_i|\leq C_i)\Big]+C\\ &\leq C\sum_{i=1}^{\infty}(\frac{C_i}{b_i})^pE\Big[(\frac{|a_iX_i|}{b_i})^pI(|a_iX_i|\leq b_i)\Big]+C\\ &\leq C\sum_{i=1}^{\infty}(\frac{C_i}{b_i})^pE\frac{g_i(|X_i|)}{g_i(b_i/|a_i|)}+C<\infty.\end{aligned}$$

By Jensen's inequality, we have

$$\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\Big(\sum_{i=1}^{m_k}ET_i^2\Big)^2\le 2\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}\Big(ET_i^2\sum_{j=1}^{i-1}ET_j^2\Big)+\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=1}^{m_k}ET_i^4.$$

By (1) and the condition (i), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{b_{m_k}^4}\sum_{i=2}^{m_k}\Big(ET_i^2\sum_{j=1}^{i-1}ET_j^2\Big)&=\sum_{i=2}^{\infty}\Big(ET_i^2\sum_{j=1}^{i-1}ET_j^2\Big)\sum_{k:m_k\ge i}\frac{1}{b_{m_k}^4}\\
&\le C\sum_{i=2}^{\infty}\frac{1}{b_i^4}\Big(ET_i^2\sum_{j=1}^{i-1}ET_j^2\Big)\\
&\le C\sum_{i=2}^{\infty}\frac{1}{b_i^4}E|a_iX_i|^2I(|a_iX_i|\le b_i)\sum_{j=1}^{i-1}E|a_jX_j|^2I(|a_jX_j|\le b_j)\\
&\le C\sum_{i=2}^{\infty}\frac{1}{b_i^2}\frac{Eg_i(|X_i|)}{g_i(b_i/|a_i|)}\sum_{j=1}^{i-1}b_j^2\frac{Eg_j(|X_j|)}{g_j(b_j/|a_j|)}\\
&=C\sum_{i=2}^{\infty}\frac{1}{b_i^p}\frac{Eg_i(|X_i|)}{g_i(b_i/|a_i|)}\sum_{j=1}^{i-1}b_j^p(\frac{b_j}{b_i})^{2-p}\frac{Eg_j(|X_j|)}{g_j(b_j/|a_j|)}\\
&\le C\sum_{i=2}^{\infty}b_i^{-p}\frac{Eg_i(|X_i|)}{g_i(b_i/|a_i|)}\sum_{j=1}^{i-1}b_j^p\frac{Eg_j(|X_j|)}{g_j(b_j/|a_j|)}\\
&<\infty.\end{aligned}$$

Therefore, (5) holds. □

Corollary 1 *Let $1<p\le 2$ and $\{X_n,n\ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n=0$ $(n\ge 1)$. Let $\{b_n,n\ge 1\}$ be sequences of constants such that $0<b_n\nearrow\infty$. Suppose that*

(i) $\sum_{n=2}^{\infty}b_n^{-2p}E|X_n|^p\sum_{i=1}^{n-1}E|X_i|^p<\infty$;

(ii) $\lim_{n\to\infty}b_n^{-p}\sum_{i=1}^{n}E|X_i|^p=0$;

(iii) $\sum_{n=1}^{\infty}P(|X_n|>C_n)<\infty$, *for some sequence $\{C_n,n\ge 1\}$ of positive numbers such that*

(iv) $\sum_{n=1}^{\infty}\frac{C_n^p}{b_n^{2p}}E|X_n|^p<\infty$.

Then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}X_i=0\ \ a.s.. \tag{6}$$

Proof Let $g_n(x)=|x|^p, a_n=1, n\ge 1$. By Theorem 1, we have

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}[X_i-EX_iI(|X_i|\le b_i)]=0\ \ \text{a.s..}$$

From the proof of Theorem 1, we have

$$\sum_{i=1}^{\infty}P(|X_i|>b_i)<\infty.$$

Since $EX_n = 0$, $n \geq 1$, by Hölder inequality and condition (ii), we have

$$\begin{aligned}\frac{1}{b_n}\Big|\sum_{i=1}^{n} EX_i I(|X_i| \leq b_i)\Big| &\leq \frac{1}{b_n}\sum_{i=1}^{n} E|X_i| I(|X_i| > b_i)\\ &\leq \frac{1}{b_n}\sum_{i=1}^{n}(E|X_i|^p)^{1/p}(P(|X_i| > b_i))^{1-1/p}\\ &\leq \Big(\frac{\sum_{i=1}^{n} E|X_i|^p}{b_n^p}\Big)^{1/p}\Big(\sum_{i=1}^{\infty} P(|X_i| > b_i)\Big)^{1-1/p} \to 0, \quad n \to \infty. \end{aligned} \tag{7}$$

Therefore, (6) holds. □

Remark Let $p = 2$, $b_n = n$, $n \geq 1$. Then Theorem B follows from Corollary 1, thus Corollary 1 extends Theorem B for the case of $\tilde{\rho}$-mixing random variables.

Furthermore, form Corollary 1 we get

Corollary 2 *Let $1 < p \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n = 0$ $(n \geq 1)$ and be stochastically dominated by a random variable X. That is*

$$P(|X_n| > t) \leq DP(D|X| > t), \text{ for all } t \geq 0 \text{ and } n \geq 1,$$

where D is a positive constant. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n \neq 0$, $n \geq 1$, $0 < b_n \nearrow \infty$. Suppose that

(i) $E|X|^p < \infty$;

(ii) $\frac{n|a_n|}{b_n} = O(1)$;

(iii) $\sum_{i=1}^{n} |a_i|^p = O(n|a_n|^p)$;

(iv) $\sum_{n=1}^{\infty} P(|a_n X| > b_n) < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i X_i = 0 \quad \text{a.s.}.$$

Theorem 2 *Let $\{X_n, n \geq 1\}$ and $\{A_n, n \geq 1\}$ be sequences of random variables satisfying $A_n \neq 0, n \geq 1$ and $\{A_n X_n, n \geq 1\}$ is a sequence $\tilde{\rho}$-mixing random variables. Let $\{b_n, n \geq 1\}$ be a sequence of constants such that $0 < b_n \nearrow \infty$. Assume that $\{g_n(x), n \geq 1\}$ is a sequence of positive and Borel measurable functions on $[0, +\infty)$ satisfying (1). Suppose that*

(i) $\sum_{n=2}^{\infty} b_n^{-p} E\frac{g_n(|X_n|)}{g_n(b_n/|A_n|)+g_n(|X_n|)}\sum_{i=1}^{n-1} b_i^p E\frac{g_i(|X_i|)}{g_i(b_i/|A_i|)+g_i(|X_i|)} < \infty$;

(ii) $\sum_{n=1}^{\infty} P(|A_n X_n| > C_n) < \infty$, *for some sequence $\{C_n, n \geq 1\}$ of positive numbers such that*

(iii) $\sum_{n=1}^{\infty} E[g_n(\frac{C_n}{|A_n|})\frac{g_n(|X_n|)}{g_n^2(b_n/|A_n|)+g_n^2(|X_n|)}] < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} A_i[X_i - EX_i I(|A_i X_i| \leq b_i)] = 0 \quad \text{a.s.}. \tag{8}$$

Proof Let $Y_n = A_n X_n I(|A_n X_n| \leq b_n)$, $T_n = Y_n - EY_n$, $n \geq 1$. To prove (8), by similar method in the proof of Theorem 1, it suffices to show the following statements (9)–(11) hold,

$$\sum_{n=1}^{\infty} P(|A_nX_n| > b_n) < \infty, \tag{9}$$

$$\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} ET_i^4 < \infty, \tag{10}$$

$$\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=2}^{m_k} \Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \Big) < \infty, \tag{11}$$

where $\{m_k, k \geq 1\} \subset \mathbb{N}$ as in Theorem 1. By (1) and conditions (i)–(iii), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P(|A_nX_n| > b_n) \\
&\quad = \sum_{n=1}^{\infty} E\left[I(|A_nX_n| > b_n)I(|A_nX_n| > C_n) + I(|A_nX_n| > b_n)I(|A_nX_n| \leq C_n) \right] \\
&\quad \leq \sum_{n=1}^{\infty} P(|A_nX_n| > C_n) + \sum_{n=1}^{\infty} E\left\{ I(2g_n^2(|X_n|) > g_n^2(b_n/|A_n|) + g_n^2(|X_n|))I(|A_nX_n| \leq C_n) \right\} \\
&\quad \leq C + 2\sum_{n=1}^{\infty} E\frac{g_n^2(|X_n|)I(|A_nX_n| \leq C_n)}{g_n^2(b_n/|A_n|) + g_n^2(|X_n|)} \\
&\quad \leq C + 2\sum_{n=1}^{\infty} E\Big[g_n(\frac{C_n}{|A_n|}) \frac{g_n(|X_n|)}{g_n^2(b_n/|A_n|) + g_n^2(|X_n|)} \Big] < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} ET_i^4 \leq C \sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} E|A_iX_i|^4 I(|A_iX_i| \leq b_i) \\
&\quad \leq C \sum_{i=1}^{\infty} \frac{1}{b_i^4} E|A_iX_i|^4 I(|A_iX_i| \leq b_i) \\
&\quad \leq C \sum_{i=1}^{\infty} \frac{1}{b_i^{2p}} E\left[|A_iX_i|^{2p} I(|A_iX_i| \leq b_i) I(|A_iX_i| \leq C_i) \right] + C \sum_{i=1}^{\infty} P(|A_iX_i| > C_i) \\
&\quad \leq C \sum_{i=1}^{\infty} E\frac{g_i^2(|X_i|)I(|A_iX_i| \leq C_i)}{g_i^2(b_i/|A_i|)} + C \\
&\quad \leq C \sum_{i=1}^{\infty} E\Big[g_i(\frac{C_i}{|A_i|}) \frac{g_i(|X_i|)}{g_i^2(b_i/|A_i|) + g_i^2(|X_i|)} \Big] + C < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=2}^{m_k} \Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \Big) \leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \Big) \\
&\quad = C \sum_{i=2}^{\infty} \frac{1}{b_i^4} E|A_iX_i|^2 I(|A_iX_i| \leq b_i) \sum_{j=1}^{i-1} E|A_jX_j|^2 I(|A_jX_j| \leq b_j)
\end{aligned}$$

$$\leq C\sum_{n=2}^{\infty} b_n^{-p} E\frac{g_n(|X_n|)}{g_n(b_n/|A_n|)+g_n(|X_n|)}\sum_{i=1}^{n-1} b_i^p E\frac{g_i(|X_i|)}{g_i(b_i/|A_i|)+g_i(|X_i|)} < \infty.$$

Therefore, (8) holds. □

Now by Hölder's inequality for $q > 1$, we have

$$E|A_nX_n|^p \leq (E|X_n|^{pq})^{1/q}(E|A_n|^{pq/(q-1)})^{(q-1)/q},$$

or

$$E|A_nX_n|^p \leq (E|A_n|^{pq})^{1/q}(E|X_n|^{pq/(q-1)})^{(q-1)/q}.$$

Thus, the arguments in the Theorem 2 and (7) of Corollary 1 allow us to give the following results.

Theorem 3 *Let $\{X_n, n \geq 1\}$ and $\{A_n, n \geq 1\}$ be sequences of random variables satisfying $A_n \neq 0, n \geq 1$ and $\{A_nX_n, n \geq 1\}$ is a sequence $\tilde{\rho}$-mixing random variables with mean zero. Let $\{b_n, n \geq 1\}$ be a sequence of constants such that $0 < b_n \nearrow \infty$. If for some constants p and q, $1 < p \leq 2, q > 1$,*

(i) $\sup_{n\geq 1} E|A_n|^{pq/(q-1)} < \infty$;

(ii) $\sum_{n=2}^{\infty} b_n^{-2p}(E|X_n|^{pq})^{1/q}\sum_{i=1}^{n-1}(E|X_i|^{pq})^{1/q} < \infty$;

(iii) $b_n^{-p}\sum_{i=1}^{n}(E|X_i|^{pq})^{1/q} = o(1)$;

(iv) $\sum_{n=1}^{\infty} P(|A_nX_n| > C_n) < \infty$, *for some sequence $\{C_n, n \geq 1\}$ of positive numbers such that*

(v) $\sum_{n=1}^{\infty}(\frac{C_n}{b_n^2})^p(E|X_n|^{pq})^{1/q} < \infty$. *Then*

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n} A_iX_i = 0 \quad \text{a.s.}. \tag{12}$$

From Theorem 3 we get

Corollary 3 *Let $\{X_n, n \geq 1\}, \{A_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be as in Theorem 3. If for some p and q, $1 < p \leq 2$, $q > 1$,*

(i) $\sup_{n\geq 1} E|A_n|^{pq/(q-1)} < \infty$;

(ii) $\sum_{n=2}^{\infty} b_n^{-p}(E|X_n|^{pq})^{1/q} < \infty$. *Then (12) holds.*

Theorem 4 *Let $\{X_n, n \geq 1\}$, $\{A_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be as in Theorem 3. If for some p and q, $1 < p \leq 2$, $q > 1$,*

(i) $\sup_{n\geq 1} E|X_n|^{pq/(q-1)} < \infty$;

(ii) $\sum_{n=2}^{\infty} b_n^{-2p}(E|A_n|^{pq})^{1/q}\sum_{i=1}^{n-1}(E|A_i|^{pq})^{1/q} < \infty$;

(iii) $b_n^{-p}\sum_{i=1}^{n}(E|A_i|^{pq})^{1/q} = o(1)$;

(iv) $\sum_{n=1}^{\infty} P(|A_nX_n| > C_n) < \infty$, *for some sequence $\{C_n, n \geq 1\}$ of positive numbers such that*

(v) $\sum_{n=1}^{\infty}(\frac{C_n}{b_n^2})^p(E|A_n|^{pq})^{1/q} < \infty$. *Then (12) holds.*

Corollary 4 *Let $\{X_n, n \geq 1\}$,$\{A_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be as in Theorem 3. If for some p and q, $1 < p \leq 2$, $q > 1$,* *(i)* $\sup_{n\geq 1} E|X_n|^{pq/(q-1)} < \infty$;

(ii) $\sum_{n=2}^{\infty} b_n^{-p}(E|A_n|^{pq})^{1/q} < \infty$. *Then (12) holds.*

Theorem 5 *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables and let $\{C_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $C_n > 0$, $n \geq 1$, $0 < b_n \nearrow \infty$. Assume that $\{g_n(x), n \geq 1\}$ is a sequence of positive and Borel measurable functions on $[0, +\infty)$ satisfying (1). Suppose that*

(i) $\sum_{n=2}^{\infty} \frac{C_n^2 Eg_n(|X_n|)}{b_n^4 g_n(C_n)} \sum_{i=1}^{n-1} \frac{C_i^2 Eg_i(|X_i|)}{g_i(C_i)} < \infty$;

(ii) $\sum_{n=1}^{\infty} P(|X_n| > C_n) < \infty$;

(iii) $\sum_{n=1}^{\infty} \frac{C_n^4 Eg_n(|X_n|)}{b_n^4 g_n(C_n)} < \infty$. *Then*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} [X_i - EX_i I(|X_i| \leq C_i)] = 0 \quad \text{a.s..} \tag{13}$$

Proof Let $Y_n = X_n I(|X_n| \leq C_n)$, $T_n = Y_n - EY_n$, $n \geq 1$. Since

$$\frac{1}{b_n}\Big|\sum_{i=1}^{n}[X_i - EX_i I(|X_i| \leq C_i)]\Big| \leq \frac{1}{b_n}\sum_{i=1}^{n}|X_i| I(|X_i| > C_i) + \frac{1}{b_n}\Big|\sum_{i=1}^{n} T_i\Big|, \tag{14}$$

from the condition (ii) and Borel-Cantelli Lemma, we have

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} |X_i| I(|X_i| > C_i) = 0 \quad \text{a.s..} \tag{15}$$

To prove (13), by (14) and (15), it suffices to show that

$$\frac{1}{b_n}\sum_{i=1}^{n} T_i \to 0 \quad \text{a.s. } n \to \infty. \tag{16}$$

To prove (16), by similar method in the proof of Theorem 1, it suffices to show the following statements (17) and (18) hold,

$$\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} ET_i^4 < \infty, \tag{17}$$

$$\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=2}^{m_k} \Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2\Big) < \infty, \tag{18}$$

where $\{m_k, k \geq 1\} \subset \mathbb{N}$ as in Theorem 1. By C_r-inequality, Jensen's inequality, (1) and the condition (iii), we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} ET_i^4 &\leq C\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=1}^{m_k} E|X_i|^4 I(|X_i| \leq C_i) \leq C\sum_{i=1}^{\infty}\frac{1}{b_i^4} E|X_i|^4 I(|X_i| \leq C_i)\\ &\leq C\sum_{i=1}^{\infty}\frac{1}{b_i^4} E\left[C_i^{4-p}|X_i|^p I(|X_i| \leq C_i)\right] \leq C\sum_{i=1}^{\infty}\frac{C_i^4 Eg_i(|X_i|)}{b_i^4 g_i(C_i)} < \infty.\end{aligned}$$

By (1) and the condition (i), we have

$$\sum_{k=1}^{\infty} \frac{1}{b_{m_k}^4} \sum_{i=2}^{m_k} \Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2\Big) \leq C\sum_{i=2}^{\infty}\frac{1}{b_i^4}\Big(ET_i^2 \sum_{j=1}^{i-1} ET_j^2\Big)$$

$$\leq C\sum_{i=2}^{\infty}\frac{1}{b_i^4}E|X_i|^2 I(|X_i|\leq C_i)\sum_{j=1}^{i-1}E|X_j|^2 I(|X_j|\leq C_j)$$
$$\leq \sum_{n=2}^{\infty}\frac{C_n E g_n(|X_n|)}{b_n^4 g_n(C_n)}\sum_{i=1}^{n-1}\frac{C_i^2 E g_i(|X_i|)}{g_i(C_i)}<\infty.$$

Therefore, (13) holds. □

Corollary 5 *Let $\{X_n, n\geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n=0, n\geq 1$, and let $\{b_n, n\geq 1\}$ be sequences of constants such that $0<b_n\nearrow\infty$. Assume that $\{g_n(x), n\geq 1\}$ is a sequence of positive and Borel measurable functions on $[0,+\infty)$ satisfying (1). If*

$$\sum_{n=1}^{\infty}\frac{E g_n(|X_n|)}{g_n(b_n)}<\infty,$$

then

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}X_i=0 \quad \text{a.s.}. \tag{19}$$

Proof Let $C_n=b_n, n\geq 1$. The conditions (i) and (iii) of Theorem 5 are satisfied. Note that

$$\sum_{n=1}^{\infty}P(|X_n|>C_n)\leq\sum_{n=1}^{\infty}P(g_n(|X_n|)\geq g_n(|C_n|)\leq\sum_{n=1}^{\infty}\frac{E g_n(|X_n|)}{g_n(|C_n)}<\infty.$$

Therefore, the condition (ii) of Theorem 5 is satisfied. By Theorem 5, we have

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}[X_i-EX_iI(|X_i|\leq b_i)]=0 \quad \text{a.s.}. \tag{20}$$

Since $EX_n=0, n\geq 1$, by (1) and Kronecker Lemma, we have

$$\frac{1}{b_n}\Big|\sum_{i=1}^{n}EX_iI(|X_i|\leq b_i)\Big|\leq\frac{1}{b_n}\sum_{i=1}^{n}E|X_i|I(|X_i|>b_i)$$
$$\leq\frac{1}{b_n}\sum_{i=1}^{n}\frac{b_iEg_i(|X_i|)I(|X_i|>b_i)}{g_i(b_i)}\leq\frac{1}{b_n}\sum_{i=1}^{n}b_i\frac{Eg_i(|X_i|)}{g_i(b_i)}\to 0,\quad n\to\infty. \tag{21}$$

From (20) and (21), (19) holds. □

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