The Strong Law of Large Numbers for $\tilde{\rho}$-Mixing Random Variables

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Abstract In this paper we present some results for the general strong laws of large numbers of $\tilde{\rho}$-mixing random variables by a maximal inequality of Utev and Peligrad. These results extend and improve the related known works in the literature.

Keywords strong laws of large numbers; $\tilde{\rho}$-mixing random variables; Utev and Peligrad’s maximal inequality.

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1. Introduction

Throughout this paper, we suppose that $\{\Omega, \mathcal{F}, P\}$ is a probability space, and all random variables are assumed to be defined on $\{\Omega, \mathcal{F}, P\}$. For a sequence of random variables $\{X_n, n \geq 1\}$, we denote $\mathcal{F}_S = \sigma(X_n : n \in S \subset \mathbb{N})$. Given two $\sigma$-subalgebras $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$, denote

$$\rho(\mathcal{F}_1, \mathcal{F}_2) = \sup \{|\text{corr}(\zeta, \eta)|, \zeta \in L_2(\mathcal{F}_1), \eta \in L_2(\mathcal{F}_2)\},$$

where the correlation coefficient is defined in the usual way

$$\text{corr}(\zeta, \eta) = \frac{E(\zeta \eta) - E\zeta E\eta}{\sqrt{\text{Var}(\zeta) \text{Var}(\eta)}}$$

and by $L_2(\mathcal{F})$ we denote the space of all $\mathcal{F}$-measurable random variables $\zeta$ such that $E(\zeta^2) < \infty$.

Stein [1] introduced the following coefficients of dependence (with slightly different notations):

$$\tilde{\rho}(k) = \sup \{\rho(\mathcal{F}_S, \mathcal{F}_T) : \text{all finite subsets } S, T \subset \mathbb{N} \text{ such that } \text{dist}(S, T) \geq k\}, k \geq 0.$$

Obviously, $0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1, k \geq 0$, and $\tilde{\rho}(0) = 1$.

Definition A sequence of random variables $\{X_n, n \geq 1\}$ are said to be a $\tilde{\rho}$-mixing sequence if there exists $k \in \mathbb{N}$ such that $\tilde{\rho}(k) < 1$.

The notion of $\tilde{\rho}$-mixing assumption is similar to $\rho$-mixing, but they are quite different from each other. A number of publications are devoted to $\tilde{\rho}$-mixing sequence. We refer to Bradley [2, 3] for the central limit theorem, Bryc and Smolenski [4] for moment inequalities and almost sure convergence, Gan [5], Gut and Peligrad [6] and Wu [7, 8] for almost sure convergence, Qiu

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Various limit properties for sums of independent random variables have been studied by many authors, more precisely, Chung [14] proved the following result:

**Theorem A** Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables with \( \mathbb{E}X_n = 0, n \geq 1 \). Let \( g \) be a positive, even and continuous function on \( \mathbb{R} \) such that
\[
\frac{g(x)}{x}, \quad \frac{g(x)}{x^2} \downarrow, \quad \text{as } |x| \to \infty.
\]

If
\[
\sum_{n=1}^{\infty} \mathbb{E}g(|X_n|)/g(n) < \infty,
\]
then
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} X_i}{n} = 0 \quad \text{a.s.}
\]

Teicher [15] proved that:

**Theorem B** Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables with \( \mathbb{E}X_n = 0, n \geq 1 \). Suppose that
\begin{enumerate}
\item[(i)] \( \sum_{n=2}^{\infty} (\mathbb{E}X_n^2/n^4) \sum_{j=1}^{n-1} \mathbb{E}X_j^2 < \infty \); \\
\item[(ii)] \( \sum_{i=1}^{n} \mathbb{E}X_i^2/n^2 \to 0 \); \\
\item[(iii)] \( \sum_{n=1}^{\infty} P(|X_n| > C_n) < \infty \), for some positive constants \( \{C_n, n \geq 1\} \) with
\[
\sum_{n=1}^{\infty} \frac{C_n^2 \mathbb{E}X_n^2}{n^4} < \infty.
\]
\end{enumerate}
Then
\[
\sum_{i=1}^{n} X_i/n \to 0 \quad \text{a.s.}
\]

In this paper, we study strong law of large numbers for \( \tilde{\rho} \)-mixing random variables inspired by Kuczmaszewska and Szynal [16] and Sung [17], and present some sufficient conditions for the general strong law of large numbers which extend and improve Theorems A and B to \( \tilde{\rho} \)-mixing random variables.

Throughout this paper, we assume that \( C \) is a positive constant which may vary from one place to another.

**2. Main results**

In order to prove our main result, we need the following lemmas. The proof of the first lemma could be found in [12].

**Lemma 1** For a positive integer \( J \) and \( 0 \leq r < 1 \) and \( u \geq 2 \), there exists a positive constant
Let \( C = C(u, J, r) \) such that if \( \{X_n, n \geq 1\} \) is a sequence of random variables with \( \rho(J) \leq r, EX_k = 0, \) and \( E|X_k|^u < \infty \) for every \( k \geq 1, \) then for all \( n \geq 1, \)

\[
E \max_{1 \leq i \leq n} \left| \sum_{k=1}^i X_k \right|^u \leq C \left\{ \sum_{k=1}^n E|X_k|^u + \left( \sum_{k=1}^n EX_k^2 \right)^{u/2} \right\}.
\]

The following lemma can be found in [18].

**Lemma 2** Let \( \{b_n, n \geq 1\} \) be a sequence of positive numbers with \( b_n \not\to \infty. \) For any \( M > 0, \) there exists a sequence \( \{m_k, k \geq 1\} \subset \mathbb{N} \)

\[
Mb_{m_k} \leq b_{m_{k+1}} \leq M^3b_{m_{k+1}}, \quad k = 1, 2, \ldots
\]

With the lemmas, we now state and prove one of our main results.

**Theorem 1** Let \( \{X_n, n \geq 1\} \) be a sequence of \( \rho \)-mixing random variables and let \( \{a_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) be sequences of constants such that \( a_n \neq 0, n \geq 1, 0 < b_n \not\to \infty. \) Assume that \( \{g_n(x), n \geq 1\} \) is a sequence of positive and Borel measurable functions on \([0, +\infty)\) such that

\[
g_n(x) / x \to \infty, \quad \text{as } x \to \infty \text{ for some } 1 < p \leq 2, \quad \forall n \geq 1. \quad (1)
\]

Suppose that

(i) \( \sum_{n=2}^{\infty} b_n^{-p} \frac{Eg_n(|X_1|)}{g_n(b_n/|a_n|)} \sum_{i=1}^{n-1} b_i^{-p} \frac{Eg_n(|X_i|)}{g_n(b_i/|a_i|)} < \infty; \)

(ii) \( \sum_{n=1}^{\infty} P(|a_n X_n| > C_n) < \infty, \) for some sequence \( \{C_n, n \geq 1\} \) of positive numbers such that

(iii) \( \sum_{n=1}^{\infty} \left( \frac{C_n}{b_n} \right)^p \frac{Eg_n(|X_n|)}{g_n(b_n/|a_n|)} < \infty. \) Then

\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i |X_i - EX_i I(|a_i X_i| \leq b_i)| = 0 \quad \text{a.s..} \quad (2)
\]

**Proof** Let \( Y_n = a_n X_n I(|a_n X_n| \leq b_n), T_n = Y_n - EY_n, n \geq 1. \) Since

\[
\frac{1}{b_n} \left| \sum_{i=1}^{n} a_i |X_i - EX_i I(|a_i X_i| \leq b_i)| \right| \leq \frac{1}{b_n} \sum_{i=1}^{n} |a_i X_i| I(|a_i X_i| > b_i) + \frac{1}{b_n} \sum_{i=1}^{n} |T_i|, \quad (3)
\]

noting that \( g_n(x) \) is nondecreasing on \([0, +\infty), \) by conditions (ii) and (iii), we have

\[
\sum_{n=1}^{\infty} P(|a_n X_n| > b_n)
\]

\[
= \sum_{n=1}^{\infty} E[I(|a_n X_n| > b_n)I(|a_n X_n| > C_n) + I(|a_n X_n| > b_n)I(|a_n X_n| \leq C_n)]
\]

\[
\leq \sum_{n=1}^{\infty} [P(|a_n X_n| > C_n) + E[I(|a_n X_n|^p g_n(|X_n|) > b_n^p g_n(b_n/|a_n|))]I(|a_n X_n| \leq C_n)]
\]

\[
\leq C + \sum_{n=1}^{\infty} E \left[ \frac{|a_n X_n|^p g_n(|X_n|)}{b_n^p g_n(b_n/|a_n|)} I(|a_n X_n| \leq C_n) \right]
\]

\[
\leq C + \sum_{n=1}^{\infty} \left( \frac{C_n}{b_n} \right)^p \frac{Eg_n(|X_n|)}{g_n(b_n/|a_n|)} < \infty.
\]
Therefore, by Borel-Cantelli lemma, we have
\[ \frac{1}{b_n} \sum_{i=1}^{n} |a_i X_i| I(|a_i X_i| > b_i) \to 0 \text{ a.s. } n \to \infty. \]

To prove (2), by (3), it suffices to show that
\[ \frac{1}{b_n} \sum_{i=1}^{n} T_i \to 0 \text{ a.s. } n \to \infty. \]  

(4)

By virtue of Lemma 2, we can choose \( \{m_k, k \geq 1\} \subset \mathbb{N} \) such that
\[ 2b_{m_k} \leq b_{m_{k+1}} \leq 8b_{m_{k+1}}, \quad k = 1, 2, \ldots. \]

Note that \( 0 < b_n \not\to \infty \) and there exists a corresponding positive integer number \( k \) such that \( m_k < n \leq m_{k+1} \) for every \( n \in \mathbb{N} \), then
\[ \frac{1}{b_n} \sum_{i=1}^{n} T_i \leq \max \left\{ \frac{1}{b_{m_k}} \sum_{1 \leq j \leq m_k} T_j, \frac{1}{b_{m_{k+1}} m_k < j \leq m_{k+1}} \sum_{i=1}^{j} T_i \right\}. \]

Thus, to prove (4), by Borel-Cantelli Lemma, it suffices to show that
\[ \sum_{k=1}^{\infty} P \left( \frac{1}{b_{m_k}} \sum_{1 \leq j \leq m_k} T_j > \epsilon \right) < \infty. \]

(5)

By Markov’s inequality and Lemma 1, we get
\[ \sum_{k=1}^{\infty} P \left( \frac{1}{b_{m_k}} \sum_{1 \leq j \leq m_k} T_j > \epsilon \right) \leq C \sum_{k=1}^{\infty} \frac{1}{b_{m_k}} E \max_{1 \leq j \leq m_k} \sum_{i=1}^{j} T_i = 1 \]

\[ \leq C \sum_{k=1}^{\infty} \frac{1}{b_{m_k}} \sum_{i=1}^{m_k} E T_i^4 + C \sum_{k=1}^{\infty} \frac{1}{b_{m_k}} \left( \sum_{i=1}^{m_k} ET_i^2 \right)^2. \]

By \( C_r \)-inequality, Jensen’s inequality and the conditions (ii) and (iii) and (1), we have
\[ \sum_{k=1}^{\infty} \frac{1}{b_{m_k}} \sum_{i=1}^{m_k} E T_i^4 \leq C \sum_{k=1}^{\infty} \frac{1}{b_{m_k}} \sum_{i=1}^{m_k} E|a_i X_i|^4 I(|a_i X_i| \leq b_i) \]

\[ = C \sum_{i=1}^{\infty} E|a_i X_i|^4 I(|a_i X_i| \leq b_i) \sum_{k: m_k \geq i} \frac{1}{b_{m_k}} \]

\[ \leq C \sum_{i=1}^{\infty} \frac{1}{b_{m_k}} E|a_i X_i|^4 I(|a_i X_i| \leq b_i) \]

\[ \leq C \sum_{i=1}^{\infty} \frac{1}{b_{m_k}} E \left[ \frac{|a_i X_i|^4}{b_i} I(|a_i X_i| \leq b_i) I(|a_i X_i| \leq C_i) \right] + C \sum_{i=1}^{\infty} P(|a_i X_i| > C_i) \]

\[ \leq C \sum_{i=1}^{\infty} \frac{C_i}{b_i} E \left[ \frac{|a_i X_i|^4}{b_i} I(|a_i X_i| \leq b_i) I(|a_i X_i| \leq C_i) \right] + C \]

\[ \leq C \sum_{i=1}^{\infty} \frac{C_i}{b_i} E \left[ \frac{|a_i X_i|^4}{b_i} I(|a_i X_i| \leq b_i) \right] + C \]

\[ \leq C \sum_{i=1}^{\infty} \frac{C_i}{b_i} E \left[ \frac{g_i(|X_i|)}{g_i(b_i/a_i)} \right] + C < \infty. \]
By Jensen’s inequality, we have
\[
\sum_{k=1}^{\infty} \frac{1}{b_k^4} \left( \sum_{i=1}^{m_k} ET_i^2 \right)^2 \leq 2 \sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=2}^{m_k} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) + \sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=1}^{m_k} ET_i^4.
\]

By (1) and the condition (i), we have
\[
\sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=2}^{m_k} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) = \sum_{i=2}^{\infty} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) \sum_{k:m_k \geq i} \frac{1}{b_k^4} \leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right)
\leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} E|a_iX_i|^2 I(|a_iX_i| \leq b_i) \sum_{j=1}^{i-1} E|a_jX_j|^2 I(|a_jX_j| \leq b_j)
\leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \frac{E g_i(|X_i|)}{g_i(b_i/|a_i|)} \sum_{j=1}^{i-1} \frac{b_j^2}{g_j(b_j/|a_j|)} E g_j(|X_j|) \frac{b_j}{g_j(b_j/|a_j|)}
= C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \frac{E g_i(|X_i|)}{g_i(b_i/|a_i|)} \sum_{j=1}^{i-1} b_j b_j^{2-p} \frac{E g_j(|X_j|)}{g_j(b_j/|a_j|)}
\leq C \sum_{i=2}^{\infty} b_i^{-p} \frac{E g_i(|X_i|)}{g_i(b_i/|a_i|)} \sum_{j=1}^{i-1} b_j \frac{E g_j(|X_j|)}{g_j(b_j/|a_j|)} < \infty.
\]

Therefore, (5) holds. \(\square\)

**Corollary 1** Let \(1 < p \leq 2\) and \(\{X_n, n \geq 1\}\) be a sequence of \(\tilde{p}\)-mixing random variables with \(EX_n = 0\) \((n \geq 1)\). Let \(\{b_n, n \geq 1\}\) be sequences of constants such that \(0 < b_n \not\to \infty\). Suppose that

(i) \(\sum_{n=2}^{\infty} b_n^{-2p} E|X_n|^p \sum_{i=1}^{n-1} E|X_i|^p < \infty;\)

(ii) \(\lim_{n \to \infty} b_n^{-p} \sum_{1}^{n} E|X_i|^p = 0;\)

(iii) \(\sum_{n=1}^{\infty} P(|X_n| > C_n) < \infty,\) for some sequence \(\{C_n, n \geq 1\}\) of positive numbers such that

(iv) \(\sum_{n=1}^{\infty} \frac{C_n^p}{b_n^p} E|X_n|^p < \infty.\)

Then
\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i = 0 \quad a.s.
\]

**Proof** Let \(g_n(x) = |x|^p, a_n = 1, n \geq 1\). By Theorem 1, we have
\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} [X_i - EX, I(|X_i| \leq b_i)] = 0 \quad a.s.
\]

From the proof of Theorem 1, we have
\[
\sum_{i=1}^{\infty} P(|X_i| > b_i) < \infty.
\]
Since $EX_n = 0$, $n \geq 1$, by Hölder inequality and condition (ii), we have

$$
\frac{1}{b_n} \left| \sum_{i=1}^{n} E[X_i I(|X_i| \leq b_i)] \right| \leq \frac{1}{b_n} \sum_{i=1}^{n} E[X_i I(|X_i| > b_i)]
$$

$$
\leq \frac{1}{b_n} \sum_{i=1}^{n} (E|X_i|^p)^{1/p} (P(|X_i| > b_i))^{1-1/p}
$$

$$
\leq \left( \sum_{i=1}^{n} \frac{E|X_i|^p}{b_n^p} \right)^{1/p} \left( \sum_{i=1}^{\infty} P(|X_i| > b_i) \right)^{1-1/p} \to 0, \quad n \to \infty.
$$

Therefore, (6) holds. □

**Remark** Let $p = 2$, $b_n = n$, $n \geq 1$. Then Theorem B follows from Corollary 1, thus Corollary 1 extends Theorem B for the case of $\tilde{\rho}$-mixing random variables.

Furthermore, form Corollary 1 we get

**Corollary 2** Let $1 < p \leq 2$ and $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n = 0$ ($n \geq 1$) and be stochastically dominated by a random variable $X$. That is

$$
P(|X_n| > t) \leq DP(D|X| > t), \quad \text{for all} \ t \geq 0 \quad \text{and} \quad n \geq 1,
$$

where $D$ is a positive constant. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants such that $a_n \neq 0$, $n \geq 1$, $0 < b_n \not\to \infty$. Suppose that

(i) $E|X|^p < \infty$;

(ii) $\frac{n|a_n|}{b_n} = O(1)$;

(iii) $\sum_{i=1}^{n} |a_i|^p = O(n|a_n|^p)$;

(iv) $\sum_{i=1}^{\infty} P(|a_n X_i| > b_i) < \infty$. Then

$$
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i X_i = 0 \quad \text{a.s.}
$$

**Theorem 2** Let $\{X_n, n \geq 1\}$ and $\{A_n, n \geq 1\}$ be sequences of random variables satisfying $A_n \neq 0$, $n \geq 1$ and $\{A_n X_n, n \geq 1\}$ is a sequence $\tilde{\rho}$-mixing random variables. Let $\{b_n, n \geq 1\}$ be a sequence of constants such that $0 < b_n \not\to \infty$. Assume that $\{g_n(x), n \geq 1\}$ is a sequence of positive and Borel measurable functions on $[0, +\infty)$ satisfying (1). Suppose that

(i) $\sum_{n=2}^{\infty} b_n^{-p} E[g_n(\frac{A_n}{|A_n|}) + g_n(\frac{X_n}{|X_n|})] \sum_{i=1}^{n-1} b_i^p E[\frac{g_i(|X_i|)}{g_i(b_i/|A_i|) + g_i(|X_i|)}] < \infty$;

(ii) $\sum_{n=1}^{\infty} P(|A_n X_n| > C_n) < \infty$, for some sequence $\{C_n, n \geq 1\}$ of positive numbers such that

(iii) $\sum_{n=1}^{\infty} E[g_n(\frac{C_n}{|A_n|}) \frac{g_n(|X_n|)}{g_n(b_n/|A_n|) + g_n(|X_n|)}] < \infty$. Then

$$
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} A_i [X_i - EX_i I(|A_i X_i| \leq b_i)] = 0 \quad \text{a.s.}
$$

**Proof** Let $Y_n = A_n X_n I(|A_n X_n| \leq b_n)$, $T_n = Y_n - EY_n$, $n \geq 1$. To prove (8), by similar method
in the proof of Theorem 1, it suffices to show the following statements (9)–(11) hold,

\[
\sum_{n=1}^{\infty} P(|A_n X_n| > b_n) < \infty, \tag{9}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{b_i^4 m_k^i} \sum_{i=1}^{m_k} E T_i^4 < \infty, \tag{10}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{b_i^4 m_k^i} \sum_{i=2}^{m_k} \left( E T_i^2 \sum_{j=1}^{i-1} E T_j^2 \right) < \infty, \tag{11}
\]

where \(\{m_k, k \geq 1\} \subset \mathbb{N}\) as in Theorem 1. By (1) and conditions (i)–(iii), we have

\[
\sum_{n=1}^{\infty} P(|A_n X_n| > b_n) \\
= \sum_{n=1}^{\infty} \mathbb{E} \left[ I(|A_n X_n| > b_n) I(|A_n X_n| > C_n) + I(|A_n X_n| > b_n) I(|A_n X_n| \leq C_n) \right] \\
\leq \sum_{n=1}^{\infty} P(|A_n X_n| > C_n) + \sum_{n=1}^{\infty} \mathbb{E} \left\{ I(2g_n^2(|X_n|) > g_n^2(b_n/|A_n|)) + g_n^2(|X_n|) I(|A_n X_n| \leq C_n) \right\} \\
\leq C + 2 \sum_{n=1}^{\infty} \mathbb{E} \left[ g_n^2(|X_n|) / g_n^2(b_n/|A_n|) + g_n^2(|X_n|) \right] I(|A_n X_n| \leq C_n) \\
\leq C + 2 \sum_{n=1}^{\infty} \mathbb{E} \left[ g_n^2(|X_n|) / g_n^2(b_n/|A_n|) + g_n^2(|X_n|) \right] < \infty,
\]

and

\[
\sum_{k=1}^{\infty} \frac{1}{b_i^4 m_k^i} \sum_{i=1}^{m_k} E T_i^4 \leq C \sum_{k=1}^{\infty} \frac{1}{b_i^4 m_k^i} \sum_{i=1}^{m_k} E |A_i X_i|^4 I(|A_i X_i| \leq b_i) \\
\leq C \sum_{i=1}^{\infty} \frac{1}{b_i^4} E |A_i X_i|^4 I(|A_i X_i| \leq b_i) \\
\leq C \sum_{i=1}^{\infty} \frac{1}{b_i^4} \mathbb{E} \left[ |A_i X_i|^2 I(|A_i X_i| \leq b_i) + I(|A_i X_i| \leq C_i) \right] + C \sum_{i=1}^{\infty} P(|A_i X_i| > C_i) \\
\leq C \sum_{i=1}^{\infty} \mathbb{E} \left[ g_i^2(|X_i|) / g_i^2(b_i/|A_i|) + C \right] \left[ g_i^2(|X_i|) / g_i^2(b_i/|A_i|) + g_i^2(|X_i|) \right] + C < \infty,
\]

and

\[
\sum_{k=1}^{\infty} \frac{1}{b_i^4 m_k^i} \sum_{i=2}^{m_k} \left( E T_i^2 \sum_{j=1}^{i-1} E T_j^2 \right) \leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \left( E T_i^2 \sum_{j=1}^{i-1} E T_j^2 \right) \\
= C \sum_{i=2}^{\infty} \frac{1}{b_i^4} E |A_i X_i|^2 I(|A_i X_i| \leq b_i) \sum_{j=1}^{i-1} E |A_j X_j|^2 I(|A_j X_j| \leq b_j)
\]
\[ \leq C \sum_{n=2}^{\infty} b_n^p E \frac{g_n(|X_n|)}{g_n(b_n/|A_n|) + g_n(|X_n|)} \sum_{i=1}^{n-1} b_i^p E \frac{g_i(|X_i|)}{g_i(b_i/|A_i|) + g_i(|X_i|)} < \infty. \]

Therefore, (8) holds. \(\square\)

Now by Hölder’s inequality for \(q > 1\), we have

\[ E|A_n X_n|^p \leq (E|X_n|^{pq})^{1/q} (E|A_n|^{pq/(q-1)})^{(q-1)/q}, \]

or

\[ E|A_n X_n|^p \leq (E|A_n|^{pq})^{1/q} (E|X_n|^{pq/(q-1)})^{(q-1)/q}. \]

Thus, the arguments in the Theorem 2 and (7) of Corollary 1 allow us to give the following results.

**Theorem 3** Let \(\{X_n, n \geq 1\}\) and \(\{A_n, n \geq 1\}\) be sequences of random variables satisfying \(A_n \neq 0, n \geq 1\) and \(\{A_nX_n, n \geq 1\}\) is a sequence \(\hat{p}\)-mixing random variables with mean zero. Let \(\{b_n, n \geq 1\}\) be a sequence of constants such that \(0 < b_n \not< \infty\). If for some constants \(p\) and \(q\), \(1 < p \leq 2, q > 1\),

(i) \(\sup_{n \geq 1} E|A_n|^{pq/(q-1)} < \infty\);

(ii) \(\sum_{n=2}^{\infty} b_n^{-2p} (E|X_n|^{pq})^{1/q} \sum_{i=1}^{n-1} (E|X_i|^{pq})^{1/q} < \infty\);

(iii) \(\sum_{n=2}^{\infty} b_n^{-p} \sum_{i=1}^{n-1} (E|X_i|^{pq})^{1/q} = o(1)\);

(iv) \(\sum_{n=1}^{\infty} P(|A_nX_n| > C_n) < \infty\), for some sequence \(\{C_n, n \geq 1\}\) of positive numbers such that

(v) \(\sum_{n=1}^{\infty} (\frac{C_n}{b_n})^p (E|X_n|^{pq})^{1/q} < \infty\). Then

\[ \lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} A_iX_i = 0 \text{ a.s.} \tag{12} \]

From Theorem 3 we get

**Corollary 3** Let \(\{X_n, n \geq 1\}\), \(\{A_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\) be as in Theorem 3. If for some \(p\) and \(q\), \(1 < p \leq 2, q > 1\),

(i) \(\sup_{n \geq 1} E|A_n|^{pq/(q-1)} < \infty\);

(ii) \(\sum_{n=2}^{\infty} b_n^{-2p} (E|A_n|^{pq})^{1/q} \sum_{i=1}^{n-1} (E|A_i|^{pq})^{1/q} < \infty\);

(iii) \(\sum_{n=2}^{\infty} b_n^{-p} \sum_{i=1}^{n-1} (E|A_i|^{pq})^{1/q} = o(1)\);

(iv) \(\sum_{n=1}^{\infty} P(|A_nX_n| > C_n) < \infty\), for some sequence \(\{C_n, n \geq 1\}\) of positive numbers such that

(v) \(\sum_{n=1}^{\infty} (\frac{C_n}{b_n})^p (E|A_n|^{pq})^{1/q} < \infty\). Then (12) holds.

**Theorem 4** Let \(\{X_n, n \geq 1\}\), \(\{A_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\) be as in Theorem 3. If for some \(p\) and \(q\), \(1 < p \leq 2, q > 1\),

(i) \(\sup_{n \geq 1} E|X_n|^{pq/(q-1)} < \infty\);

(ii) \(\sum_{n=2}^{\infty} b_n^{-2p} (E|A_n|^{pq})^{1/q} \sum_{i=1}^{n-1} (E|A_i|^{pq})^{1/q} < \infty\);

(iii) \(\sum_{n=2}^{\infty} b_n^{-p} \sum_{i=1}^{n-1} (E|A_i|^{pq})^{1/q} = o(1)\);

(iv) \(\sum_{n=1}^{\infty} P(|A_nX_n| > C_n) < \infty\), for some sequence \(\{C_n, n \geq 1\}\) of positive numbers such that

(v) \(\sum_{n=1}^{\infty} (\frac{C_n}{b_n})^p (E|A_n|^{pq})^{1/q} < \infty\). Then (12) holds.

**Corollary 4** Let \(\{X_n, n \geq 1\}\), \(\{A_n, n \geq 1\}\) and \(\{b_n, n \geq 1\}\) be as in Theorem 3. If for some \(p\) and \(q\), \(1 < p \leq 2, q > 1\),
(i) \( \sup_{n \geq 1} E|X_n|^{pq/(q-1)} < \infty; \)
(ii) \( \sum_{n=0}^{\infty} b_n^p (E|A_n|^{pq})^{1/q} < \infty. \) Then (12) holds.

**Theorem 5** Let \( \{X_n, n \geq 1\} \) be a sequence of \( \tilde{\rho} \)-mixing random variables and let \( \{C_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) be sequences of constants such that \( C_n > 0, n \geq 1, 0 < b_n \not\to \infty. \) Assume that \( \{g_n(x), n \geq 1\} \) is a sequence of positive and Borel measurable functions on \([0, +\infty)\) satisfying (1). Suppose that

\[
\sum_{n=1}^{\infty} \frac{C_n^2 E_g((X_n))}{b_n^2 g_n(C_n)} < \infty; \\
\sum_{n=1}^{\infty} \frac{C_n^2 E_{g_n}((X_n))}{b_n^2 g_n(C_n)} \cdot \left\langle \sum_{i=1}^{\infty} \frac{C_n^2 g_i((X_n))}{b_n^2 g_n(C_n)} \right\rangle < \infty. 
\]

Then

\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} |X_i - E X_i I(|X_i| \leq C_i)| = 0 \quad a.s. 
\] (13)

**Proof** Let \( Y_n = X_n I(|X_n| \leq C_n), T_n = Y_n - EY_n, n \geq 1. \) Since

\[
\frac{1}{b_n} \left| \sum_{i=1}^{n} |X_i - E X_i I(|X_i| \leq C_i)| \right| \leq \frac{1}{b_n} \sum_{i=1}^{n} |X_i| I(|X_i| > C_i) + \frac{1}{b_n} \left| \sum_{i=1}^{n} T_i \right|. 
\] (14)

from the condition (ii) and Borel-Cantelli Lemma, we have

\[
\lim_{n \to \infty} \frac{1}{b_n} \sum_{i=1}^{n} |X_i| I(|X_i| > C_i) = 0 \quad a.s. 
\] (15)

To prove (13), by (14) and (15), it suffices to show that

\[
\frac{1}{b_n} \sum_{i=1}^{n} T_i \to 0 \quad a.s. \quad n \to \infty. \] (16)

To prove (16), by similar method in the proof of Theorem 1, it suffices to show the following statements (17) and (18) hold,

\[
\sum_{k=1}^{\infty} \frac{1}{b_k} \sum_{i=1}^{m_k} ET_i^4 < \infty, \] (17)

\[
\sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=2}^{m_k} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) < \infty. \] (18)

where \( \{m_k, k \geq 1\} \subset \mathbb{N} \) as in Theorem 1. By \( C_r \)-inequality, Jensen’s inequality, (1) and the condition (iii), we have

\[
\sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=1}^{m_k} ET_i^4 \leq C \sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=1}^{m_k} E|X_i|^4 I(|X_i| \leq C_i) \leq C \sum_{i=1}^{\infty} \frac{1}{b_i^4} E|X_i|^4 I(|X_i| \leq C_i) \\
\leq C \sum_{i=1}^{\infty} \frac{1}{b_i^4} E \left[ C_i^{4-p} |X_i|^p I(|X_i| \leq C_i) \right] \leq C \sum_{i=1}^{\infty} \frac{C_i^4 E_g(|X_i|)}{b_i^4 g_i(C_i)} < \infty. 
\]

By (1) and the condition (i), we have

\[
\sum_{k=1}^{\infty} \frac{1}{b_k^4} \sum_{i=2}^{m_k} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) \leq C \sum_{i=2}^{\infty} \frac{1}{b_i^4} \left( ET_i^2 \sum_{j=1}^{i-1} ET_j^2 \right) 
\]
Since

Therefore, the condition (ii) of Theorem 5 is satisfied. By Theorem 5, we have

Let \( \{X_n, n \geq 1\} \) be a sequence of \( \tilde{\rho} \)-mixing random variables with \( EX_n = 0, n \geq 1 \), and let \( \{b_n, n \geq 1\} \) be sequences of constants such that \( 0 < b_n \not\rightarrow \infty \). Assume that \( \{g_n(x), n \geq 1\} \) is a sequence of positive and Borel measurable functions on \([0, +\infty)\) satisfying (1). If

then

\[
\lim_{n \rightarrow \infty} \frac{1}{b_n} \sum_{i=1}^{n} X_i = 0 \text{ a.s.} \tag{19}
\]

Proof Let \( C_n = b_n, n \geq 1 \). The conditions (i) and (iii) of Theorem 5 are satisfied. Note that

Therefore, the condition (ii) of Theorem 5 is satisfied. By Theorem 5, we have

Since \( EX_n = 0, n \geq 1 \), by (1) and Kronecker Lemma, we have

\[
\frac{1}{b_n} \left| \sum_{i=1}^{n} EX_i I(|X_i| \leq b_i) \right| \leq \frac{1}{b_n} \sum_{i=1}^{n} E|X_i| I(|X_i| > b_i) \\
\leq \frac{1}{b_n} \sum_{i=1}^{n} \frac{b_i E g_i(|X_i|)}{g_i(b_i)} I(|X_i| > b_i) \leq \frac{1}{b_n} \sum_{i=1}^{n} \frac{b_i E g_i(|X_i|)}{g_i(b_i)} \rightarrow 0, \quad n \rightarrow \infty. \tag{21}
\]

From (20) and (21), (19) holds. \( \Box \)

References


