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C^1 Solutions of the Iterative Equation $G(x, f(x), \ldots, f^n(x)) = F(x)$

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Abstract In this paper we consider the iterative equation $G(x, f(x), \ldots, f^n(x)) = F(x)$ on $\mathbb{R}$, and give the existence of C^1 solutions near the fixed point of F, which generalize some results on the leading coefficient problem from the form of the polynomial-like iterative equations to the general form.

Keywords iterative equation; Schröder transformation; contractive solution; leading coefficient problem.

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1. Introduction

Iterative equations, a class of functional equations involving iteration of the unknown function, attract extensive interests [1, 2, 4, 12]. The problem of iterative roots [4] is to find the unknown function f such that $f^n(x) = F(x)$ and the well-known Feigenbaum equation $f(x) = -\frac{1}{\lambda} f(f(-\lambda x))$ are both iterative equations [8]. The general form of iterative equation can be represented as

$$G(x, f(x), f^2(x), \ldots, f^n(x)) = 0,$$

where f^n denotes the nth iterate of f, defined by $f^n(x) := f(f^{n-1}(x))$ and $f^0(x) = x$ inductively.

A special form of iterative equation is the polynomial-like iterative equation

$$\lambda_n f^n(x) + \lambda_{n-1} f^{n-1}(x) + \ldots + \lambda_1 f(x) = F(x), \quad x \in \mathcal{D} \subset \mathcal{X}, \tag{1}$$

where $\mathcal{X}$ is a Banach space over $\mathbb{R}$, $F : \mathcal{D} \to \mathcal{D}$ is a given mapping, λ_j's are real constants and $f : \mathcal{D} \to \mathcal{D}$ is the unknown mapping. For $\mathcal{X} = \mathbb{R}$ and $\mathcal{D}$ being a compact interval, the existence and uniqueness of continuous solutions of this equation were discussed in [16] in 1987. Later, C^1 and C^r smoothness of those solutions and some generalizations to high dimensional cases and to the general form

$$G(x, f(x), f^2(x), \ldots, f^n(x)) = F(x), \quad \in \mathcal{D} \subset \mathcal{X}, \tag{2}$$

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where G is a given multivariate function, were given in [5–7, 9–11, 13, 17]. Although those results were given globally, a requirement that $\lambda_1 > 0$ or that

$$\sum_{i=1}^{n} \alpha_i |y_i - \tilde{y}_i| \le |G(x, y_1, \ldots, y_n) - G(x, \tilde{y}_1, \ldots, \tilde{y}_n)| \le \sum_{i=1}^{n} \beta_i |y_i - \tilde{y}_i|,$$

where $\beta_i \ge \alpha_i \ge 0$ for all i and $\alpha_1 > 0$, was imposed. The nonzero constant λ_1 or α_1 prevents the equation from including the problem of iterative roots. The most natural way is to impose such a nonzero assumption to the coefficient of the highest order term, i.e., $\lambda_n > 0$ or $\alpha_n > 0$. This leads the so-called "leading coefficient problem" as mentioned in [15, 19].

The first answer to the leading coefficient problem of equation (1) with $\mathcal{X} = \mathbb{R}$ was given in [18], where locally expansive C^1 solutions were obtained for locally expansive F. Some constructive results on continuous solutions were given in [14]. Following [18], more cases were discussed for C^1 solutions in [3].

In this paper we generalize some results on the leading coefficient problem from equation (1) to Eq. (2). We give locally contractive C^1 solutions for Eq. (2) near the fixed point of F in increasing and decreasing cases separately for locally contractive or expansive F on $\mathbb{R}$.

2. Main results

Let $C^1(\mathbb{R}, \mathbb{R})$ denote the set of all continuously differentiable self-mappings on $\mathbb{R}$. For convenience we let X denote $(x_0, x_1, \ldots, x_n)$ and O denote $(0, 0, \ldots, 0)$ for short. Let $G_i'(X)$ denote the partial derivative $\partial G/\partial x_i$ simply, $i = 0, 1, \ldots, n$. A main result is the following:

Theorem 1 *Let $F \in C^1(\mathbb{R}, \mathbb{R})$ satisfy $F(0) = 0$ and $G : \mathbb{R}^{n+1} \to \mathbb{R}$ be continuously differentiable such that $G(O) = 0$. Suppose that*

(H_F^+) F' is locally Lipschitzian and $F'(x) \ge F'(0) > 0$ in a neighborhood of 0,

(H_G^+) $0 \le G_i'(X) \le G_i'(O)$ for all $i = 0, 1, \ldots, n$ in a neighborhood of O in $\mathbb{R}^{n+1}$ and

$$|G_i'(x_0, x_1, \ldots, x_n) - G_i'(y_0, y_1, \ldots, y_n)| \le \sum_{j=0}^{n} L_{ij} |x_j - y_j|$$

for $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ in a neighborhood of O, where $L_{ij}s$ $(i, j = 0, 1, \ldots, n)$ are nonnegative constants.

(H) $|G_0'(O)| < |F'(0)| < \sum_{i=0}^{n} |G_i'(O)|$.

Then Eq. (2) has a locally contractive increasing C^1 solution near 0.

In this theorem, instead of hypothesis $|G_1'(O) \ge l > 0|$ (see [11, 13]) we assume that $|G_i'(O) \ge 0|$, $i = 1, 2, \ldots, n$, which means the equation in our theorem includes the iterative root problem as a special case.

In order to prove Theorem 1, we apply the Schröder transformation $f(x) = \phi(c\phi^{-1}(x))$ to change Eq. (2) into the auxiliary equation

$$G(\phi(s), \phi(cs), \phi(c^2 s), \ldots, \phi(c^n s)) = F(\phi(s)). \tag{3}$$

Lemma 1 *Under the conditions of Theorem 1, there exist constants $0 < c < 1$ and $\sigma > 0$ such* *that for arbitrarily given* $0 < \tau < 1/|F'(0)|$, *Eq. (3) with such a constant* c *has a* C^1 *solution* ϕ *on* $[-\sigma, \sigma]$ *with* $\phi(0) = 0$ *and* $\phi'(0) = \tau$.

Proof Differentiate Eq. (2) at $x = 0$. We see that if Eq. (2) has a C^1 solution f near 0, then $f'(0)$ must be a real root of polynomial

$$P(\mu) := G'_n(O)\mu^n + G'_{n-1}(O)\mu^{n-1} + \cdots + G'_1(O)\mu + G'_0(O) - F'(0).$$

The hypotheses (H), (H_F^+) and the signs of $G'_i(O)$, $i = 0, 1, \dots, n$ imply $P(0) < 0$ and $P(1) > 0$. By the continuity $P(\mu)$ has a root $0 < c < 1$. Consider Eq. (3) with such a constant c. From hypothesis (H) we note $|G'_0(O)| < |F'(0)|$, then we can choose a k satisfying $0 \le k < 1$ such that $|G'_0(O)| = k|F'(0)|$. If $|G'_0(O)| = 0$ we take $k = 0$. By (H_F^+), (H_G^+) and the continuity of F', we choose suitable positive constants σ_1, K_1 and $M \in (|F'(0)|, |F'(0)|/((1 - |c|)k + |c|))$ such that $0 < |F'(x)| \le M$ and $|F'(x) - F'(y)| \le K_1|x - y|$ for all $x, y \in [-\sigma_1, \sigma_1]$. We further choose a $\sigma \in (0, \sigma_1)$ such that $[-\sigma, \sigma] \subset F([-\sigma_1, \sigma_1])$. Then F^{-1} is well defined in $[-\sigma, \sigma]$. For arbitrary given $\tau \in (0, 1/|F'(0)|]$, let

$$K := \frac{K_1\tau^2|F'(0)| + M\tau^2 \sum_{i=0}^n \sum_{j=0}^n L_{ij}|c^{i+j}|}{|F'(0)|(|F'(0)| - M((1 - |c|)k + |c|)}.$$

Define a subset of $C^1[-\sigma, \sigma]$ by

$$\begin{aligned}\mathcal{A} :=& \{\phi \in C^1[-\sigma, \sigma] : \phi(0) = 0, \phi'(0) = \tau, 0 \le \phi'(x) \le \phi'(0) = \tau, \text{ and} \\ & |\phi'(x) - \phi'(y)| \le K|x - y|, \ \forall x, y \in I\}.\end{aligned}$$

Using Ascoli-Arzela Lemma, we can verify that $\mathcal{A}$ is uniformly bounded and equi-continuous. Hence $\mathcal{A}$ is a convex compact subset of $C^1[-\sigma, \sigma]$, a Banach space endowed with the norm $\|\cdot\|_1$, defined by $\|\phi\|_1 := \max\{\|\phi\|, \|\phi'\|\}$ and $\|\phi\| := \sup\{|\phi(x)| : x \in [-\sigma, \sigma]\}$ for $\phi \in C^1[-\sigma, \sigma]$. Define a mapping $\mathcal{G} : \mathcal{A} \to C^1[-\sigma, \sigma]$ by

$$\mathcal{G}\phi(s) := G(\phi(s), \phi(cs), \dots, \phi(c^n s)), \quad \forall \phi \in \mathcal{A}.$$

We calculate that

$$0 \le |\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{G}\phi(s)| = |\sum_{i=0}^n G'_i(\phi(s), \phi(cs), \dots, \phi(c^n s))c^i\phi'(c^i s)| \le |\sum_{i=0}^n G'_i(O)c^i\phi'(0)| \le 1,$$

which means the range of $\mathcal{G}\phi$ is included in $[-\sigma, \sigma]$.

For convenience, in what follows let $\Phi(s)$ denote the vector $(\phi(s), \phi(cs), \dots, \phi(c^n s))$. Define a map $\mathcal{T} : \mathcal{A} \to C^1[-\sigma, \sigma]$ by

$$\mathcal{T}\phi := F^{-1} \circ \mathcal{G}\phi, \quad \forall \phi \in \mathcal{A}.$$

We can check that $\mathcal{T}\phi(0) = 0$ and

$$0 \le \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}\phi(s) = \frac{\sum_{i=0}^n G'_i(\Phi(s))c^i\phi'(c^i s)}{F'(F^{-1} \circ \mathcal{G}\phi(s))} \le \frac{\sum_{i=0}^n G'_i(O)c^i\phi'(0)}{F'(0)} = \phi'(0) = \tau.$$

In order to prove $\mathcal{T}$ is a self-mapping on the set $\mathcal{A}$, for all $x, y \in [-\sigma, \sigma]$ and $i = 0, 1, \dots, n$

we give following inequalities

$$|G_i'(\Phi(x)) - G_i'(\Phi(y))| \leq \sum_{j=0}^{n} L_{ij}|\phi(c^j x) - \phi(c^j y)| \leq (\tau \sum_{j=0}^{n} L_{ij}|c^j|)|x-y|.$$

Therefore, we have

$$\begin{aligned}
&|G_i'(\Phi(x)))\phi'(c^i x) - G_i'(\Phi(y)))\phi'(c^i y)| \\
&\quad \leq |G_i{}'(\Phi(x))||\phi'(c^i x) - \phi'(c^i y)| + |\phi'(c^i y)||G_i{}'(\Phi(x)) - G_i{}'(\Phi(y))| \\
&\quad \leq (K|G_i{}'(O)c^i| + \tau^2 \sum_{j=0}^{n} L_{ij}|c^j|))|x-y|. \qquad (4)
\end{aligned}$$

By the C^1 continuity of G and the Lipschtiz constant K_1 of F',

$$\begin{aligned}
&|F'(F^{-1} \circ G(\Phi(x))) - F'(F^{-1} \circ G(\Phi(y)))| \\
&\quad \leq K_1|F^{-1}(\Phi(x)) - F^{-1}(\Phi(y))| \leq \frac{K_1}{|F'(0)|}|G(\Phi(x)) - G(\Phi(y))| \\
&\quad \leq \frac{K_1}{|F'(0)|}\tau \sum_{i=0}^{n} |G_i'(O)c^i||x-y| = K_1\tau|x-y|. \qquad (5)
\end{aligned}$$

Inequalities (4) and (5) imply that

$$\begin{aligned}
&|(\mathcal{T}\phi)'(x) - (\mathcal{T}\phi)'(y)| \\
&= |\frac{1}{F'(F^{-1} \circ G(\Phi(x)))F'(F^{-1} \circ G(\Phi(y)))}| \cdot \\
&\quad |F'(F^{-1} \circ G(\Phi(y))) \sum_{i=0}^{n} c^i G_i'(\Phi(x))\phi'(c^i x) - F'(F^{-1} \circ G(\Phi(x))) \sum_{i=0}^{n} c^i G_i'(\Phi(y))\phi'(c^i y)| \\
&\leq \frac{1}{|F'(0)|^2}\{|F'(F^{-1} \circ G(\Phi(y))| \cdot |\sum_{i=0}^{n} c^i\{G_i'(\Phi(x))\phi'(c^i x) - G_i'(\Phi(y))c^i\phi'(c^i y)\}| + \\
&\quad |\sum_{i=0}^{n} G_i'(\Phi(y))c^i\phi'(c^i y)| \cdot |F'(F^{-1} \circ G(\Phi(y)) - F'(F^{-1} \circ G(\Phi(x))|\} \\
&\leq \frac{1}{|F'(0)|^2}\{M|\sum_{i=0}^{n} c^i(G_i'(O)Kc^i + \tau^2\sum_{j=0}^{n} L_{ij}c^j)| + K_1\tau^2\sum_{i=0}^{n}|c^i G_i'(O)|\}|x-y| \\
&= \frac{1}{|F'(0)|^2}\{MK\sum_{i=0}^{n}|c^{2i}G_i'(O)| + M\tau^2\sum_{i=0}^{n}\sum_{j=0}^{n} L_{ij}|c^{i+j}| + K_1\tau^2|F'(0)|\}|x-y| \\
&= \frac{M}{|F'(0)|} \cdot \frac{(1-|c|)|G_0'(O)| + (|cG_0'(O)| + \sum_{i=1}^{n}|c^{2i}G_i'(O)|)}{|F'(0)|} \cdot K|x-y| + \\
&\quad \frac{1}{|F'(0)|^2}\{K_1\tau^2|F'(0)| + M\tau^2\sum_{i=0}^{n}\sum_{j=0}^{n} L_{ij}|c^{i+j}|\}|x-y| \\
&\leq \frac{M}{|F'(0)|} \cdot \frac{(1-|c|)k|F'(0)| + |c|(\sum_{i=0}^{n}|c^i G_i'(O)|)}{|F'(0)|} \cdot K|x-y| + \\
&\quad \frac{1}{|F'(0)|^2}\{K_1\tau^2|F'(0)| + M\tau^2\sum_{i=0}^{n}\sum_{j=0}^{n} L_{ij}|c^{i+j}|\}|x-y|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{M}{|F'(0)|} \cdot \frac{(1-|c|)k|F'(0)| + |cF'(0)|}{|F'(0)|} \cdot K|x-y| + \\
&\quad \frac{1}{|F'(0)|^2}\{K_1\tau^2|F'(0)| + M\tau^2 \sum_{i=0}^{n}\sum_{j=0}^{n} L_{ij}|c^{i+j}|\}|x-y| \\
&= K|x-y|,
\end{aligned}
$$

which implies $\mathcal{T}(\mathcal{A}) \subset \mathcal{A}$. Considering $\phi_1, \phi_2 \in \mathcal{A}$, we have

$$
\begin{aligned}
\|\mathcal{T}\phi_1 - \mathcal{T}\phi_2\| &= \max_{x\in[-\sigma,\sigma]} |F^{-1}\circ\mathcal{G}\phi_1(x) - F^{-1}\circ\mathcal{G}\phi_2(x)| \\
&\le \max_{x\in[-\sigma,\sigma]} \frac{|G(\phi_1(x), \phi_1(cx), \ldots, \phi_1(c^n x)) - G(\phi_2(x), \phi_2(cx), \ldots, \phi_2(c^n x))|}{|F'(0)|} \\
&\le \frac{\sum_{i=0}^{n}|G_i'(O)|}{|F'(0)|}\|\phi_1 - \phi_2\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}\phi_1 - \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{T}\phi_2\| \\
&= \max_{x\in[-\sigma,\sigma]} |\frac{\sum_{i=0}^{n} G_i'(\Phi_1(x))c^i{\phi'}_1(c^i x)}{F'(F^{-1}\circ\mathcal{G}\phi_1(x))} - \frac{\sum_{i=0}^{n} G_i'(\Phi_2(x))c^i{\phi'}_2(c^i x)}{F'(F^{-1}\circ\mathcal{G}\phi_2(x))}| \\
&\le \max_{x\in[-\sigma,\sigma]} \frac{1}{|F'(F^{-1}\circ\mathcal{G}\phi_1(x))F'(F^{-1}\circ\mathcal{G}\phi_2(x))|}\cdot \\
&\quad \{|F'(F^{-1}\circ\mathcal{G}\phi_2(x))|\sum_{i=0}^{n}|c^i||G_i'(\Phi_1(x)){\phi'}_1(c^i x) - G_i'(\Phi_2(x)){\phi'}_2(c^i x)| + \\
&\quad |F'(F^{-1}\circ\mathcal{G}\phi_2(x)) - F'(F^{-1}\circ\mathcal{G}\phi_1(x))|\sum_{i=0}^{n}|c^i||G_i'(\Phi_2(x)){\phi'}_2(c^i x)|\} \\
&\le \frac{1}{|F'(0)|^2}\{M\max_{x\in[-\sigma,\sigma]}\sum_{i=0}^{n}|c^i||G_i'(\Phi_1(x)){\phi'}_1(c^i x) - G_i'(\Phi_2(x)){\phi'}_1(c^i x) + \\
&\quad G_i'(\Phi_2(x)){\phi'}_1(c^i x) - G_i'(\Phi_2(x)){\phi'}_2(c^i x)| + \\
&\quad K_1\tau\sum_{i=0}^{n}|G_i'(O)c^i|\max_{x\in[-\sigma,\sigma]}|F^{-1}\circ\mathcal{G}\phi_1(x) - F^{-1}\circ\mathcal{G}\phi_2(x)|\} \\
&\le \frac{1}{|F'(0)|^2}\{M\tau\sum_{i=0}^{n}|c^i|\max_{x\in[-\sigma,\sigma]}|G_i'(\Phi_i(x)) - G_i'(\Phi_2(x))| + \\
&\quad M\sum_{i=0}^{n}|G_i'(O)c^i|\|\phi_1 - \phi_2\|_1 + K_1\tau\sum_{i=0}^{n}|G_i'(O)|\|\phi_1 - \phi_2\|\} \\
&\le \frac{1}{|F'(0)|^2}\{M\tau\sum_{i=0}^{n}\sum_{j=0}^{n}L_{ij}|c^i| + M\sum_{i=0}^{n}|G_i'(O)c^i| + \\
&\quad K_1\tau\sum_{i=0}^{n}|G_i'(O)|\}\|\phi_1 - \phi_2\|_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}\phi_1-\mathcal{T}\phi_2\| \le \max\{\frac{\sum_{i=0}^n|G_i'(O)|}{|F'(0)|}, \frac{1}{|F'(0)|^2}(M\tau\sum_{i=0}^n\sum_{j=0}^n L_{ij}|c^i|+M\sum_{i=0}^n|G_i'(O)c^i|+ K_1\tau\sum_{i=0}^n|G_i'(O)|)\}\|\phi_1-\phi_2\|_1.$$

This proves that $\mathcal{T}$ maps the convex compact subset $\mathcal{A}$ continuously into itself. By Schauder's fixed point theorem, there exists a function $\phi\in\mathcal{A}$ such that $\mathcal{T}\phi=\phi$, which means ϕ is a solution of equation (3) with derivative τ at 0. This completes the proof. □

Proof of Theorem 1 Choose $c,\sigma,\tau,\ \phi$ as in Lemma 1. Because of the continuity of ϕ', we choose a small closed subset I of $[-\sigma,\sigma]$, a neighborhood of 0, such that ϕ^{-1} exists and is C^1 on I. Moreover, $c^i\phi^{-1}(x)\in[-\sigma,\sigma]$ for $i=1,2,\ldots,n$ since $0<c<1$. Let $f(x):=\phi(c\phi^{-1}(x))$ for $x\in I$. Obviously, f is C^1 and invertible on I and satisfies $f(0)=0$ and $f'(0)=\phi'(0)c(\phi^{-1})'(0)=c$. One can check that

$$\begin{aligned}G(x,f(x),f^2(x),\ldots,f^n(x)) =&G(\phi(\phi^{-1}(x)),\phi(c\phi^{-1}(x)),\phi(c^2\phi^{-1}(x)),\ldots,\phi(c^n\phi^{-1}(x)))\\ =&F(\phi(\phi^{-1}(x)))=F(x),\quad \forall x\in I,\end{aligned}$$

implying that f is a solution of Eq. (2) near 0. The proof is completed. □

Theorem 1 gives a local contractive increasing solution for increasing F. The following theorems give locally contractive decreasing solutions for decreasing F and increasing F, respectively.

Theorem 2 *Suppose that n is odd, $F\in C^1(\mathbb{R},\mathbb{R})$ satisfies $F(0)=0$ and $G:\mathbb{R}^{n+1}\to\mathbb{R}$ is continuously differentiable such that $G(O)=0$ and (H) holds. Suppose that*

[(H_F^-)] F' is locally Lipschitzian and $F'(x)\le F'(0)<0$ in a neighborhood of 0, and

[$(H_G^\pm)$] $0\le G_i'(X)\le G_i'(O)$ for odd i and $G_i'(O)\le G_i'(X)\le 0$ for even i, $i=0,1,\ldots,n$ in a neighborhood of O in $\mathbb{R}^{n+1}$ and

$$|G_i'(x_0,x_1,\ldots,x_n)-G_i'(y_0,y_1,\ldots,y_n)|\le\sum_{j=0}^n L_{ij}|x_j-y_j|$$

for $(x_0,x_1,\ldots,x_n)$ and $(y_0,y_1,\ldots,y_n)$ in a neighborhood of O, where $L_{ij}s$ $(i,j=0,1,\ldots,n)$ are nonnegative constants.

Then Eq. (2) has a locally contractive decreasing C^1 solution near 0.

Theorem 3 *Suppose that n is even. Let $F\in C^1(\mathbb{R},\mathbb{R})$ satisfy $F(0)=0$ and F' satisfy (H_F^+). $G:\mathbb{R}^{n+1}\to\mathbb{R}$ be continuously differentiable such that $G(O)=0$ and (H) holds. Suppose that*

[$(H_G^\mp)$] $G_i'(O)\le G_i'(X)\le 0$ for odd i and $0\le G_i'(X)\le G_i'(O)$ for even i, $i=0,1,\ldots,n$ in a neighborhood of O in $\mathbb{R}^{n+1}$ and

$$|G_i'(x_0,x_1,\ldots,x_n)-G_i'(y_0,y_1,\ldots,y_n)|\le\sum_{j=0}^n L_{ij}|x_j-y_j|$$

for $(x_0,x_1,\ldots,x_n)$ and $(y_0,y_1,\ldots,y_n)$ in a neighborhood of O, where $L_{ij}s$ $(i,j=0,1,\ldots,n)$ are nonnegative constants.

Then Eq. (2) has a locally contractive decreasing C^1 solution near 0.

In order to prove Theorem 2 (resp., Theorem 3), we first notice that $P(-1) < 0$ (resp., $P(-1) > 0$) and $P(0) > 0$ (resp., $P(0) < 0$), which implies that P has a root $-1 < c < 0$ by the continuity. Then we consider Eq. (3) with this constant c. The rest of proof for Theorem 2 (resp., Theorem 3) is similar to that of Theorem 1.

Remark that we do not consider the increasing solution for decreasing F throughout our paper. If the characteristic polynomial P has a positive real root, we have the fact that at least one of $G_i'(O), i = 0, 1, \dots, n$ must be negative since $F'(0) < 0$. By this method we cannot come to analogous conclusion as above.

3. Examples

Example 1 The equation $\sin(A_0x(1+x^2)^{-\frac{1}{2}} + A_1f^2(x)) + \sin(A_2f^3(x)) = a^2x^5 + A_3x$, where x is in a neighborhood of 0, and A_i, $i = 0, 1, 2, 3$, are nonnegative constants and $A_i\,s$ satisfy $A_0 < A_3 < A_0 + A_1 + A_2$. In this equation, $G(x_0, x_1, x_2, x_3) = G(X) = \sin(A_0x_0(1+x_0^2)^{-\frac{1}{2}} + A_1x_2) + \sin(A_2x_3)$, and we calculate that $G_0'(X) = A_0\cos(A_0x_0(1+x_0^2)^{-\frac{1}{2}} + A_1x_2)((1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}})$, and $G_2'(X) = A_1\cos(A_0x_0(1+x_0^2)^{-\frac{1}{2}} + A_1x_2)$, $G_3'(X) = A_2\cos(A_2x_3)$. It is easy to verify that $G(O) = 0$, and $0 \le G_i'(X) \le G_i'(O)$ for $i = 0, 1, 2, 3$. The rest hypotheses of Theorem 1 can be verified similarly. So this equation has a locally invertible contractive increasing C^1 solution in a neighborhood of $x = 0$.

Example 2 The equation $\sin(A_0x + A_1f^4(x)) + A_2\arctan(f^5(x)) = A_3x(1-x^2)^{-\frac{1}{2}}$, where x is in a neighborhood of 0, and A_i, $i = 0, 1, 2, 3$, are real constants. A_is satisfy $A_0 \le 0$, $A_1 \le 0$, $A_2 \ge 0$, $A_3 < 0$ and $|A_0| < |A_3| < |A_0| + |A_1| + |A_2|$. It is easy to verify that $G(x_0, x_1, x_2, x_3, x_4, x_5) = G(X) = \sin(A_0x_0 + A_1x_4) + A_2\arctan x_5$, and $G_0'(X) = A_0\cos(A_0x_0 + A_1x_4)$, $G_4'(X) = A_1\cos(A_0x_0 + A_1x_4)$, $G_5'(X) = A_2(1+x_5^2)^{-1}$. Simultaneously, we notice that $F(x) = A_3x(1-x^2)^{-\frac{1}{2}}$, and $F'(x) = A_3((1-x^2)^{-\frac{1}{2}} + x^2(1-x^2)^{-\frac{3}{2}})$. Then this equation satisfies the hypotheses of Theorem 2. It has a locally invertible contractive decreasing C^1 solution in a neighborhood of $x = 0$.

Example 3 The equation $\sin(A_0\ln(x+(1+x^2)^{-\frac{1}{2}}) + A_1f^2(x)) + A_2f^4(x) = \frac{A_3}{2}\ln\frac{1+x}{1-x}$, where x is in a neighborhood of 0, A_i, $i = 0, 1, 2$, are nonnegative constants, and $A_i\,s$ satisfy $A_0 < A_3 < A_0 + A_1 + A_2$. In this equation, $G(x_0, x_1, x_2, x_3, x_4) = G(X) = \sin(A_0\ln(x_0+(1+x_0^2)^{-\frac{1}{2}}) + A_1x_2) + A_2x_4$, and $G_0'(X) = A_0\cos(A_0\ln(x_0 + (1+x_0^2)^{-\frac{1}{2}} + A_1x_2)(1+x_0^2)^{-\frac{1}{2}}$, $G_2'(X) = A_1\cos(A_0\ln(x_0 + (1+x_0^2)^{-\frac{1}{2}} + A_1x_2)$, $G_4'(X) = A_2$. $F(x) = \frac{A_3}{2}\ln\frac{1+x}{1-x}$, and $F'(x) = \frac{A_3}{1-x^2}$. By Theorem 3 it has a locally invertible contractive decreasing C^1 solution in a neighborhood of $x = 0$.

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