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Gould-Hsu Inversion Chains and Their Applications

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Abstract In this paper, by means of Gould-Hsu inverse series relations, we establish several Gould-Hsu inversion chains. As consequence, some new transformation formulae as well as some famous hypergeometric series identities are derived.

Keywords Gould-Hsu inversion; combinatorial identities; hypergeometric series.

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1. Introduction

Gould-Hsu inversions, which were established by Gould and Hsu in 1973 (see [6]), play an important role in the computation of combinatorial identities. With the help of Gould-Hsu inversions, many authors obtained a great deal of new transformation formulae, especially Chu [2, 4] and Krattenthaler [3].

The Gould-Hsu inversions are described as follows:

Let $\{a_i\}$ and $\{b_j\}$ be two complex sequences, and let the polynomials $\phi(x;n)$ be defined by

$$\phi(x;0)=1 \quad \text{and} \quad \phi(x;n)=\prod_{k=0}^{n-1}(a_k+xb_k), \quad \text{for} \quad n=1,2,\ldots .$$

Then we have the following inverse series relations:

$$\begin{cases} f(n)=\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(k;n)g(k), & n=0,1,2\ldots, \\ g(n)=\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k+kb_k}{\phi(n;k+1)}f(k), & n=0,1,2\ldots . \end{cases} \tag{1}$$

If a pair of sequences $\{f(n),g(n)\}$ satisfy the above relation, then we call $\{f(n),g(n)\}$ a Gould-Hsu inversion pair, or a G'-pair for short. Gould-Hsu inversion chain was first put forward in [7], where two chains were given by the authors. In this paper, we will establish several new Gould-Hsu inversion chains by making use of hypergeometric series transformations and the following Lemma:

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Lemma 1 *Let*

$$a_{nk} = (-1)^k \binom{n}{k} \phi(k;n), \quad b_{nk} = (-1)^k \binom{n}{k} \frac{a_k + kb_k}{\phi(n;k+1)}.$$

If $\{f(n), g(n)\}$ *is a G'-pair, so is* $\{f'(n), g'(n)\}$, *where*

$$f'(n) = c_n f(n),\ g'(n) = \sum_{k=0}^{n} d_{nk} g(k) \ \text{ and } \ d_{nk} = \sum_{j=k}^{n} b_{nj} c_j a_{jk}.$$

Proof Let matrices $A = (a_{nk})$, $B = (b_{nk})$, $D = (d_{nk})$ and a diagonal matrix $C = (c_0, c_1, \ldots, c_n)$. It is easy to see that A, B, and D are lower triangle matrices, and

$$f = Ag,\ g = Bf,\ f' = Cf,\ g' = Dg \ \text{ and } \ D = BCA.$$

Further, by means of (1), $AB = BA = I$, so we can easily verify

$$f' = Ag' \ \text{ and } \ g' = Bf',$$

which indeed proves the lemma. □

If $\{f(n), g(n)\}$ is a G'-pair, from Lemma 1, we can establish another G'-pair $\{f'(n), g'(n)\}$. Repeating this process, we can create a Gould-Hsu inversion chain:

$$\{f(n), g(n)\} \longrightarrow \{f'(n), g'(n)\} \longrightarrow \{f''(n), g''(n)\} \longrightarrow \cdots.$$

As consequence, in Section 3 some new transformation formulae are derived.

Because hypergeometric series plays an central role in the present work, we introduce its notation as follows. According to Bailey [1] and Slater [8], the hypergeometric series is defined as

$${}_{r+1}F_s \left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\Big|\, z \right] = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} z^n,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$. In this paper, we will apply the following famous hypergeometric series identities: Gauss's theorem [1, 1.3 (1)]:

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} \,\Big|\, 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{2}$$

Vandermonde's theorem [1, 1.3]:

$${}_2F_1 \left[\begin{matrix} -n, b \\ c \end{matrix} \,\Big|\, 1 \right] = \frac{(c-b)_n}{(c)_n}, \tag{3}$$

Dougall-Dixon formula [1, 4.3 (3)]:

$${}_5F_4 \left[\begin{matrix} a, & 1+a/2, & c, & d, & -m \\ a/2, & 1+a-c, & 1+a-d, & 1+a+m & \end{matrix} \,\Big|\, 1 \right] = \frac{(1+a)_m (1+a-c-d)_m}{(1+a-c)_m (1+a-d)_m}. \tag{4}$$

Whipple formula [1, 4.3 (4)]:

$${}_7F_6 \left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & e, & -m \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m & \end{matrix} \,\Big|\, 1 \right]$$

$$= \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m} \ {}_4F_3\left[\begin{matrix} 1+a-b-c, & d, & e, & -m \\ 1+a-b, 1+a-c, d+e-a-m \end{matrix} \Big| \, 1\right]. \tag{5}$$

2. Several Gould-Hsu inversion chains

Theorem 2 *If $\{f(n), g(n)\}$ is a G'-pair with $\phi(x;n)=1$, and*

$$f'(n) = x^n f(n), \quad g'(n) = \sum_{k=0}^{n} x^k \binom{n}{k}(1-x)^{n-k} g(k),$$

then $\{f'(n), g'(n)\}$ is a G'-pair too.

Proof Since $\phi(x;n)=1$, we have

$$a_{nk} = b_{nk} = (-1)^k \binom{n}{k},$$

$$c_n = x^n \quad \text{and} \quad d_{nk} = x^k \binom{n}{k}(1-x)^{n-k}.$$

Therefore

$$\begin{aligned}\sum_{j=k}^{n} b_{nj}c_j a_{jk} =& \sum_{j=k}^{n}(-1)^j \binom{n}{j} x^j (-1)^k \binom{j}{k} = \binom{n}{k} \sum_{j=0}^{n-k}(-1)^j \binom{n-k}{j} x^{j+k} \\ =& x^k \binom{n}{k}(1-x)^{n-k},\end{aligned}$$

which is just equal to d_{nk}. By Lemma 1, the proof of the theorem is completed. □

Theorem 3 *If $\{f(n), g(n)\}$ is a G'-pair with $\phi(x;n)=1$ and*

$$f'(n) = \frac{(b)_n}{(c)_n} f(n), \quad g'(n) = \sum_{k=0}^{n} \binom{n}{k} \frac{(b)_k (c-b)_{n-k}}{(c)_n} g(k),$$

then $\{f'(n), g'(n)\}$ is a G'-pair too.

Proof In this case, a_{nk} and b_{nk} are the same as those in the above theorem, and

$$c_n = \frac{(b)_n}{(c)_n}, \quad d_{nk} = \binom{n}{k} \frac{(b)_k (c-b)_{n-k}}{(c)_n}.$$

Then

$$\sum_{j=k}^{n} b_{nj}c_j a_{jk} = \sum_{j=k}^{n}(-1)^j \binom{n}{j} \frac{(b)_j}{(c)_j}(-1)^k \binom{j}{k}.$$

Replacing j by $j+k$ and then applying the Vandermonde's theorem (3), we obtain

$$\sum_{j=k}^{n} b_{nj}c_j a_{jk} = \binom{n}{k} \frac{(b)_k}{(c)_k} \ {}_2F_1\left[\begin{matrix} -n+k, b+k \\ c+k \end{matrix} \Big| \, 1\right] = \binom{n}{k} \frac{(b)_k (c-b)_{n-k}}{(c)_n},$$

which is just equal to d_{nk}, and the result follows from Lemma 1. □

Theorem 4 *If $\{f(n),g(n)\}$ is a G'-pair with $\phi(x;n)=(a+x)_n$ and*

$$f'(n)=\frac{(c)_n(d)_n}{(1+a-c)_n(1+a-d)_n}f(n),$$
$$g'(n)=\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_n(c)_k(d)_k(1+a-c-d)_{n-k}}{(a)_k(1+a-c)_n(1+a-d)_n}g(k),$$

then $\{f'(n),g'(n)\}$ is a G'-pair too.

Proof Since $\phi(x;n)=(a+x)_n$, we have

$$a_{nk}=(-1)^k\binom{n}{k}(a+k)_n,\quad b_{nk}=(-1)^k\binom{n}{k}\frac{2k+a}{(a+n)_{k+1}},$$
$$c_n=\frac{(c)_n(d)_n}{(1+a-c)_n(1+a-d)_n},\quad d_{nk}=\binom{n}{k}\frac{(a)_n(c)_k(d)_k(1+a-c-d)_{n-k}}{(a)_k(1+a-c)_n(1+a-d)_n}.$$

Replacing j by $j+k$, we obtain

$$\begin{aligned}\sum_{j=k}^{n}b_{nj}c_ja_{jk}&=\sum_{j=k}^{n}(-1)^j\binom{n}{j}\frac{2j+a}{(a+n)_{j+1}}\frac{(c)_j(d)_j}{(1+a-c)_j(1+a-d)_j}(-1)^k\binom{j}{k}(a+k)_j\\
&=\binom{n}{k}\frac{(c)_k(d)_k(a+k)_{k+1}}{(1+a-c)_k(1+a-d)_k(a+n)_{k+1}}\\
&\quad\sum_{j=0}^{n-k}\frac{(a+2k)_j(2j+2k+a)(c+k)_j(d+k)_j(-n+k)_j}{j!(2k+a)(1+a-c+k)_j(1+a-d+k)_j(1+a+n+k)_j}\\
&=\binom{n}{k}\frac{(c)_k(d)_k(a+k)_{k+1}}{(1+a-c)_k(1+a-d)_k(a+n)_{k+1}}\\
&\quad {}_5F_4\left[\begin{matrix}a+2k, & 1+a/2+k, & c+k, & d+k, & -n+k\\ a/2+k, & 1+a-c+k, & 1+a-d+k, & 1+a+n+k\end{matrix}\,\Big|\,1\right].\end{aligned}\tag{6}$$

Applying Dougall-Dixon formula (4) to the last hypergeometric series gives

$$(6)=\frac{(1+a+2k)_{n-k}(1+a-c-d)_{n-k}}{(1+a-c+k)_{n-k}(1+a-d+k)_{n-k}}.$$

Finally, we find

$$\sum_{j=k}^{n}b_{nj}c_ja_{jk}=\binom{n}{k}\frac{(a)_n(c)_k(d)_k(1+a-c-d)_{n-k}}{(a)_k(1+a-c)_n(1+a-d)_n}=d_{nk}.$$

By Lemma 1, the proof of the theorem is completed. $\Box$

Theorem 5 *If $\{f(n),g(n)\}$ is a G'-pair with $\phi(x;n)=(a+x)_n$ and*

$$f'(n)=\frac{(c)_n(d)_n(e)_n(f)_n}{(1+a-c)_n(1+a-d)_n(1+a-e)_n(1+a-f)_n}f(n),$$
$$g'(n)=\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_n(1+a-e-f)_{n-k}(c)_k(d)_k(e)_k(f)_k}{(a)_k(1+a-c)_k(1+a-d)_k(1+a-e)_n(1+a-f)_n}$$

$$
{}_4F_3\left[\begin{matrix}1+a-c-d, e+k, f+k, -n+k\\ 1+a-c+k, 1+a-d+k, e+f-a-n+k\end{matrix}\,\Big|\,1\right] g(k),
$$

then $\{f'(n), g'(n)\}$ *is a G'-pair too.*

By making use of Whipple formula (5), similarly to the proof of Theorem 4, we can easily obtain this theorem. Here, we omit the details.

3. Applications

Iterating Theorems 2 and 3 one time, we have the following results.

Theorem 6 *If* $\{f(n), g(n)\}$ *is a G'-pair with* $\phi(x;n)=1$, *then*

$$
\sum_{k=0}^{n} x^k \binom{n}{k}(1-x)^{n-k} g(k) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} x^k f(k), \tag{7}
$$

$$
\sum_{k=0}^{n} \binom{n}{k}\frac{(b)_k(c-b)_{n-k}}{(c)_n} g(k) = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\frac{(b)_k}{(c)_k} f(k). \tag{8}
$$

When $\phi(x;n)=1$, we can present many Gould-Hsu inversion pairs. For example, from binomial theorem, $\{x^n, (1-x)^n\}$ is a G'-pair; from Vandermonde theorem (3), $\{\frac{(e-d)_n}{(e)_n}, \frac{(d)_n}{(e)_n}\}$ is a G'-pair; from (4.2) in [5], $\{\binom{d+n}{e}^{-1}, \frac{e}{n+e}\binom{n+d}{d-e}^{-1}\}$ is a G'-pair; from (7.6) in [5], there exists a G'-pair $\{\binom{2n}{n}\binom{j+n}{n}^{-1}2^{-2n}, \binom{2n+2j}{n+j}\binom{2j}{j}^{-1}2^{-2n}\}$. Certainly, there exist some other G'-pairs from the identities listed in the book of Gould [5]. Here we do not present all. Inserting the above G'-pairs into Theorem 6, we can obtain some new transformation formulae.

- Inserting G'-pair $\{x^n, (1-x)^n\}$ into (8), we deduce

$$
\frac{(c-b)_n}{(c)_n}\ {}_2F_1\left[\begin{matrix}-n, b\\ 1+b-c-n\end{matrix}\,\Big|\,1-x\right] = {}_2F_1\left[\begin{matrix}-n, b\\ c\end{matrix}\,\Big|\,x\right].
$$

- Inserting G'-pair $\{\frac{(e-d)_n}{(e)_n}, \frac{(d)_n}{(e)_n}\}$ into (7) and (8), respectively, we get

$$
\begin{aligned}
&(1-x)^n\ {}_2F_1\left[\begin{matrix}-n, d\\ e\end{matrix}\,\Big|\,\frac{-x}{1-x}\right] = {}_2F_1\left[\begin{matrix}-n, e-d\\ e\end{matrix}\,\Big|\,x\right],\\
&\frac{(c-b)_n}{(c)_n}\ {}_3F_2\left[\begin{matrix}-n, b, d\\ e, 1+b-c-n\end{matrix}\,\Big|\,1\right] = {}_3F_2\left[\begin{matrix}-n, b, e-d\\ c, e\end{matrix}\,\Big|\,1\right].
\end{aligned} \tag{9}
$$

In this case, the first one in (9) is just the special form of [1, 2.4 (1)].

- Inserting G'-pair $\{\binom{d+n}{e}^{-1}, \frac{e}{n+e}\binom{n+d}{d-e}^{-1}\}$ into (7) and (8), respectively, gives

$$
\sum_{k=0}^{n}\frac{e}{k+e}\binom{n}{k}\binom{k+d}{d-e}^{-1} x^k(1-x)^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{d+k}{e}^{-1}x^k,
$$

$$
\sum_{k=0}^{n}\binom{n}{k}\frac{(b)_k(c-b)_{n-k}}{(c)_n}\frac{e}{k+e}\binom{k+d}{d-e}^{-1} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(b)_k}{(c)_k}\binom{d+k}{e}^{-1}.
$$

• Inserting G'-pair $\{\binom{2n}{n}\binom{j+n}{n}^{-1}2^{-2n}, \binom{2n+2j}{n+j}\binom{2j}{j}^{-1}2^{-2n}\}$ into (7) and (8), respectively, we obtain

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k+2j}{k+j}\binom{2j}{j}^{-1}(\frac{x}{4})^k(1-x)^{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{j+k}{k}^{-1}(\frac{x}{4})^k,$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(b)_k(c-b)_{n-k}}{(c)_n}\binom{2k+2j}{k+j}\binom{2j}{j}^{-1}2^{-2k}=\sum_{k=0}^{n}(-4)^{-k}\binom{n}{k}\binom{2k}{k}\binom{j+k}{k}^{-1}\frac{(b)_k}{(c)_k}.$$

Further, iterating Theorems 4 and 5 once again, we have the following

Theorem 7 *If $\{f(n), g(n)\}$ is a G'-pair with $\phi(x;n)=(a+x)_n$, then*

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_n(c)_k(d)_k(1+a-c-d)_{n-k}}{(a)_k(1+a-c)_n(1+a-d)_n}g(k)$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2k+a)(c)_k(d)_k}{(a+n)_{k+1}(1+a-c)_k(1+a-d)_k}f(k), \tag{10}$$

$$\sum_{k=0}^{n}\binom{n}{k}\frac{(a)_n(1+a-e-f)_{n-k}(c)_k(d)_k(e)_k(f)_k}{(a)_k(1+a-c)_k(1+a-d)_k(1+a-e)_n(1+a-f)_n}\times$$
$${}_4F_3\left[\begin{matrix}1+a-c-d, e+k, f+k, -n+k\\ 1+a-c+k, 1+a-d+k, e+f-a-n+k\end{matrix}\Big|\,1\right]g(k)$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(2k+a)(c)_k(d)_k(e)_k(f)_k}{(a+n)_{k+1}(1+a-c)_k(1+a-d)_k(1+a-e)_k(1+a-f)_k}f(k). \tag{11}$$

When $\phi(x;n)=(a+x)_n$, we get a G'-pair $\{(-1)^n\frac{(a)_n(a-b+1)_n}{(b)_n}, \frac{(a)_n}{(b)_n}\}$ from Vandermonde theorem (3). There exists a G'-pair $\{\frac{n!}{(2n+1)^2}, \binom{n+1/2}{n}^{-2}\}$ with $a=1$ by (4.17) in [5]. Inserting these two pairs into Theorem 7, we can obtain the following identities.

• Inserting G'-pair $\{(-1)^n\frac{(a)_n(a-b+1)_n}{(b)_n}, \frac{(a)_n}{(b)_n}\}$ into (10), letting $n\to\infty$, then using Gauss's theorem (2), we get

$${}_5F_4\left[\begin{matrix}a, 1+a/2,\ c,\ d, 1+a-b\\ a/2, 1+a-c, 1+a-d, b\end{matrix}\Big|\,1\right]=\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(b)\Gamma(b-c-d)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(b-c)\Gamma(b-d)},$$

which is just the generalized form of Dougall-Dixon formula [1, 4.4 (1)].

• Inserting G'-pair $\{(-1)^n\frac{(a)_n(a-b+1)_n}{(b)_n}, \frac{(a)_n}{(b)_n}\}$ into (11) and letting $n\to\infty$, we get the following identity:

$$\frac{\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-e-f)}\sum_{k=0}^{\infty}\frac{(c)_k(d)_k(e)_k(f)_k}{k!(1+a-c)_k(1+a-d)_k(b)_k}\times$$
$${}_3F_2\left[\begin{matrix}1+a-c-d, e+k, f+k\\ 1+a-c+k, 1+a-d+k\end{matrix}\Big|\,1\right]$$
$$={}_7F_6\left[\begin{matrix}a, 1+a/2,\ c,\ d,\ e,\ f,\ 1+a-b\\ a/2, 1+a-c, 1+a-d, 1+a-e, 1+a-f, b\end{matrix}\Big|\,1\right].$$

• Inserting G'-pair $\{\frac{n!}{(2n+1)^2}, \binom{n+1/2}{n}^{-2}\}$ into (10) with $a=1$, we obtain

$$ {}_5F_4\left[\begin{matrix}1, 1/2, \ c, \ d, -n\\ 3/2, 2-c, 2-d, 2+n\end{matrix}\,\Big|\, 1\right] = \frac{(n+1)!(2-c-d)_n}{(2-c)_n(2-d)_n}\ {}_4F_3\left[\begin{matrix}1, c, \ d, -n\\ 3/2, 3/2, c+d-n-1\end{matrix}\,\Big|\, 1\right], $$

which is the special form of [1, 4.3 (2)].

• Inserting G'-pair $\{\frac{n!}{(2n+1)^2}, \binom{n+1/2}{n}^{-2}\}$ into (11) with $a=1$, we obtain

$$ \sum_{k=0}^{n}\binom{n}{k}\frac{(n+1)!(2-e-f)_{n-k}(c)_k(d)_k(e)_k(f)_k}{k!(2-c)_k(2-d)_k(2-e)_n(2-f)_n}\binom{k+1/2}{k}^{-2}\times $$

$$ {}_4F_3\left[\begin{matrix}2-c-d, e+k, f+k, -n+k\\ 2-c+k, 2-d+k, e+f-1-n+k\end{matrix}\,\Big|\, 1\right] $$

$$ = {}_7F_6\left[\begin{matrix}1, \ 1/2, \ c, \ d, \ e, \ f, \ -n\\ 3/2, 2-c, 2-d, 2-e, 2-f, 2+n\end{matrix}\,\Big|\, 1\right]. $$

We can also iterate Theorems 2–5 i ($i>1$) times, and insert the above inversion pairs into them, then many other transformation formulae can be derived.

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