Journal of Mathematical Research with Applications
Mar., 2012, Vol. 32, No. 2, pp. 181–188
DOI:10.3770/j.issn:2095-2651.2012.02.006
Http://jmre.dlut.edu.cn

The Stabilization and Idempotent Completion of a Left Triangulated Category

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Abstract Let $(\mathscr{C},\Omega,\Delta)$ be a left triangulated category with a fully faithful endofunctor Ω. We show a triangle-equivalence $(S(\overline{\mathscr{C}}),\widetilde{\overline{\Omega}},\widetilde{\overline{\Delta}}) \cong (\overline{S(\mathscr{C})},\overline{\widetilde{\Omega}},\overline{\widetilde{\Delta}})$, where $(S(\overline{\mathscr{C}}),\widetilde{\overline{\Omega}},\widetilde{\overline{\Delta}})$ denotes the stabilization of the idempotent completion of $(\mathscr{C},\Omega,\Delta)$ and $(\overline{S(\mathscr{C})},\overline{\widetilde{\Omega}},\overline{\widetilde{\Delta}})$ denotes the idempotent completion of the stabilization of $(\mathscr{C},\Omega,\Delta)$.

Keywords left triangulated category; idempotent completion; stabilization; triangle-equivalence.

MR(2010) Subject Classification 18E30; 18A35

1. Introduction

Let $\mathscr{C}$ be an additive category. An idempotent morphism e is said to be *split* if there exist morphisms $B \xrightarrow{p} A \xrightarrow{q} B$ with $qp = 1_B$ and $pq = e$ (see [5, Prop.18.15]). The additive category $\mathscr{C}$ is said to be idempotent complete (or idemsplit) provided that every idempotent morphism splits.

For every additive category $\mathscr{C}$ there is a fully faithful embedding $l : \mathscr{C} \to \overline{\mathscr{C}}$ into an idempotent complete additive category. Moreover, the functor l induces an equivalence $\mathrm{Hom}_{add}(\overline{\mathscr{C}},\mathscr{L}) \cong \mathrm{Hom}_{add}(\mathscr{C},\mathscr{L})$ for each idempotent complete additive category $\mathscr{L}$, where Hom_{add} denotes the (large) category of additive functors [7]. Many natural triangulated categories, such as the derived categories of perfect complexes over a quasi-separated, quasi-compact scheme, the bounded derived categories of abelian categories and the triangulated categories satisfying [TR5$\aleph_1$], are idempotent complete [4]. But not all the triangulated categories are idempotent complete. Fortunately any additive category can be idempotent completed and the idempotent completion of a triangulated category is still a triangulated category [7]. Recently, Chen and Tang in [6] have shown that the idempotent completion of a right or left recollement of a triangulated category is still a right or left recollement.

Motivated by the idempotent completion of a triangulated category, we study the analogous aspects of the idempotent completion of a one-sided triangulated category. For details and more information on one-sided triangulated categories we refer to [1, 2, 9]. Let $\mathscr{C}$ be an additive

Received June 20, 2010; Accepted September 6, 2011
Supported by the National Natural Science Foundation of China (Grant No. 11071040) and the Natural Science Foundation of Fujian Province (Grant No. 2011J01004).
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category equipped with an additive endofunctor $\Omega : \mathscr{C} \to \mathscr{C}$. Consider the category $LT(\mathscr{C}, \Omega)$ whose objects are diagrams of the form $\Omega(A) \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A$ where the morphisms are indicated by the following commutative diagram:

$$\begin{tikzcd}
\Omega(A) \arrow[r, "u"] \arrow[d, "\Omega(\gamma)"] & B \arrow[r, "v"] \arrow[d, "\alpha"] & C \arrow[r, "w"] \arrow[d, "\beta"] & A \arrow[d, "\gamma"] \\
\Omega(A') \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & A'
\end{tikzcd}$$

Figure 1 Triangulation Morphism

A left triangulation of the category $(\mathscr{C}, \Omega)$ is a full subcategory Δ of $LT(\mathscr{C}, \Omega)$ which satisfies all the axioms of a triangulated category, except that Ω is not necessarily an equivalence. Then the triple $(\mathscr{C}, \Omega, \Delta)$ is called a left triangulated category and the diagrams in Δ are *the left triangles*. For a left triangulated category $(\mathscr{C}, \Omega, \Delta)$, there exists a triangulated category $(S(\mathscr{C}), \widetilde{\Omega}, \widetilde{\Delta})$ called the stabilization of $\mathscr{C}$. The category $S(\mathscr{C})$ is the universal triangulated category for exact functors starting at $\mathscr{C}$, that is, $S : \mathscr{C} \to S(\mathscr{C})$ is an exact functor such that for any exact functor $F : \mathscr{C} \to \mathscr{D}$ with a triangulated category $\mathscr{D}$, there is a unique exact functor $F^* : S(\mathscr{C}) \to \mathscr{D}$ such that $F^* \circ S = F$. The existence of $(S(\mathscr{C}), \widetilde{\Omega}, \widetilde{\Delta})$ can be found in [3].

Throughout this paper, if $\mathscr{C}$ is an additive category and A, B are objects in $\mathscr{C}$, then the set of morphisms from A to B in $\mathscr{C}$ is denoted by $\mathscr{C}(A, B)$ instead of $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and the set of integers is denoted by $\mathbb{Z}$.

2. Preliminaries

Before giving the main results we first provide elementary definitions and constructions in this section.

2.1. Stabilization

Let $(\mathscr{C}, \Omega, \Delta)$ be a left triangulated category. Heller in [3] and Beligiannis in [1] constructed a triangulated category $S(\mathscr{C})$ as follows.

Objects of $S(\mathscr{C})$ are pairs (A, n) where A is an object of $\mathscr{C}$ and $n \in \mathbb{Z}$. If $n, m \in \mathbb{Z}$, we consider the directed set $I_{n,m} = \{k \in \mathbb{Z} | k \geq n, k \geq m\}$. The set of morphisms between (A, n), $(B, m) \in S(\mathscr{C})$ is defined by $S(\mathscr{C})((A, n), (B, m)) = \varinjlim_{k \in I_{n,m}} \mathscr{C}(\Omega^{k-n}(A),\ \Omega^{k-m}(B))$. Then $S(\mathscr{C})$ is an additive category and there exists an equivalence $\widetilde{\Omega} : S(\mathscr{C}) \to S(\mathscr{C})$ defined as follows. $(A, n) \in S(\mathscr{C})$, $\widetilde{\Omega}(A, n) = (A, n-1)$ and if $\alpha : (A, n) \to (B, m)$, then $\widetilde{\Omega}(\alpha) = i_{(k-1:A,n-1;B,m-1)}(\alpha_{k-1})$, where $i_{(k,A,n;B,m)} : \mathscr{C}(\Omega^{k-n}(A), \Omega^{k-m}(B)) \to S(\mathscr{C})((A, n), (B, m))$ is the canonical morphism and $\alpha_k : \Omega^{k-n}(A) \to \Omega^{k-m}(B)$ is representative such that $i_{(k,A,n;B,m)}(\alpha_k) = \alpha$ for some $k \in I_{n,m}$. Moreover, $\alpha_{k-1} = \alpha_k : \Omega^{(k-1)-(n-1)}(A) \to \Omega^{(k-1)-(m-1)}(B)$. The inverse of $\widetilde{\Omega}$ is defined by $\widetilde{\Omega}^{-1}(A, n) = (A, n+1)$, $\widetilde{\Omega}^{-1}(\alpha) = i_{(k+1:A,n+1;B,m+1)}(\alpha_{k+1})$. For $l \geq k$, let $f_{kl} = \Omega^{l-k}(-) : \mathscr{C}(\Omega^{k-n}(A), \Omega^{k-m}(B)) \to \mathscr{C}(\Omega^{l-n}(A), \Omega^{l-m}(B))$ be the canonical morphism such that $i_{(k:A,n;B,m)} = i_{(l:A,n;B,m)} \cdot f_{kl}$. If $\alpha_k : \Omega^{k-n}(A) \to \Omega^{k-m}(B)$ and $\beta_k : \Omega^{k-m}(B) \to \Omega^{k-l}(C)$, then

$$i_{(k:A,n;C,l)}(\beta_k \alpha_k) = i_{(k:B,m;C,l)}(\beta_k) i_{(k:A,n;B,m)}(\alpha_k)$$

For brevity we also write $i_k = i_{(k:-,-;-,-)}$.

There also exists a natural additive functor $S : \mathscr{C} \to S(\mathscr{C})$ defined as follows. $S(A) = (A, 0)$ and if $\alpha : A \to B$ is a morphism in $\mathscr{C}$, then $S(\alpha) = i_0(\alpha) : (A, 0) \to (B, 0)$. Thus we have a natural isomorphism $\theta : \widetilde{\Omega} S(A) = (A, -1) \xrightarrow{i_0(1_{\Omega(A)})} (\Omega(A), 0) = S\Omega(A)$. So $\widetilde{\Omega} i_0(\alpha) \cong S\Omega(\alpha) = i_0 \Omega(\alpha)$, for every morphism $\alpha : A \to B$ in $\mathscr{C}$.

Using the functor $S : \mathscr{C} \to S(\mathscr{C})$ and the left triangulation Δ of $\mathscr{C}$, we define a triangulation $\widetilde{\Delta}$ of the pair $(S(\mathscr{C}), \widetilde{\Omega})$ as follows. A diagram

$$\widetilde{\Omega}(C, l) \xrightarrow{\alpha} (A, n) \xrightarrow{\beta} (B, m) \xrightarrow{\gamma} (C, l)$$

belongs to $\widetilde{\Delta}$ if there exists $k \in 2\mathbb{Z}$, $k \geq \max\{l, n, m\}$ and a triangle of representatives

$$\Omega(\Omega^{k-l}(C)) \xrightarrow{\alpha_k} \Omega^{k-n}(A) \xrightarrow{\beta_k} \Omega^{k-m}(B) \xrightarrow{\gamma_k} \Omega^{k-l}(C)$$

in $\mathscr{C}$ where $\alpha = i_k(\alpha_k), \beta = i_k(\beta_k), \gamma = i_k(\gamma_k)$. Then the triple $(S(\mathscr{C}), \widetilde{\Omega}, \widetilde{\Delta})$ is a triangulated category and $S : \mathscr{C} \to S(\mathscr{C})$ is the stabilization of $\mathscr{C}$ in the sense of [1] or [3, Th.9.2].

2.2. Idempotent completion

Let $\mathscr{C}$ be an additive category. Balmer and Schlichting in [7] constructed an idempotent completion $\overline{\mathscr{C}}$ of $\mathscr{C}$ as follows.

Objects of $\overline{\mathscr{C}}$ are pairs (A, e) where A is an object of $\mathscr{C}$ and $e : A \to A$ is an idempotent morphism in $\mathscr{C}$. A morphism in $\overline{\mathscr{C}}$ from (A_1, e_1) to (A_2, e_2) is a morphism $\alpha : A_1 \to A_2$ in $\mathscr{C}$ with $\alpha e_1 = e_2 \alpha = \alpha$. So $\overline{\mathscr{C}}((A_1, e_1), (A_2, e_2)) = e_2 \circ \mathscr{C}(\mathscr{A}, \mathscr{A}) \circ e_1$.

The assignment $A \mapsto (A, 1_A)$ defines a fully faithful functor l from $\mathscr{C}$ to $\overline{\mathscr{C}}$. So we can think of $\mathscr{C}$ as a full subcategory of $\overline{\mathscr{C}}$. If $(\mathscr{C}, \Omega, \Delta)$ is a triangulated category, define $\overline{\Omega} : \overline{\mathscr{C}} \to \overline{\mathscr{C}}$ by $\overline{\Omega}(A, e) = (\Omega(A), \Omega(e))$. For every morphism $\alpha : (A_1, e_1) \to (A_2, e_2)$ in $\overline{\mathscr{C}}$, $\overline{\Omega}(\alpha) = \Omega(\alpha)$. So $\overline{\Omega} \circ l = l \circ \Omega$. We define a triangulation $\overline{\Delta}$ of the pair $(\overline{\mathscr{C}}, \overline{\Omega})$ as follows. A diagram $\overline{\Omega}(A_1, e_1) \xrightarrow{\alpha} (A_2, e_2) \xrightarrow{\beta} (A_3, e_3) \xrightarrow{\gamma} (A_1, e_1)$ belongs to $\overline{\Delta}$ if it is a direct factor of a triangle of $\mathscr{C}$, that is, if there is a diagram $\overline{\Omega}(B_1, d_1) \xrightarrow{u} (B_2, d_2) \xrightarrow{v} (B_3, d_3) \xrightarrow{w} (B_1, d_1)$ such that $\overline{\Omega}((A_1, e_1) \oplus (B_1, d_1)) \xrightarrow{\alpha \oplus u} (A_2, e_2) \oplus (B_2, d_2) \xrightarrow{\beta \oplus v} (A_3, e_3) \oplus (B_3, d_3) \xrightarrow{\gamma \oplus w} (A_1, e_1) \oplus (B_1, d_1)$ is isomorphic to a triangle in $\mathscr{C}$. Balmer and Schlichting in [7, Th.1.12] proved the triple $(\overline{\mathscr{C}}, \overline{\Omega}, \overline{\Delta})$ is a triangulated category.

3. Main theorem

In this section we assume that $(\mathscr{C}, \Omega, \Delta)$ is a left triangulated category with a fully faithful endofunctor Ω. We denote its idempotent completion by $(\overline{\mathscr{C}}, \overline{\Omega}, \overline{\Delta})$ and its stabilization by $(S(\mathscr{C}), \widetilde{\Omega}, \widetilde{\Delta})$.

Note that the proof of [7, Th.1.12] can be transferred to the case of a left triangulated category. Thus we have the following theorem.

Theorem 3.1 $(\overline{\mathscr{C}},\overline{\Omega},\overline{\Delta})$ *is a left triangulated category.*

Denote the stabilization of $(\overline{\mathscr{C}},\overline{\Omega},\overline{\Delta})$ by $(S(\overline{\mathscr{C}}),\widetilde{\overline{\Omega}},\widetilde{\overline{\Delta}})$ and the idempotent completion of $(S(\mathscr{C}),\widetilde{\Omega},\widetilde{\Delta})$ by $(\overline{S(\mathscr{C})},\overline{\widetilde{\Omega}},\overline{\widetilde{\Delta}})$. Our main theorem is

Theorem 3.2 *There is a triangle-equivalence* $(S(\overline{\mathscr{C}}),\widetilde{\overline{\Omega}},\widetilde{\overline{\Delta}})\cong(\overline{S(\mathscr{C})},\overline{\widetilde{\Omega}},\overline{\widetilde{\Delta}})$.

Before proving the theorem, we need some preparations.

An object of $S(\overline{\mathscr{C}})$ is $((A,e),n)$ where $A\in\mathscr{C}$, $e^2=e:A\to A$ and $n\in\mathbb{Z}$. For brevity, let (A,e,n) denote $((A,e),n)$. For (A_1,e_1,n_1), $(A_2,e_2,n_2)\in S(\overline{\mathscr{C}})$,

$$S(\overline{\mathscr{C}})((A_1,e_1,n_1),(A_2,e_2,n_2))=\varinjlim_{k\in I_{n_1,n_2}}\overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1,e_1),\overline{\Omega}^{k-n_2}(A_2,e_2)).$$

Denote by $j_k:\overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1,e_1),\overline{\Omega}^{k-n_2}(A_2,e_2))\to S(\overline{\mathscr{C}})((A_1,e_1,n_1),(A_2,e_2,n_2))$ the canonical morphism and

$$g_{kl}=\overline{\Omega}^{l-k}(-):\overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1,e_1),\overline{\Omega}^{k-n_2}(A_2,e_2))\to\overline{\mathscr{C}}(\overline{\Omega}^{l-n_1}(A_1,e_1),\overline{\Omega}^{l-n_2}(A_2,e_2))$$

such that $j_k=j_lg_{kl}$ whenever $l\geq k$. If $\alpha:(A_1,e_1,n_1)\to(A_2,e_2,n_2)$ is a morphism in $S(\overline{\mathscr{C}})$, by the definition of direct limit, there exists an integer $k\in I_{n_1,n_2}$ and a morphism $\alpha_k:\overline{\Omega}^{k-n_1}(A_1,e_1)\to\overline{\Omega}^{k-n_2}(A_2,e_2)$ in $\overline{\mathscr{C}}$ such that $j_k(\alpha_k)=\alpha$. Thus $\alpha_k:\Omega^{k-n_1}(A_1)\to\Omega^{k-n_2}(A_2)$ in $\mathscr{C}$ and $\alpha_k\Omega^{k-n_1}(e_1)=\Omega^{k-n_2}(e_2)\alpha_k=\alpha_k$.

For the canonical morphism

$$i_k=i_{(k:A,n;B,m)}:\mathscr{C}(\Omega^{k-n}(A),\Omega^{k-m}(B))\to S(\mathscr{C})((A,n),(B,m))$$

we have

Lemma 3.3 *Let* $(A,e,n)\in S(\overline{\mathscr{C}})$ *and* $i_n=i_{(n:A,n;A,n)}$. *Then*

(i) $e'=i_n(e)$ *is idempotent.*

(ii) If $k\geq n$ *and* $e_k=\Omega^{k-n}(e):\Omega^{k-n}(A)\to\Omega^{k-n}(A)$, *then* $i_k(e_k)=e'$.

Proof (i) Since that $e:A\to A$ is idempotent, $e'e'=i_n(e)i_n(e)=i_n(e)=e'$.

(ii) For $k\geq n, i_k(e_k)=i_k\Omega^{k-n}(e)=i_n(e)=e'$. $\square$

Lemma 3.4 *If* $(A,n,e')\in\overline{S(\mathscr{C})}$, *then there exists a unique idempotent morphism* $e\in\mathscr{C}(A,A)$ *such that* $i_n(e)=e'$ *and for* $k\geq n$, $i_k(\Omega^{k-n}(e))=e'$.

Proof Since $e':(A,n)\to(A,n)$ is idempotent, there exists an integer $t\geq n$ and $e_t\in\mathscr{C}(\Omega^{t-n}(A),\Omega^{t-n}(A))$ such that $i_t(e_t)=e'$. So $i_t(e_t^2)=i_t(e_t)$. By [8, Prop.24.3(1)], $(\Omega^{k-t}(e_t))^2=\Omega^{k-t}(e_t)$ for some integer $k\geq t$. Thus $e_k=\Omega^{k-t}(e_t):\Omega^{k-n}(A)\to\Omega^{k-n}(A)$ is idempotent and $i_k(e_k)=i_t(e_t)=e'$. Since endofunctor Ω is fully faithful, it follows that there exists a unique idempotent morphism $e:A\to A$ such that $e_k=\Omega^{k-n}(e)$. So $i_n(e)=i_k\Omega^{k-n}(e)=i_k(e_k)=e'$. $\square$

Lemma 3.5 $i_k(1_{\Omega^{k-n}(A)}) = 1_{(A,n)}$.

Proof Let $\alpha_k = 1_{\Omega^{k-n}(A)}$ and $\alpha = i_k(\alpha_k)$. For any morphism $\beta : (A, n) \to (B, m)$ in $S(\mathscr{C})$, we have $\beta_t : \Omega^{t-n}(A) \to \Omega^{t-m}(B)$ for some $t \in I_{n,m}$ such that $\beta = i_t(\beta_t)$. Without loss of generality, we can assume $k \geq t$. So $\beta\alpha = i_t(\beta_t)i_k(\alpha_k) = i_k(\Omega^{k-t}(\beta_t)) \cdot i_k(\alpha_k) = i_k(\cdot\Omega^{k-t}(\beta_t)(\alpha_k)) = i_k(\Omega^{k-t}(\beta_t)) = i_t(\beta_t) = \beta$. Similarly, we have $\alpha\gamma = \gamma$ for each γ in $S(\mathscr{C})((\mathscr{C},),(\mathscr{A},))$. So $i_k(1_{\Omega^{k-n}(A)}) = 1_{(A,n)}$. $\square$

For any object (A, e, n) in $S(\overline{\mathscr{C}})$, $\tilde{\overline{\Omega}}(A, e, n) = (A, e, n-1)$. For any morphism $\alpha : (A_1, e_1, n_1) \to (A_2, e_2, n_2)$ in $S(\overline{\mathscr{C}})$, there exists an integer $k \in I_{n_1,n_2}$ and $\alpha_k \in \overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1, e_1), \overline{\Omega}^{k-n_2}(A_2, e_2))$ such that $j_k(\alpha_k) = \alpha$. So $\alpha_{k-1} = \alpha_k : \overline{\Omega}^{(k-1)-(n_1-1)}(A_1, e_1) \to \overline{\Omega}^{(k-1)-(n_2-1)}(A_2, e_2)$, $\tilde{\overline{\Omega}}(\alpha) = i_{k-1}(\alpha_{k-1})$. A diagram

$$\tilde{\overline{\Omega}}(A_1, e_1, n_1) \xrightarrow{\alpha} (A_2, e_2, n_2) \xrightarrow{\beta} (A_3, e_3, n_3) \xrightarrow{\gamma} (A_1, e_1, n_1)$$

belongs to $\tilde{\overline{\Delta}}$ if there exists $k \in 2\mathbb{Z}$, $k \geq \max\{n_1, n_2, n_3\}$ and a left triangle of representatives $(*)$

$$\overline{\Omega}(\overline{\Omega}^{k-n_1}(A_1, e_1)) \xrightarrow{\alpha_k} \overline{\Omega}^{k-n_2}(A_2, e_2) \xrightarrow{\beta_k} \overline{\Omega}^{k-n_3}(A_3, e_3) \xrightarrow{\gamma_k} \overline{\Omega}^{k-n_1}(A_1, e_1)$$

in $\overline{\mathscr{C}}$ with $\alpha = j_k(\alpha_k), \beta = j_k(\beta_k), \gamma = j_k(\gamma_k)$. Thus $(*)$ is a direct factor of a triangle of $\mathscr{C}$. Objects of $\overline{S(\mathscr{C})}$ are $((A, n), e')$ where $A \in \mathscr{C}$, $n \in \mathbb{Z}$, $e' : (A, n) \to (A, n)$ is an idempotent morphism. For brevity, let (A, n, e') denote $((A, n), e')$. A morphism $\alpha' : (A_1, n_1, e'_1) \to (A_2, n_2, e'_2)$ in $\overline{S(\mathscr{C})}$ is a morphism $\alpha' : (A_1, n_1) \to (A_2, n_2)$ in $S(\mathscr{C})$ with $\alpha' e'_1 = e'_2\alpha' = \alpha'$. There exist $\alpha_k : \Omega^{k-n_1}(A_1) \to \Omega^{k-n_2}(A_2)$ in $\mathscr{C}$ such that $i_k(\alpha_k) = \alpha'$ and idempotent morphisms $e_{tk} : \Omega^{k-n_t}(A_t) \to \Omega^{k-n_t}(A_t)$ such that $i_k(e_{tk}) = e'_t$ for some integer $k \in I_{n_1,n_2}, t = 1, 2$. By Lemma 3.4, there exist unique idempotent morphisms $e_1 : A_1 \to A_1$ and $e_2 : A_2 \to A_2$ such that $e_{1k} = \Omega^{k-n_1}(e_1)$ and $e_{2k} = \Omega^{k-n_2}(e_2)$. So $i_k(\Omega^{k-n_t}(e_t)) = e'_t$ and $\tilde{\Omega}(e'_t) = \tilde{\Omega}(i_k(\Omega^{k-n_t}(e_t))) = i_{k-1}(\Omega^{(k-1)-(n_t-1)}(e_t))$, for $t = 1, 2$.

Proof of Theorem 3.2 Using the notations as above, we define $F : S(\overline{\mathscr{C}}) \to \overline{S(\mathscr{C})}$ as follows. For any object $(A, e, n) \in S(\overline{\mathscr{C}})$, $F(A, e, n) = (A, n, e')$, where $e' = i_{(n:A,n;A,n)}(e)$. For any morphism $\alpha : (A_1, e_1, n_1) \to (A_2, e_2, n_2)$, there exists an integer $k \geq n_1, n_2$ and a morphism $\alpha_k : \overline{\Omega}^{k-n_1}(A_1, e_1) \to \overline{\Omega}^{k-n_2}(A_2, e_2)$ in $\overline{\mathscr{C}}$ such that $j_k(\alpha_k) = \alpha$. So $\alpha_k : \Omega^{k-n_1}(A_1) \to \Omega^{k-n_2}(A_2)$ is a morphism in $\mathscr{C}$ and $\alpha_k\Omega^{k-n_1}(e_1) = \Omega^{k-n_2}(e_2)\alpha_k = \alpha_k$. Let $e'_1 = i_{(n_1:A_1,n_1;A_1,n_1)}(e_1)$ and $e'_2 = i_{(n_2:A_2,n_2;A_2,n_2)}(e_2)$. Then for the canonical morphism

$$i_k = i_{(k:A_1,n_1;A_2,n_2)} : \mathscr{C}(\Omega^{k-n_1}(A_1), \Omega^{k-n_2}(A_2)) \to S(\mathscr{C})((A_1, n_1), (A_2, n_2))$$

we have $i_k(\alpha_k) = \alpha' : (A_1, n_1) \to (A_2, n_2)$, $\alpha' e'_1 = e'_2\alpha' = \alpha'$. Thus $\alpha' \in \overline{S(\mathscr{C})}((A_1, n_1, e'_1), (A_2, n_2, e'_2))$, we define $F(\alpha) = \alpha'$.

Assume that $F(\alpha) = \alpha'$, where $\alpha = j_k(\alpha_k) = j_t(\alpha_t)$, $\alpha' = i_k(\alpha_k)$ for some $k, t \in I_{n_1,n_2}$. It is clear that we can assume that $k \geq t$. Thus $j_t = j_k \cdot g_{tk}$, $j_k(\alpha_k) = \alpha = j_t(\alpha_t) = j_k g_{tk}(\alpha_t) = j_k(\overline{\Omega}^{k-t}(\alpha_t)) = j_k(\Omega^{k-t}(\alpha_t))$. So $j_k(\alpha_k - \Omega^{k-t}(\alpha_t)) = 0$. Thus there exists an integer $l \geq k$ such that $g_{kl}(\alpha_k - \Omega^{k-t}(\alpha_t)) = 0$ (see [8, Prop.24.3]). That is, $\overline{\Omega}^{l-k}(\alpha_k - \Omega^{k-t}(\alpha_t)) = 0$ and then $\Omega^{l-k}(\alpha_k) = \Omega^{l-t}(\alpha_t)$ which implies that $i_k(\alpha_k) = i_l(\Omega^{l-k}(\alpha_k)) = i_l(\Omega^{l-t}(\alpha_t)) = i_t(\alpha_t)$.

Let $\alpha : (A_1, e_1, n_1) \to (A_2, e_2, n_2)$ and $\beta : (A_2, e_2, n_2) \to (A_3, e_3, n_3)$. Then there exist an integer $k \geq \max\{n_1, n_2, n_3\}$, $\alpha_k \in \overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1, e_1), \overline{\Omega}^{k-n_2}(A_2, e_2))$ and $\beta_k \in \overline{\mathscr{C}}(\overline{\Omega}^{k-n_2}(A_2, e_2), \overline{\Omega}^{k-n_3}(A_3, e_3))$ with $\alpha = j_k(\alpha_k)$ and $\beta = j_k(\beta_k)$. Thus $F(\beta)F(\alpha) = i_k(\beta_k)i_k(\alpha_k) = i_k(\beta_k\alpha_k) = F(j_k(\beta_k\alpha_k)) = F(j_k(\beta_k)j_k(\alpha_k)) = F(\beta\alpha)$.

For each $(A, e, n) \in S(\overline{\mathscr{C}})$ and $\beta' \in \overline{S(\mathscr{C})}((A, n, e'), (B, m, e'_1))$, then $\beta' \in S(\mathscr{C})((A, n), (B, m))$ and $\beta' e' = e'_1\beta' = \beta'$. Hence there exists an integer $k \geq n, m$ and $\beta_k \in \mathscr{C}(\Omega^{k-n}(A), \Omega^{k-m}(B))$ such that $\beta' = i_k(\beta_k)$. Thus $j_k(\beta_k) = \beta \in S(\overline{\mathscr{C}})((A, e, n), (B, e_1, m))$ with $F(\beta) = \beta'$. Hence we have $\beta' F(1_{(A,e,n)}) = F(\beta)F(1_{(A,e,n)}) = F(\beta 1_{(A,e,n)}) = F(\beta) = \beta'$. Similarly, we have $F(1_{(A,e,n)})\gamma' = F(1_{(A,e,n)})F(\gamma) = F(1_{(A,e,n)}\gamma) = F(\gamma) = \gamma'$ for each $\gamma' \in \overline{S(\mathscr{C})}((C, l, e'_2), (A, n, e'))$. So $F(1_{(A,e,n)}) = 1_{F(A,e,n)}$.

It remains to show that F is a triangle-equivalence.

It is clear that F is dense from Lemma 3.4 and full. To show that F is faithful, we let two morphisms $\alpha, \beta \in S(\overline{\mathscr{C}})((A_1, e_1, n_1), (A_2, e_2, n_2))$ with $F(\alpha) = \alpha' = F(\beta)$. So there exist $\alpha_k \in \overline{\mathscr{C}}(\overline{\Omega}^{k-n_1}(A_1, e_1), \overline{\Omega}^{k-n_2}(A_2, e_2))$ and $\beta_t \in \overline{\mathscr{C}}(\overline{\Omega}^{t-n_1}(A_1, e_1), \overline{\Omega}^{t-n_2}(A_2, e_2))$ for some $k, t \in I_{n_1,n_2}$, such that $j_k(\alpha_k) = \alpha$, $j_t(\beta_t) = \beta$, $i_k(\alpha_k) = i_t(\beta_t) = \alpha'$. Without loss of generality, suppose $k \geq t$, clearly, $i_t(\beta_t) = i_k\Omega^{k-t}(\beta_t)$. Then there exists an integer $l \geq k$ such that $\Omega^{l-k}(\alpha_k - \Omega^{k-t}(\beta_t)) = 0$, i.e., $\Omega^{l-k}(\alpha_k) = \Omega^{l-t}(\beta_t)$. Therefore, $\alpha = j_k(\alpha_k) = j_l\Omega^{l-k}(\alpha_k) = j_l\Omega^{l-t}(\beta_t) = j_t(\beta_t) = \beta$. Finally, we will show that F is exact.

For any object $(A, e, n) \in S(\overline{\mathscr{C}})$, $F\widetilde{\overline{\Omega}}(A, e, n) = F(A, e, n-1) = (A, n-1, e')$, where $e' = i_{(n-1:A,n-1;A,n-1)}(e)$. $\overline{\widetilde{\Omega}}F(A, e, n) = \overline{\widetilde{\Omega}}(A, n, e'') = (\widetilde{\Omega}(A, n), \widetilde{\Omega}(e'')) = (A, n-1, e')$, where $e'' = i_{(n:A,n;A,n)}(e)$. For any $\alpha \in S(\overline{\mathscr{C}})((A_1, e_1, n), (A_2, e_2, n_2))$, $\overline{\widetilde{\Omega}}F(\alpha) = \overline{\widetilde{\Omega}}(\alpha') = \widetilde{\Omega}(\alpha')$ where $j_k(\alpha_k) = \alpha$, $i_k(\alpha_k) = \alpha'$. So $\widetilde{\Omega}(\alpha') = i_{k-1}(\alpha_{k-1})$ and $j_{k-1}(\alpha_{k-1}) = \widetilde{\overline{\Omega}}(\alpha)$ implies $F\widetilde{\overline{\Omega}}(\alpha) = i_{k-1}(\alpha_{k-1}) = \overline{\widetilde{\Omega}}F(\alpha)$. Hence $F\widetilde{\overline{\Omega}} = \overline{\widetilde{\Omega}}F$.

Suppose that

$$\widetilde{\overline{\Omega}}(A_3, e_3, n_3) \xrightarrow{\alpha} (A_1, e_1, n_1) \xrightarrow{\beta} (A_2, e_2, n_2) \xrightarrow{\gamma} (A_3, e_3, n_3) \tag{1}$$

is a triangle in $S(\overline{\mathscr{C}})$, applying the functor F yields a diagram in $\overline{S(\mathscr{C})}$

$$\overline{\widetilde{\Omega}}(A_3, n_3, e'_3) \xrightarrow{\alpha'} (A_1, n_1, e'_1) \xrightarrow{\beta'} (A_2, n_2, e'_2) \xrightarrow{\gamma'} (A_3, n_3, e'_3). \tag{2}$$

It follows from (1) that there exists a left triangle in $\overline{\mathscr{C}}$

$$\overline{\Omega}(\overline{\Omega}^{k-n_3}(A_3, e_3)) \xrightarrow{\alpha_k} \overline{\Omega}^{k-n_1}(A_1, e_1) \xrightarrow{\beta_k} \overline{\Omega}^{k-n_2}(A_2, e_2) \xrightarrow{\gamma_k} \overline{\Omega}^{k-n_3}(A_3, e_3) \tag{3}$$

for some $k \in 2\mathbb{Z}$, $k \geq \max\{n_1, n_2, n_3\}$, where $\alpha = j_k(\alpha_k)$, $\beta = j_k(\beta_k)$, $\gamma = j_k(\gamma_k)$. We let $d_i = 1_{A_i} - e_i$, $i = 1, 2, 3$. Since there is $(A_i, e_i) \oplus (A_i, 1 - e_i) \cong (A_i, 1_{A_i})$ in $\overline{\mathscr{C}}$ for $i = 1, 2, 3$ by [6, Lemma 12], there is a diagram in $\overline{\mathscr{C}}$

$$\overline{\Omega}(\overline{\Omega}^{k-n_3}(A_3, d_3)) \xrightarrow{u_k} \overline{\Omega}^{k-n_1}(A_1, d_1) \xrightarrow{v_k} \overline{\Omega}^{k-n_2}(A_2, d_2) \xrightarrow{w_k} \overline{\Omega}^{k-n_3}(C_3, d_3) \tag{4}$$

such that the direct sum of (3) and (4) is isomorphic to a left triangle in $\mathscr{C}$

$$\Omega^{k+1-n_3}(A_3) \xrightarrow{\xi_k} \Omega^{k-n_1}(A_1) \xrightarrow{\eta_k} \Omega^{k-n_2}(A_2) \xrightarrow{\zeta_k} \Omega^{k-n_3}(A_3) \tag{5}$$

where $\xi_k \cong \begin{pmatrix} \alpha_k & 0 \\ 0 & u_k \end{pmatrix}$, $\eta_k \cong \begin{pmatrix} \beta_k & 0 \\ 0 & v_k \end{pmatrix}$, $\zeta_k \cong \begin{pmatrix} \gamma_k & 0 \\ 0 & w_k \end{pmatrix}$. Then we have a triangle in $S(\mathscr{C})$

$$\widetilde{\Omega}(A_3, n_3) \xrightarrow{\xi} (A_1, n_1) \xrightarrow{\eta} (A_2, n_2) \xrightarrow{\zeta} (A_3, n_3) \tag{6}$$

where $\xi = i_k(\xi_k), \eta = i_k(\eta_k), \zeta = i_k(\zeta_k)$. On the other hand, we can get a diagram in $\mathscr{C}$ from (4)

$$\Omega^{k+1-n_3}(A_3) \xrightarrow{u_k} \Omega^{k-n_1}(A_1) \xrightarrow{v_k} \Omega^{k-n_2}(A_2) \xrightarrow{w_k} \Omega^{k-n_3}(A_3) \tag{7}$$

and

$$u_k(\Omega^{k+1-n_3}(d_3)) = (\Omega^{k-n_1}(d_1))u_k = u_k,\ v_k(\Omega^{k-n_1}(d_1)) = (\Omega^{k-n_2}(d_2))v_k = v_k,$$

$$w_k(\Omega^{k-n_2}(d_2)) = (\Omega^{k-n_3}(d_3))w_k = w_k,$$

which implies a diagram in $S(\mathscr{C})$

$$\widetilde{\Omega}(A_3, n_3) \xrightarrow{u} (A_1, n_1) \xrightarrow{v} (A_2, n_2) \xrightarrow{w} (A_3, n_3) \tag{8}$$

where $u = i_{(k:A_3,n_3-1,A_1,n_1)}(u_k), v = i_{(k:A_1,n_1,A_2,n_2)}(v_k), w = i_{(k:A_2,n_2,A_3,n_3)}(w_k)$. Let $d_3' = i_{(n_3:A_3,n_3,A_3,n_3)}(d_3)$, $d_1' = i_{(n_1:A_1,n_1,A_1,n_1)}(d_1), d_2' = i_{(n_2:A_2,n_2,A_2,n_2)}(d_2)$. So

$$\begin{aligned} u(\widetilde{\Omega}(d_3{}')) &= i_{(k:A_3,n_3-1;A_1,n_1)}(u_k) \cdot i_{(n_3-1:A_3,n_3-1;A_3,n_3-1)}(d_3) \\ &= i_{(k:A_3,n_3-1;A_1,n_1)}(u_k) \cdot i_{(k:A_3,n_3-1;A_3,n_3-1)}\Omega^{k-n_3+1}(d_3) \\ &= i_{(k:A_3,n_3-1;A_1,n_1)}(u_k \cdot \Omega^{k-n_3+1}(d_3)) \\ &= i_{(k:A_3,n_3-1;A_1,n_1)}((\Omega^{k-n_1}(d_1))u_k) \\ &= i_{(k:A_1,n_1;A_1,n_1)}(\Omega^{k-n_1}(d_1)) \cdot i_{(k:A_3,n_3-1;A_1,n_1)}(u_k) \\ &= i_{(n_1:A_1,n_1;A_1,n_1)}(d_1) \cdot i_{(k:A_3,n_3-1;A_1,n_1)}(u_k) \\ &= d_1' \cdot u = u. \end{aligned}$$

Similarly, we have $v \cdot d_1' = d_2' \cdot v = v$, $w \cdot d_2' = d_3' \cdot w = w$. Thus we have a diagram in $\overline{S(\mathscr{C})}$

$$\overline{\widetilde{\Omega}}(A_3, n_3, d_3') \xrightarrow{u} (A_1, n_1, d_1') \xrightarrow{v} (A_2, n_2, d_2') \xrightarrow{w} (A_3, n_3, d_3')\,. \tag{9}$$

Now $e_3' \oplus d_3' = i_{(n_3:A_3,n_3;A_3,n_3)}(e_3) \cdot i_{(n_3:A_3,n_3;A_3,n_3)}(d_3) = i_{(n_3:A_3,n_3;A_3,n_3)}(1_{A_3}) = 1_{(A_3,n_3)}$ by Lemma 3.5. Similarly, there are $e_1' \oplus d_1' = 1_{(B_1,n_1)}$ and $e_2' \oplus d_2' = 1_{(B_2,n_2)}$. So the direct sum of (2) and (9) is isomorphic to the diagram in $S(\mathscr{C})$

$$\widetilde{\Omega}(A_3, n_3) \xrightarrow{\xi'} (A_1, n_1) \xrightarrow{\eta'} (A_2, n_2) \xrightarrow{\zeta'} (A_3, n_3) \tag{10}$$

where $\xi' \cong \begin{pmatrix} \alpha' & 0 \\ 0 & u \end{pmatrix}$, $\eta' \cong \begin{pmatrix} \beta' & 0 \\ 0 & v \end{pmatrix}$, $\zeta' \cong \begin{pmatrix} \gamma' & 0 \\ 0 & w \end{pmatrix}$. Thus

$$\xi' \cong \begin{pmatrix} \alpha' & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} i_k(\alpha_k) & 0 \\ 0 & i_k(u_k) \end{pmatrix} = i_k \begin{pmatrix} \alpha_k & 0 \\ 0 & u_k \end{pmatrix} \cong i_k(\xi_k) = \xi.$$

Similarly, $\eta' \cong \eta$, $\zeta' \cong \zeta$. It follows that (10) $\cong$ (6). So (10) is a triangle in $S(\mathscr{C})$, and then (2) is a triangle in $\overline{S(\mathscr{C})}$. This completes the proof. □

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