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# Approximate Inertial Manifolds for Chemotaxis-Growth System

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**Abstract** The long-time behaviour of solution to chemotaxis-growth system with Neumann condition is considered in this paper. The approximate inertial manifolds of such equations are constructed based on the contraction principle, and the orders of approximations of the manifolds to the global attractor are derived.

**Keywords** approximate inertial manifolds; chemotaxis-growth system; long-time behaviour; contraction principle.

**MR(2010) Subject Classification** 35B40; 35B42; 35K57; 92C17

## 1. Introduction

This paper is concerned with the following initial-boundary problem for a quasi-linear parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \nabla\cdot\{u\nabla\chi(\rho)\} + f(u), & \text{in } \Omega\times(0,\infty),\\ \frac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + \mathrm{d}u, & \text{in } \Omega\times(0,\infty),\\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0, & \text{in } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0,\ \rho(x,0)=\rho_0, & \text{in } \Omega, \end{cases} \tag{1}$$

where $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x\in\Omega\subset R^2$ and a time $t\in[0,\infty)$, respectively. This problem arises in biology. The mobility of individuals consists of two effects: one is random walking, and the other is the directed movement in a sense that they have a tendency to move toward higher concentration of the chemical substance. This is called chemotaxis in biology [1–4]. $a>0$ and $b>0$ are the diffusion rates of $u$ and $\rho$, respectively. $c>0$ and $d>0$ are the degradation and production rates of $\rho$, respectively. $\chi(\rho)$ is the sensitivity function due to chemotaxis. $f(u)$ is a growth term of $u$.

In order to study aggregating patterns due to chemotaxis and growth, there are several contributions not only from experiments but also from mathematical analysis. Budrene and Berg [5] experimentally observed that bacteria called Escherichia coli form complex spatio-temporal

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colony patterns. In order to understand theoretically such chemotactic pattern formation, several models have been proposed in [6–9]. Among them, Mimura and Tsujikawa [10] presented a model (1) which is based on the chemotaxis and growth of bacteria. In the absence of the growth term $f(u)$, (1) reduces to the Keller-Segel equations [11] to explain the initiation of aggregating pattern of slime mold.

Some authors have already studied the existence of the global attractor for (1) [12–18]. The global attractor is strictly defined as $\omega$-limit set of the ball, which is nonempty, compact, and invariant under additional assumptions [19, 20]. While it is known in certain cases that the set has a finite Hausdorff dimension, it may be quite complicated topologically and attract solutions very slowly. The theory of inertial manifolds allows us to reduce the long-time behavior of PDE to that of a finite-dimensional dynamical system. So inertial manifold has been introduced by defining as finite-dimensional, positively invariant Lipschitz manifold which exponentially attracts all trajectories, and thus contains the global attractor. The flow restricted to such a manifold is equivalend to that of a finite system of ordinary differential equations called an inertial form. The existence of inertial manifolds has been established for a growing list of dissipative PDEs modeling physical systems [20, 21]. But the theory does not provide inertial manifold in an explicit form even when its existence can be established. Thus, in order to implement an inertial form computationally, an approximation is necessary. An approximate inertial manifold has been introduced, which has been used independent of the existence of an inertial manifold [20, 22–26]. In this paper, the approximate inertial manifolds of such equations are constructed based on the contraction principle, and the orders of approximations of the manifolds to the global attractor are derived.

The rest of the paper is organized as follows. In Section 2, we present preliminary results. In particular, we shall recall the existence of unique global solution and global attractor in the certain space. In Section 3, we construct two kinds of approximate inertial manifolds for such equations.

## 2. Preliminaries

The precise assumptions are the followings. $\Omega \subset R^2$ is a bounded domain of $C^3$ class. $a; b; c$ and $d$ are positive constants. $\chi(\rho)$ is a real smooth function of $\rho \in [0, \infty)$ with uniformly bounded derivatives up to the third order

$$\sup_{\rho \geq 0} |\frac{d^i \chi}{d\rho^i}(\rho)| < \infty, \quad i = 1, 2, 3. \tag{2}$$

For example, the possible normalized forms of $\chi(\rho)$ are $\rho$, $\log(\rho + 1)$ and so on. $f(u)$ is a real smooth function of $u \in [0, \infty)$ such that $f(0) = 0$ and

$$f(u) = (-\mu u + \nu)u, \quad \text{for sufficiently large } u \tag{3}$$

with $\mu > 0$ and $-\infty < \nu < \infty$. Let $f(u) = f_1(u)u$. Then $f_1(u)$ is a smooth function of $u \in [0, \infty)$ such that $f_1(u) = -\mu u + \nu$ for sufficiently large $u$. A typical form of $f(u)$ is a function which coincides with a cubic function $-\gamma(u-\alpha)(u-\beta)u$ when $u$ varies in some bounded subinterval of $[0,\infty)$, where $0<\alpha<\beta<\infty$ and $\gamma>0$, and coincides with a quadratic function as (3) for sufficiently large $u$. Clearly, derivatives of $f(u)$ are continuous.

Let $L^2(\Omega)$ denote the $L^2$ space of real valued measurable functions in $\Omega$, whose norm is denoted by $\|\cdot\|$, and whose inner product is denoted by $(\cdot,\cdot)$. $H^k(\Omega), k=1;2;\ldots$, denotes the real Sobolev space in $\Omega$. $H^m_N=\{u\in H^m|\frac{\partial u}{\partial n}|_{\partial\Omega}=0\}$ denotes the subspace of the Sobolev space $H^m$. Let $H$ be a Hilbert space and let $I$ be an interval of $R$. $L^2(I;H)$ denotes the space of $H$ valued $L^2$ functions defined in $I$. $H^1(I;H)$ denotes the space of functions in $L^2(I;H)$ whose first derivatives are also in $L^2(I;H)$. $C(I;H)$ and $C^m(I;H), m=1;2;3;\ldots$, denote the space of $H$ valued continuous functions and that of $H$ valued $m$-times continuously differentiable functions, respectively.

For simplicity, we shall use a universal notation $M, M_0, t_*, t_{**}$ to denote various constants which are determined in each occurrence by $\Omega;a;b;c;d;\chi(\cdot);f(\cdot)$ and so on in a specific way.

The existence of unique global solution of chemotaxis-growth system has been established [13–16, 18].

**Lemma 1** ([18]) *Let $0\le u_0\in H^2_N(\Omega)$ and $0\le\rho_0\in H^3_N(\Omega)$. Then (1) possesses a unique global solution such that*

$$0\le u\in C^1([0,\infty);L^2(\Omega))\cap C([0,\infty);H^2_N(\Omega)),$$

$$0\le \rho\in C^1([0,\infty);H^1(\Omega))\cap C([0,\infty);H^3_N(\Omega)).$$

**Lemma 2** ([14, 15, 18]) *For each bounded ball $B_r=\{u_0\in L^2,\rho_0\in H^2:\|u_0\|+\|\rho_0\|_{H^2}\le r\}$, there exist constant $M_0$ and $t_r$ dependent on $a,b,c,d,\Omega,B_r$, such that*

$$\|u\|_{H^2}\le M_0,\ \|\rho\|_{H^3}\le M_0,\quad t\ge t_r.$$

**Remark** Lemma 2 tells us that chemotaxis-growth system admits a global attractor in the product space $L^2(\Omega)\times H^1(\Omega)$.

We give the functional setting of chemotaxis-growth system. Let $Au=-\Delta u+\frac{c}{b}u, H= L^2(\Omega), D(A)=\{u\in H^2(\Omega)|\frac{\partial u}{\partial n}|_{\partial\Omega}=0\}$, and $J(u,\rho)=\nabla\cdot\{u\nabla\chi(\rho)\}$, $J_1(u,\rho)=\nabla\cdot\{u\nabla\chi'(\rho)\}$. $A^s(\cdot)(s>0)$ denotes fractional power of $A$, whose norm is denoted by $\|A^s\cdot\|$. Chemotaxis and growth system (1) can be renormalized as the following equation:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t}=-aAu+\frac{ac}{b}u-J(u,\rho)+f(u),\\ \frac{\mathrm{d}\rho}{\mathrm{d}t}=-bA\rho+\mathrm{d}u,\\ u(x,0)=u_0,\quad \rho(x,0)=\rho_0. \end{cases}\tag{4}$$

**Lemma 3** ([14, 15, 18, 20]) *Let $0\le u_0\in H^2_N(\Omega)$, $0\le\rho_0\in H^3_N(\Omega)$, and $\|u_0\|\le R_0,\|A\rho_0\|\le R_0$. Then there exist constant $M_0$ and $t_*$ dependent on $a,b,c,d,\Omega,R_0$, such that*

$$\|Au\|,\ \|\frac{\mathrm{d}u}{\mathrm{d}t}\|,\ \|A^{\frac{3}{2}}\rho\|,\ \|A^{\frac{1}{2}}\frac{\mathrm{d}\rho}{\mathrm{d}t}\|\le M_0.\ \ \forall t\ge t_*.$$

Since $A$ is a self-adjoint positive operator whose inverse is compact, it follows that the space $H$ has a complete orthonormal basis consisting of the eigenfunctions of $A$, $\{\omega_j\}_{j=1}^{\infty}$, where $\omega_j(x)$

corresponds to the eigenvalue $\lambda_j$ for $j = 1, 2, 3, \ldots$ . Let $P = P_m$ be the orthogonal projection from $H$ onto Span$\{\omega_1, \omega_2, \ldots, \omega_m\}$ and $Q = Q_m = I - P$. Since $P$ and $Q$ commute with $A$ and its powers. We may split (4) as

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} + aAu_1 - \frac{ac}{b}u_1 + PJ(u,\rho) - Pf(u) = 0, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} + aAu_2 - \frac{ac}{b}u_2 + QJ(u,\rho) - Qf(u) = 0, \\ \frac{\mathrm{d}\rho_1}{\mathrm{d}t} + bA\rho_1 - \mathrm{d}u_1 = 0, \\ \frac{\mathrm{d}\rho_2}{\mathrm{d}t} + bA\rho_2 - \mathrm{d}u_2 = 0, \\ u_1(0) = Pu_0, \quad \rho_1(0) = P\rho_0. \end{cases} \tag{5}$$

**Lemma 4** *Let $0 \le u_0 \in H_N^2(\Omega)$, $0 \le \rho_0 \in H_N^3(\Omega)$, and $\|u_0\| \le R_0$, $\|A\rho_0\| \le R_0$. Then there exist constant $M_0$ and $t_{**}$ dependent on $a, b, c, d, \Omega, R_0$, such that*

$$\|u_2\|,\ \|A^{\frac{1}{2}}u_2\|,\ \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|,\ \|A^{\frac{1}{2}}\rho_2\|,\ \|A\rho_2\|,\ \|A^{\frac{1}{2}}\frac{\mathrm{d}\rho_2}{\mathrm{d}t}\| \le M_0\lambda_{m+1}^{-\frac{1}{2}},\quad \forall t \ge t_{**}.$$

**Proof** 1) Multiplying the second equation of (5) by $u_2$ and integrating the product in $\Omega$ gives

$$(\frac{\mathrm{d}u_2}{\mathrm{d}t}, u_2) + a(Au_2, u_2) - \frac{ac}{b}(u_2, u_2) + (QJ(u,\rho), u_2) - (Qf(u), u_2) = 0,$$

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + a\|A^{\frac{1}{2}}u_2\|^2 &\le \|u\|_{L^6}\|A^{\frac{1}{2}}\rho\|_{L^3}\|A^{\frac{1}{2}}u_2\| + \|f(u)\|\|u_2\| + \frac{ac}{b}(u_2, u_2) \\ &\le M_1\|A^{\frac{1}{2}}u\|\|A\rho\|\|A^{\frac{1}{2}}u_2\| + \lambda_{m+1}^{-\frac{1}{2}}\|f(u)\|\|A^{\frac{1}{2}}u_2\| + \frac{ac}{b}\|u_2\|^2 \\ &\le M\|A^{\frac{1}{2}}u_2\| + \frac{ac}{b}\lambda_{m+1}^{-1}\|A^{\frac{1}{2}}u_2\|^2 \le \frac{a}{2}\|A^{\frac{1}{2}}u_2\|^2 + M, \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + a\lambda_{m+1}\|u_2\|^2 \le M.$$

Applying Gronwall inequality, we have

$$\begin{aligned} \|u_2(t)\|^2 &\le \|u_2(t_*)\|^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1},\quad t \ge t_*, \\ &\le M^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1} \le M\lambda_{m+1}^{-1},\quad t \ge t_{**}. \\ \|u_2(t)\| &\le M\lambda_{m+1}^{-\frac{1}{2}},\quad t \ge t_{**}. \end{aligned}$$

2) Multiplying the second equation of (5) by $Au_2$ and integrating the product in $\Omega$ yields

$$(\frac{\mathrm{d}u_2}{\mathrm{d}t}, Au_2) + a(Au_2, Au_2) - \frac{ac}{b}(u_2, Au_2) + (QJ(u,\rho), Au_2) - (Qf(u), Au_2) = 0,$$

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u_2\|^2 + a\|Au_2\|^2 &\le \|Au\|\|A\chi(\rho)\|\|Au_2\| + \|f(u)\|\|Au_2\| + \frac{ac}{b}\|A^{\frac{1}{2}}u_2\|^2, \\ &\le \|Au\|(|\chi''(\rho)|\|A^{\frac{1}{2}}\rho\|^2 + |\chi'(\rho)|\|A\rho\|)\|Au_2\| + \|f(u)\|\|Au_2\| + \frac{ac}{b}\lambda_{m+1}^{-1}\|Au_2\|^2 \\ &\le M\|Au_2\| + \frac{ac}{b}\lambda_{m+1}^{-1}\|Au_2\|^2, \\ &\le \frac{a}{2}\|Au_2\|^2 + M, \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u_2\|^2 + a\|Au_2\|^2 \le M,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}u_2\|^2 + a\lambda_{m+1}\|A^{\frac{1}{2}}u_2\|^2 \le M.$$

Applying Gronwall inequality, we find that

$$\begin{aligned}\|A^{\frac{1}{2}}u_2(t)\|^2 &\le \|A^{\frac{1}{2}}u_2(t_*)\|^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1}, \quad t \ge t_*,\\ &\le M^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1} \le M\lambda_{m+1}^{-1}, \quad t \ge t_{**}.\end{aligned}$$

$$\|A^{\frac{1}{2}}u_2(t)\| \le M\lambda_{m+1}^{-\frac{1}{2}}, \quad t \ge t_{**}.$$

3) Differentiate the second equation of (5) with respect to $t$. Let $u_{2t}$ denote $\frac{\mathrm{d}u_2}{\mathrm{d}t}$. We have

$$\frac{\mathrm{d}u_{2t}}{\mathrm{d}t} + aAu_{2t} - \frac{ac}{b}u_{2t} + QJ(u_t,\rho) + QJ_1(u,\rho)\rho_t - Qf^{'}(u)u_t = 0,$$

Multiply the above equation by $u_{2t}$ and integrate the product in $\Omega$. It follows that

$$(\frac{\mathrm{d}u_{2t}}{\mathrm{d}t}, u_{2t}) + a(Au_{2t}, u_{2t}) - \frac{ac}{b}(u_{2t}, u_{2t}) + (QJ(u_t,\rho), u_{2t}) + (QJ_1(u,\rho)\rho_t, u_{2t}) - (Qf'(u)u_t, u_{2t}) = 0,$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{2t}\|^2 + a\|A^{\frac{1}{2}}u_{2t}\|^2\\ &\quad\le |(QJ(u_t,\rho), u_{2t}| + |(QJ_1(u,\rho)\rho_t, u_{2t})| + |(Qf'(u)u_t, u_{2t})| + |\frac{ac}{b}(u_{2t}, u_{2t})|\\ &\quad\le |(J(u_t,\rho), u_{2t}| + |(J_1(u,\rho)\rho_t, u_{2t})| + |(f'(u)u_t, u_{2t})| + \frac{ac}{b}\|u_{2t}\|^2\\ &\quad\le \|u_t A^{\frac{1}{2}}\chi(\rho)\|\|A^{\frac{1}{2}}u_{2t}\| + \|uA^{\frac{1}{2}}\chi(\rho)\rho_t\|\|A^{\frac{1}{2}}u_{2t}\| + \|(f'(u)u_t\|\|u_{2t}\| + \frac{ac}{b}\|u_{2t}\|^2\\ &\quad\le \|u_t\chi'(\rho)A^{\frac{1}{2}}\rho\|\|A^{\frac{1}{2}}u_{2t}\| + \|u\chi'(\rho)A^{\frac{1}{2}}\rho\rho_t\|\|A^{\frac{1}{2}}u_{2t}\| + \|(f'(u)u_t\|\|u_{2t}\| + \frac{ac}{b}\|u_{2t}\|^2\\ &\quad\le M\|A^{\frac{1}{2}}u_{2t}\| + M\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{1}{2}}u_{2t}\| + M\lambda_{m+1}^{-1}\|A^{\frac{1}{2}}u_{2t}\|^2\\ &\quad\le M\|A^{\frac{1}{2}}u_{2t}\| + M\lambda_{m+1}^{-1}\|A^{\frac{1}{2}}u_{2t}\|^2\\ &\quad\le \frac{a}{2}\|A^{\frac{1}{2}}u_{2t}\|^2 + M,\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{2t}\|^2 + a\|A^{\frac{1}{2}}u_{2t}\|^2 \le M,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{2t}\|^2 + a\lambda_{m+1}\|u_{2t}\|^2 \le M.$$

Applying Gronwall inequality, we find that

$$\begin{aligned}\|u_{2t}(t)\|^2 &\le \|u_{2t}(t_*)\|^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1}, \quad t \ge t_*,\\ &\le M^2 e^{-a\lambda_{m+1}(t-t_*)} + Ma\lambda_{m+1}^{-1} \le M\lambda_{m+1}^{-1}, \quad t \ge t_{**},\end{aligned}$$

$$\|u_{2t}(t)\| \le M\lambda_{m+1}^{-\frac{1}{2}}, \quad t \ge t_{**}.$$

4) Multiplying the forth equation of (5) by $A\rho_2$ and integrating the product in $\Omega$ gives

$$(\frac{\mathrm{d}\rho_2}{\mathrm{d}t}, A\rho_2) + b(A\rho_2, A\rho_2) - d(u_2, A\rho_2) = 0,$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\rho_2\|^2 + b\|A\rho_2\|^2 &\le d|(u_2, A\rho_2)|\\ &\le d\|u_2\|\|A\rho_2\| \le \frac{b}{2}\|A\rho_2\|^2 + M\|u_2\|^2,\end{aligned}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\rho_2\|^2+b\|A\rho_2\|^2\leq M\|u_2\|^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\rho_2\|^2+b\lambda_{m+1}\|A^{\frac{1}{2}}\rho_2\|^2\leq M\lambda_{m+1}^{-1}.$$

Applying Gronwall inequality, we find that

$$\|A^{\frac{1}{2}}\rho_2(t)\|^2\leq\|A^{\frac{1}{2}}\rho_2(t_*)\|^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2},\quad t\geq t_*,$$
$$\leq M^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2}\leq M\lambda_{m+1}^{-2},\quad t\geq t_{**},$$
$$\|A^{\frac{1}{2}}\rho_2(t)\|\leq M\lambda_{m+1}^{-1}\leq M\lambda_{m+1}^{-\frac{1}{2}},\quad t\geq t_{**}.$$

5) Multiplying the forth equation of (5) by $A^2\rho_2$ and integrating the product in $\Omega$ gives

$$(\frac{\mathrm{d}\rho_2}{\mathrm{d}t},A^2\rho_2)+b(A\rho_2,A^2\rho_2)-d(u_2,A^2\rho_2)=0,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A\rho_2\|^2+b\|A^{\frac{3}{2}}\rho_2\|^2\leq d|(A^{\frac{1}{2}}u_2,A^{\frac{3}{2}}\rho_2)|$$
$$\leq d\|A^{\frac{1}{2}}u_2\|\|A^{\frac{3}{2}}\rho_2\|\leq\frac{b}{2}\|A^{\frac{3}{2}}\rho_2\|^2+M\|A^{\frac{1}{2}}u_2\|^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|A\rho_2\|^2+b\|A^{\frac{3}{2}}\rho_2\|^2\leq M\|A^{\frac{1}{2}}u_2\|^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|A\rho_2\|^2+b\lambda_{m+1}\|A\rho_2\|^2\leq M\lambda_{m+1}^{-1}.$$

Applying Gronwall inequality, we find that

$$\|A\rho_2(t)\|^2\leq\|A\rho_2(t_*)\|^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2},\quad t\geq t_*,$$
$$\leq M^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2}\leq M\lambda_{m+1}^{-2},\quad t\geq t_{**},$$
$$\|A\rho_2(t)\|\leq M\lambda_{m+1}^{-\frac{1}{2}},\quad t\geq t_{**}.$$

6) Differentiate the last equation of (5) with respect to $t$. Let $\rho_{2t}$ denote $\frac{\mathrm{d}\rho_2}{\mathrm{d}t}$. We have

$$\frac{\mathrm{d}\rho_{2t}}{\mathrm{d}t}+bA\rho_{2t}-\mathrm{d}u_{2t}=0.$$

Multiply the above equation by $A\rho_{2t}$ and integrate the product in $\Omega$. It follows that

$$(\frac{\mathrm{d}\rho_{2t}}{\mathrm{d}t},A\rho_{2t})+b(A\rho_{2t},A\rho_{2t})-d(u_{2t},A\rho_{2t})=0,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\rho_{2t}\|^2+b\|A\rho_{2t}\|^2\leq d|(u_{2t},A\rho_{2t})|$$
$$\leq d\|u_{2t}\|\|A\rho_{2t}\|\leq\frac{b}{2}\|A\rho_{2t}\|^2+M\|u_{2t}\|^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|A^{\frac{1}{2}}\rho_{2t}\|^2+b\lambda_{m+1}\|A^{\frac{1}{2}}\rho_{2t}\|^2\leq M\|u_{2t}\|^2\leq M\lambda_{m+1}^{-1}.$$

Applying Gronwall inequality, we find that

$$\|A^{\frac{1}{2}}\rho_{2t}(t)\|^2\leq\|A^{\frac{1}{2}}\rho_{2t}(t_*)\|^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2},\quad\geq t_*,$$
$$\leq M^2e^{-b\lambda_{m+1}(t-t_*)}+M\lambda_{m+1}^{-2}\leq M\lambda_{m+1}^{-2},\quad t\geq t_{**},$$

$$\|A^{\frac{1}{2}}\rho_{2t}(t)\| \le M\lambda_{m+1}^{-\frac{1}{2}}, \quad t \ge t_{**}.$$

The proof of Lemma 4 is completed. □

## 3 Main results

In this section we show that every orbit for chemotaxis-growth system eventually enters a thin neighborhood of the linear manifold $PH$. This implies that in particular the global attractor lies within this neighborhood. In the following theorems we estimate the thickness of this neighborhood and the rate of its exponential attraction.

3.1. Firstly, we construct the approximate solution $(u_1, A^{\frac{1}{2}}\rho_1) \in PH \times PH$ by nonlinear Galerkin method, which satisfies

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} + aAu_1 - \frac{ac}{b}u_1 + PJ(u_1,\rho_1) + PJ(u_1,\varphi_2) + PJ(\varphi_1,\rho_1) - Pf(u_1) = 0, \\ aA\varphi_1 - \frac{ac}{b}\varphi_1 + QJ(u_1,\rho_1) - Qf(u_1) = 0, \\ \frac{\mathrm{d}\rho_1}{\mathrm{d}t} + bA\rho_1 - \mathrm{d}u_1 = 0, \\ bA\varphi_2 - d\varphi_1 = 0, \end{cases} \tag{6}$$

where $\varphi_1, \varphi_2 \in QH$. The above equation defines a nonlinear mapping $F$: $PH \times PH \to QH \times QH$ satisfying $F(u_1, \rho_1) = (\varphi_1, \varphi_2)$ for $\forall (u_1, A^{\frac{1}{2}}\rho_1) \in PH \times PH$. $\Sigma_1 = \mathrm{Graph}(F)$ is an approximate inertial manifold.

**Theorem 1** *Let $0 \le u_0 \in H_N^2(\Omega)$, $0 \le \rho_0 \in H_N^3(\Omega)$, and $\|u_0\| \le R_0, \|A\rho_0\| \le R_0$. Then there exist constants $m_0, M_0$ and $t_*$ dependent on $a, b, c, d, \Omega, R_0$, such that*

$$\mathrm{dist}_{H\times H}((u, A^{\frac{1}{2}}\rho), \Sigma_1) \le M\lambda_{m+1}^{-\frac{3}{2}}, \quad m \ge m_0, \ t \ge t_*,$$

*where $u, \rho$ are the solutions of the equation (1).*

**Proof** From (5) and (6), we have

$$aA(\varphi_1 - u_2) - \frac{ac}{b}(\varphi_1 - u_2) + QJ(u_1,\rho_1) - QJ(u,\rho) - Qf(u_1) + Qf(u) - \frac{\mathrm{d}u_2}{\mathrm{d}t} = 0,$$

$$aA(\varphi_1 - u_2) - \frac{ac}{b}(\varphi_1 - u_2) + QJ(u_1,\rho_1) - QJ(u,\rho_1) + QJ(u,\rho_1) - QJ(u,\rho) + Qf(u_1) + Qf(u) - \frac{\mathrm{d}u_2}{\mathrm{d}t} = 0,$$

$$\begin{aligned} a\|A(\varphi_1 - u_2)\| \le& \|A^{\frac{1}{2}}\{u_2A^{\frac{1}{2}}\chi(\rho_1)\}\| + \|A^{\frac{1}{2}}\{uA^{\frac{1}{2}}[\chi(\rho) - \chi(\rho_1)]\}\| + \|f(u) - f(u_1)\| + \\ & \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \frac{ac}{b}\|\varphi_1 - u_2\| \\ \le& \|A^{\frac{1}{2}}u_2A^{\frac{1}{2}}\chi(\rho_1)\| + \|u_2A\chi(\rho_1)\| + \|A^{\frac{1}{2}}uA^{\frac{1}{2}}[\chi(\rho) - \chi(\rho_1)]\| + \|uA[\chi(\rho) - \chi(\rho_1)]\| + \\ & |f'(\widetilde{u})|\|u_2\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \frac{ac}{b}\|\varphi_1 - u_2\| \\ \le& \|A^{\frac{1}{2}}u_2\||\chi'(\rho_1)|\|A^{\frac{1}{2}}\rho_1\| + \|u_2\|(|\chi''(\rho)|\|A^{\frac{1}{2}}\rho_1\|^2 + |\chi'(\rho_1)|\|A\rho\|) + \\ & \|A^{\frac{1}{2}}u\||\chi'(\widetilde{\rho})|\|A^{\frac{1}{2}}\rho_2\| + \|u\|[|\chi''|(\|A^{\frac{1}{2}}\rho\| + \|A^{\frac{1}{2}}\rho_1\|)\|A^{\frac{1}{2}}\rho_2\| + |\chi'(\widetilde{\rho}|)\|A\rho_2\|] + \\ & |f'(\widetilde{u})|\|u_2\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \frac{ac}{b}\|\varphi_1 - u_2\| \\ \le& M\|A^{\frac{1}{2}}u_2\| + M\|u_2\| + M\|A^{\frac{1}{2}}\rho_2\| + M\|A\rho_2\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \frac{ac}{b}\|\varphi_1 - u_2\| \end{aligned}$$

$$\le M\lambda_{m+1}^{-\frac{1}{2}} + \frac{ac}{b}\lambda_{m+1}^{-1}\|A(\varphi_1 - u_2)\|,$$

$$(a - \frac{ac}{b}\lambda_{m+1}^{-1})\|A(\varphi_1 - u_2)\| \le M\lambda_{m+1}^{-\frac{1}{2}},$$

$$\|\varphi_1 - u_2\| \le M\lambda_{m+1}^{-\frac{3}{2}}.$$

From (5) and (6), we have

$$bA^{\frac{3}{2}}(\varphi_2 - \rho_2) - dA^{\frac{1}{2}}(\varphi_1 - u_2) - \frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2 = 0,$$

$$b\|A^{\frac{3}{2}}(\varphi_2 - \rho_2)\| \le d\|A^{\frac{1}{2}}(\varphi_1 - u_2)\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\| \le dM\lambda_{m+1}^{-1} + M\lambda_{m+1}^{-\frac{1}{2}},$$

$$b\|A^{\frac{3}{2}}(\varphi_2 - \rho_2)\| \le M\lambda_{m+1}^{-\frac{1}{2}},$$

$$\|A^{\frac{1}{2}}(\varphi_2 - \rho_2)\| \le M\lambda_{m+1}^{-\frac{3}{2}}.$$

Thus

$$\mathrm{dist}_{H\times H}((u, A^{\frac{1}{2}}\rho), \Sigma_1) \le \|\varphi_1 - u_2\| + \|A^{\frac{1}{2}}(\varphi_2 - \rho_2)\| \le M\lambda_{m+1}^{-\frac{3}{2}}, \quad m \ge m_0,\ t \ge t_*. \quad \square$$

3.2. Secondly, we adopt another method to introduce approximate inertial manifold for chemotaxis-growth system. Let

$$B_m = \{u_1 \in PH : \|A^{\frac{1}{2}}u_1\| \le 2M_0\},$$
$$O_m = \{A^{\frac{1}{2}}\rho_1 \in PH : \|A\rho_1\| \le 2M_0\},$$
$$B_m^{\perp} = \{g \in QH : \|A^{\frac{1}{2}}g\| \le 2M_0\},$$
$$O_m^{\perp} = \{A^{\frac{1}{2}}h \in QH : \|Ah\| \le 2M_0\}.$$

Define a mapping $G: B_m \times O_m \to B_m^{\perp} \times O_m^{\perp}$ such that $G(u_1, \rho_1) = (g, h)$ for each $(u_1, \rho_1) \in B_m \times O_m$, where $(g, h)$ satisfies

$$aAg - \frac{ac}{b}g + QJ(u_1 + g, \rho_1 + h) - Qf(u_1 + g) = 0, \tag{7}$$

$$bAh - dg = 0. \tag{8}$$

**Lemma 5** *Let $0 \le u_0 \in H_N^2(\Omega)$, $0 \le \rho_0 \in H_N^3(\Omega)$, and $\|u_0\| \le R_0, \|A\rho_0\| \le R_0$. Then there exists constant $m_0$ dependent on $a, b, c, d, \Omega, R_0$, such that the equations (7),(8) have a unique solution $(g, h) \in B_m^{\perp} \times O_m^{\perp}$ for $\forall (u_1, \rho_1) \in B_m \times O_m$ when $m \ge m_0$.*

**Proof** Let $(u_1, \rho_1) \in B_m \times O_m$. Define a mapping $\widetilde{G}: B_m^{\perp} \times O_m^{\perp} \to Q_mH \times Q_mH$ such that $(g, h) = \widetilde{G}(g_1, h_1)$, for $(g_1, h_1) \in B_m^{\perp} \times O_m^{\perp}$, by the following equations

$$aAg - \frac{ac}{b}g + QJ(u_1 + g_1, \rho_1 + h_1) - Qf(u_1 + g_1) = 0, \tag{9}$$

$$bAh - dg_1 = 0. \tag{10}$$

1) $\widetilde{G}$ maps $B_m^{\perp} \times O_m^{\perp}$ to itself. From (9), we have

$$a\|Ag\| \le \|A^{\frac{1}{2}}\{(u_1 + g_1)A^{\frac{1}{2}}\chi(\rho_1 + h_1)\}\| + \|f(u_1 + g_1)\| + \frac{ac}{b}\|g\|$$

$$\le \|A(u_1+g_1)\|\|A\chi(\rho_1+h_1)\|+\frac{ac}{b}\|g\|+M$$
$$\le \|A(u_1+g_1)\|(|\chi''(\rho_1+h_1)|\|A^{\frac{1}{2}}(\rho_1+h_1)\|^2+|\chi'(\rho_1+h_1)|\|A(\rho_1+h_1)\|)+ \frac{ac}{b}\lambda_{m+1}^{-1}\|Ag\|+M$$
$$\le \frac{ac}{b}\lambda_{m+1}^{-1}\|Ag\|+M,$$

$$(a-\frac{ac}{b}\lambda_{m+1}^{-1})\|Ag\|\le M,$$

$$\|A^{\frac{1}{2}}g\|\le M\lambda_{m+1}^{-\frac{1}{2}}. \tag{11}$$

From (10), we have

$$bA^{\frac{3}{2}}h-dA^{\frac{1}{2}}g_1=0,$$

$$b\|A^{\frac{3}{2}}h\|\le d\|A^{\frac{1}{2}}g_1\|\le M,$$

$$\|Ah\|\le M\lambda_{m+1}^{-\frac{1}{2}}. \tag{12}$$

By (11) and (12), it is true that there exists $m_0$ such that $(g,h)\in B_m^\perp\times O_m^\perp$ when $m\ge m_0$.

2) $\widetilde{G}$ is contraction mapping in $B_m^\perp\times O_m^\perp$.

Let $(g_1,h_1),(g_2,h_2)\in B_m^\perp\times O_m^\perp$. From (9), we have

$$aAg(g_1,h_1)-aAg(g_2,h_2)-\frac{ac}{b}[g(g_1,h_1)-g(g_2,h_2)]+QJ(u_1+g_1,\rho_1+h_1)-QJ(u_1+g_2,\rho_1+h_2)+$$
$$Q[f(u_1+g_1)-f(u_1+g_2)]=0,$$

$$aAg(g_1,h_1)-aAg(g_2,h_2)-\frac{ac}{b}[g(g_1,h_1)-g(g_2,h_2)]+QJ(u_1+g_1,\rho_1+h_1)-QJ(u_1+g_2,\rho_1+h_1)+$$
$$QJ(u_1+g_2,\rho_1+h_1)-QJ(u_1+g_2,\rho_1+h_2)-Q[f(u_1+g_1)-f(u_1+g_2)]=0,$$

$$a\|Ag(g_1,h_1)-Ag(g_2,h_2)\|\le\|A^{\frac{1}{2}}\{(g_1-g_2)A^{\frac{1}{2}}\chi(\rho_1+h_1)\}\|+$$
$$\|A^{\frac{1}{2}}\{(u_1+g_2)A^{\frac{1}{2}}[\chi(\rho_1+h_1)\}-\chi(\rho_1+h_2)]\}\|+\|f(u_1+g_1)-f(u_1+g_2)\|+\frac{ac}{b}\|g(g_1,h_1)-g(g_2,h_2)\|$$

$$\le\|(g_1-g_2)A\chi(\rho_1+h_1)\|+\|A^{\frac{1}{2}}(g_1-g_2)A^{\frac{1}{2}}\chi(\rho_1+h_1)\|+$$
$$\|A^{\frac{1}{2}}(u_1+g_2)\||\chi'(\widetilde{\rho})|\|A^{\frac{1}{2}}(h_1-h_2)\|+|f'(\widetilde{u})|\|g_1-g_2\|+\frac{ac}{b}\|g(g_1,h_1)-g(g_2,h_2)\|$$

$$\le\|g_1-g_2\|\|A\chi(\rho_1+h_1)\|+\|A^{\frac{1}{2}}(g_1-g_2)\|\|A^{\frac{1}{2}}\chi(\rho_1+h_1)\|+$$
$$\|A^{\frac{1}{2}}(u_1+g_2)\||\chi'(\widetilde{\rho})|\|A^{\frac{1}{2}}(h_1-h_2)\|+|f'(\widetilde{u})|\|g_1-g_2\|+\frac{ac}{b}\|g(g_1,h_1)-g(g_2,h_2)\|$$

$$\le M_1\|g_1-g_2\|+M_2\|A^{\frac{1}{2}}(g_1-g_2)\|+M_3\|A^{\frac{1}{2}}(h_1-h_2)\|+\frac{ac}{b}\|g(g_1,h_1)-g(g_2,h_2)\|$$

$$\le M_1\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{1}{2}}(g_1-g_2)\|+M_2\|A^{\frac{1}{2}}(g_1-g_2)\|+M_3\lambda_{m+1}^{-\frac{1}{2}}\|A(h_1-h_2)\|+\frac{ac}{b}\lambda_{m+1}^{-1}\|A[g(g_1,h_1)-g(g_2,h_2)]\|,$$

$$(a-\frac{ac}{b}\lambda_{m+1}^{-1})\|Ag(g_1,h_1)-Ag(g_2,h_2)\|\le M_4\|A^{\frac{1}{2}}(g_1-g_2)\|+M_3\lambda_{m+1}^{-\frac{1}{2}}\|A(h_1-h_2)\|,$$

$$\|A^{\frac{1}{2}}[g(g_1,h_1)-g(g_2,h_2)]\|\le M_4\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{1}{2}}(g_1-g_2)\|+M_3\lambda_{m+1}^{-1}\|A(h_1-h_2)\|. \tag{13}$$

From (10), we have

$$b[A^{\frac{3}{2}}h(g_1,h_1)-A^{\frac{3}{2}}h(g_2,h_2)]-dA^{\frac{1}{2}}(g_1-g_2)=0,$$

$$b\|A^{\frac{3}{2}}h(g_1,h_1)-A^{\frac{3}{2}}h(g_2,h_2)\|\le d\|A^{\frac{1}{2}}(g_1-g_2)\|,$$

$$\|A[h(g_1,h_1)-h(g_2,h_2)]\|\le d\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{1}{2}}(g_1-g_2)\|. \tag{14}$$

From (13) and (14), it follows that

$$\begin{aligned}&\|A^{\frac{1}{2}}[g(g_1,h_1)-g(g_2,h_2)]\|+\|A[h(g_1,h_1)-h(g_2,h_2)]\|\\ &\le M_5\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{1}{2}}(g_1-g_2)\|+M_6\lambda_{m+1}^{-1}\|A(h_1-h_2)\|.\end{aligned}$$

By contraction principle, it is true that $\widetilde{G}$ has a unique fixed point and the equations (7),(8) have a unique solution $(g,h)\in B_m^{\perp}\times O_m^{\perp}$ for $\forall(u_1,\rho_1)\in B_m\times O_m$ when $m\ge m_0$. $\Box$

Clearly, $\Sigma_2$=Graph($G$) is an approximate inertial manifold of the equations.

**Theorem 2** *Let $0\le u_0\in H_N^2(\Omega)$, $0\le\rho_0\in H_N^3(\Omega)$, and $\|u_0\|\le R_0, \|A\rho_0\|\le R_0$. Then there exist constants $m_0, M$ and $t_*$ dependent on $a,b,c,d,\Omega,R_0$, such that*

$$\mathrm{dist}_{H\times H}((u,A^{\frac{1}{2}}\rho),\Sigma_2)\le M\lambda_{m+1}^{-\frac{3}{2}},\quad m\ge m_0,\ t\ge t_*,$$

*where $u,\rho$ are the solution of the equation (1).*

**Proof** From (7) and (5), we have

$$aA(g-u_2)-\frac{ac}{b}(g-u_2)+QJ(u_1+g,\rho_1+h)-QJ(u,\rho)-Qf(u)+Qf(u_1+g)-\frac{\mathrm{d}u_2}{\mathrm{d}t}=0,$$

$$aA(g-u_2)-\frac{ac}{b}(g-u_2)+QJ(u_1+g,\rho_1+h)-QJ(u,\rho_1+h)+QJ(u,\rho_1+h)-QJ(u,\rho)$$

$$-Qf(u)+Qf(u_1+g)-\frac{\mathrm{d}u_2}{\mathrm{d}t}=0,$$

$$a\|A(g-u_2)\|\le\|A^{\frac{1}{2}}\{(u_1+g)A^{\frac{1}{2}}\chi(\rho_1+h)\}-A^{\frac{1}{2}}\{uA^{\frac{1}{2}}\chi(\rho_1+h)\}\|+$$

$$\|A^{\frac{1}{2}}\{uA^{\frac{1}{2}}\chi(\rho_1+h)\}-A^{\frac{1}{2}}\{uA^{\frac{1}{2}}\chi(\rho)\}\|+\|f(u)-f(u_1+g)\|+\|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|+\frac{ac}{b}\|g-u_2\|$$

$$\le\|A^{\frac{1}{2}}\{(g-u_2)A^{\frac{1}{2}}\chi(\rho_1+h)\}\|+\|A^{\frac{1}{2}}\{uA^{\frac{1}{2}}[\chi(\rho_1+h)-\chi(\rho)]\}\|+\|f(u)-f(u_1+g_1)\|+\|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|+\frac{ac}{b}\|g-u_2\|$$

$$\le\|A^{\frac{1}{2}}(g-u_2)\|\|A^{\frac{1}{2}}\chi(\rho_1+h)\|+\|g-u_2\|\|A\chi(\rho_1+h)\|+\|A^{\frac{1}{2}}u\|\|A^{\frac{1}{2}}[\chi(\rho_1+h)-\chi(\rho)]\|+$$

$$\|u\|\|A[\chi(\rho_1+h)-\chi(\rho)]\|+\|f(u)-f(u_1+g)\|+\|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|+\frac{ac}{b}\|g-u_2\|$$

$$\le\|A^{\frac{1}{2}}(g-u_2)\||\chi'(\rho)|\|A^{\frac{1}{2}}(\rho_1+h)\|+\|g-u_2\|[|\chi''|\|A^{\frac{1}{2}}(\rho_1+h)\|^2+|\chi'|\|A(\rho_1+h)\|]+$$

$$\|A^{\frac{1}{2}}u\||\chi'|\|A^{\frac{1}{2}}(h-\rho_2)\|+\|u\|[|\chi''|(\|A^{\frac{1}{2}}(\rho_1+h)+A^{\frac{1}{2}}\rho\|\|A^{\frac{1}{2}}(h-\rho_2)\|)]+\|u\||\chi'|\|A(h-\rho_2)\|+$$

$$|f'(u)|\|g-u_2\|+\|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|+\frac{ac}{b}\|g-u_2\|$$

$$\le M\|A^{\frac{1}{2}}(g-u_2)\|+M\|g-u_2\|+M\|A^{\frac{1}{2}}(h-\rho_2)\|+M\|A(h-\rho_2)\|+\|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|+\frac{ac}{b}\|g-u_2\|$$

$$\leq M\lambda_{m+1}^{-\frac{1}{2}}\|A(g-u_2)\| + M\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{3}{2}}(h-\rho_2)\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \frac{ac}{b}\lambda_{m+1}^{-1}\|A(g-u_2)\|,$$

$$\|A(g-u_2)\| \leq M\lambda_{m+1}^{-\frac{1}{2}}\|A(g-u_2)\| + M\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{3}{2}}(h-\rho_2)\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\|. \tag{15}$$

From (8) and (5), we have

$$bA^{\frac{3}{2}}(h-\rho_2) - dA^{\frac{1}{2}}(g-u_2) - \frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2 = 0,$$

$$b\|A^{\frac{3}{2}}(h-\rho_2)\| \leq d\|A^{\frac{1}{2}}(g-u_2)\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\|$$

$$\leq d\lambda_{m+1}^{-\frac{1}{2}}\|A(g-u_2)\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\|,$$

$$\|A^{\frac{3}{2}}(h-\rho_2)\| \leq M\lambda_{m+1}^{-\frac{1}{2}}\|A(g-u_2)\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\|. \tag{16}$$

From (15) and (16), it follows that

$$\|A(g-u_2)\| + \|A^{\frac{3}{2}}(h-\rho_2)\|$$

$$\leq M\lambda_{m+1}^{-\frac{1}{2}}\|A^{\frac{3}{2}}(h-\rho_2)\| + M\lambda_{m+1}^{-\frac{1}{2}}\|A(g-u_2)\| + \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\|,$$

$$(1-M\lambda_{m+1}^{-\frac{1}{2}})(\|A(g-u_2)\| + \|A^{\frac{3}{2}}(h-\rho_2)\|) \leq \|\frac{\mathrm{d}u_2}{\mathrm{d}t}\| + \|\frac{\mathrm{d}}{\mathrm{d}t}A^{\frac{1}{2}}\rho_2\| \leq M\lambda_{m+1}^{-\frac{1}{2}}.$$

There exists $m_0$ such that when $m \geq m_0$, it follows that

$$\|A(g-u_2)\| + \|A^{\frac{3}{2}}(h-\rho_2)\| \leq M\lambda_{m+1}^{-\frac{1}{2}},$$

$$\|g-u_2\| + \|A^{\frac{1}{2}}(h-\rho_2)\| \leq M\lambda_{m+1}^{-\frac{3}{2}}.$$

Thus

$$\mathrm{dist}_{H\times H}((u, A^{\frac{1}{2}}\rho), \Sigma_2) \leq \|u-(u_1+g)\| + \|A^{\frac{1}{2}}[\rho-(\rho_1+h)]\|$$

$$\leq \|u_2-g\| + \|A^{\frac{1}{2}}(\rho_2-h\| \leq M\lambda_{m+1}^{-\frac{3}{2}}, \quad m \geq m_0,\ t \geq t_*. \ \square$$

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